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METRIC DIMENSION OF INDU-BALA PRODUCT OF GRAPHS

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Abstract. In a simple connected graph A, a set of vertices A' resolves A if every vertex of A is uniquely represented by its vector of distances to the vertices in A'. A resolving set containing the smallest number of vertices is known as basis for A and its cardinality is called metric dimension of A. The Indu-Bala product $A_1 \blacktriangledown A_2$ of graphs A_1 and A_2 is obtained from two disjoint copies of $A_1 + A_2$ by joining the corresponding vertices in the two copies of A_2. In this paper, we derive the metric dimension of Indu-Bala product of some families of graphs.

1. Introduction

Throughout the article, all examined graphs are connected and simple. For a graph A, the vertex and edge sets are denoted as $\mathcal{V}(A)$ and $\mathcal{E}(A)$, respectively. For $a_t, a_s \in \mathcal{V}(A)$, the distance among two vertices is represented by $d_A(a_t, a_s)$ and defined as the length of the shortest path in A from a_t to a_s. The graphs $\mathcal{P}_n$ and $\mathcal{C}_n$ present the path and the cycle, respectively, with n vertices. A pair of vertices $a_t, a_s \in \mathcal{V}(A)$ resolved by a vertex a' of A if $d_A(a', a_s) \neq d_A(a', a_t)$. For a set of vertices $A' = \{a'_1, a'_2, \ldots, a'_k\} \subseteq \mathcal{V}(A)$, the metric representation of $a_t \in \mathcal{V}(A)$ with reference to A' is the k-tuple

$$\mathrm{r}(a_t|A') = (d_A(a_t, a'_1), d_A(a_t, a'_2), \ldots, d_A(a_t, a'_k)).$$

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A set A' is recognized as a resolving set for A if $\mathrm{r}(a_t|A') \neq \mathrm{r}(a_s|A')$ for every pair of distinct vertices $a_t, a_s \in \mathcal{V}(A)$. The metric dimension of A is the smallest cardinality of any resolving set for A, and denoted as $\dim(A)$. If $\dim(A) = k$, then A is said to be a k-dimensional.

The idea of metric dimension was introduced by Slater [?], where the resolving set was called locating set. Later Harary and Melter [10] studied the resolving sets and they introduced the term metric dimension rather than location number. Khuller et al. [16] discussed applications of metric dimension to the navigation of robots in networks. Applications of metric dimension in chemistry are discussed by Johnson [13, 14]. Several variations of metric dimension have been discussed in the literature, including resolving dominating sets [2], independent resolving sets [7], local metric sets [18], resolving partitions [8], and strong metric generators [?].

Many graph operations show a major part in the computer science, the applied and the pure mathematics, and many other fields of science. A novel graph can be constructed from a given graph by the help of different graph operations, and also a number of chemical graphs can be formed from these graph operations. In these graph operations, Indu-Bala product of different graphs is a very important and novel graph operation. Let A_1 and A_2 be two vertex-disjoint graphs of order n_1 and n_2, and size m_1 and m_2, respectively. The union $A_1 \cup A_2$ of graphs A_1 and A_2 is a graph with $\mathcal{V}(A_1 \cup A_2) = \mathcal{V}(A_1) \cup \mathcal{V}(A_2)$ and $\mathcal{E}(A_1 \cup A_2) = \mathcal{E}(A_1) \cup \mathcal{E}(A_2)$. The order and size of $A_1 \cup A_2$ are $n_1 + n_2$ and $m_1 + m_2$, respectively. The join $A_1 + A_2$ of A_1 and A_2 is a graph union $A_1 \cup A_2$ where all the vertices of A_1 are joining with every vertex of $\mathcal{V}(A_2)$. The order and size of $A_1 + A_2$ are $n_1 + n_2$ and $m_1 + m_2 + n_1 n_2$, respectively. Recently, Indulal and Balakrishnan [11] introduced a new graph operation named Indu-Bala product of graphs. The Indu-Bala product $A_1 \blacktriangledown A_2$ of graphs A_1 and A_2 is obtained from two disjoint copies of $A_1 + A_2$ by joining the corresponding vertices in the two copies of A_2. The order and size of $A_1 \blacktriangledown A_2$ are $2(n_1 + n_2)$ and $2(m_1 + m_2 + n_1 n_2) + n_2$, respectively. The Indu-Bala product of $\mathcal{P}_3$ and $\mathcal{P}_4$ is depicted in Figure 1. Let A'_1

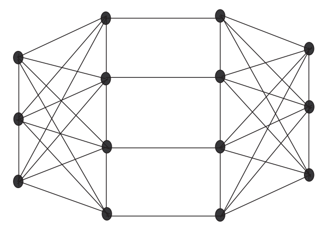

FIGURE 1. $\mathcal{P}_3 \blacktriangledown \mathcal{P}_4$.

and A'_2 be the copies of graphs A_1 and A_2, respectively. Let $\mathcal{V}(A_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $\mathcal{V}(A_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ be the sets of vertices of A_1 and A_2, respectively and $\mathcal{V}(A'_1) = \{v'_1, v'_2, \ldots, v'_{n_1}\}$ and $\mathcal{V}(A'_2) = \{u'_1, u'_2, \ldots, u'_{n_2}\}$ be the sets of vertices of A'_1 and A'_2, respectively. The vertex set of $A_1 \blacktriangledown A_2$ is $\mathcal{V}(A_1) \cup \mathcal{V}(A_2) \cup \mathcal{V}(A'_1) \cup \mathcal{V}(A'_2)$. The distances between all pair of vertices of $A_1 \blacktriangledown A_2$ are given by:

$$d_{A_1 \blacktriangledown A_2}(v_i, v'_j) = 3, \tag{1.1}$$

$$d_{A_1 \blacktriangledown A_2}(v_i, u_k) = d_{A_1 \blacktriangledown A_2}(v'_i, u'_k) = 1, \tag{1.2}$$

$$d_{A_1 \blacktriangledown A_2}(v_i, u'_k) = d_{A_1 \blacktriangledown A_2}(v'_i, u_k) = 2, \tag{1.3}$$

$$d_{A_1 \blacktriangledown A_2}(u_k, u_l) = d_{A_1 \blacktriangledown A_2}(u'_k, u'_l) = \min\{2, d_{A_2}(u_k, u_l)\}, \tag{1.4}$$

$$d_{A_1 \blacktriangledown A_2}(v_i, v_j) = d_{A_1 \blacktriangledown A_2}(v'_i, v'_j) = \min\{2, d_{A_1}(v_i, v_j)\}. \tag{1.5}$$

The distance between the vertices of A_2 and A'_2 in $A_1 \blacktriangledown A_2$ is given by:

$$d_{A_1 \blacktriangledown A_2}(u_k, u'_l) = \begin{cases} 1 & \text{if } k = l, \\ 2 & \text{if } u_k u_l \in \mathcal{E}(A_2), \\ 3 & \text{otherwise,} \end{cases} \tag{1.6}$$

where $i, j \in \{1, 2, \ldots, n_1\}$ and $k, l \in \{1, 2, \ldots, n_2\}$.

Yero et al. [5, **?**] computed the metric dimension of Cartesian product and some applications of metric dimensions. Jannesari et al. [12] computed the metric dimension of composition of graphs. Metric dimension have been studied for corona product of graphs [17, ?], Hamming graphs [15], join of graphs [?] and comb product of graphs [?]. For the depth study of metric dimension, we recommend the reader to see [1, 3, 4, 9, 19, ?, ?, ?]. In this paper, we study the metric dimension of Indu-Bala product of some families of graphs.

2. Metric dimension of $\mathcal{P}_{n_1}\blacktriangledown\mathcal{P}_{n_2}$

In this section, we compute the metric dimension of Indu-Bala product of paths. Let $\{p_1, p_2, \ldots, p_{n_1}\}$ and $\{q_1, q_2, \ldots, q_{n_2}\}$ be the sets of vertices of $\mathcal{P}_{n_1}$ and $\mathcal{P}_{n_2}$, respectively. Let $\mathcal{P}'_{n_1}$ and $\mathcal{P}'_{n_2}$ be the copies of paths $\mathcal{P}_{n_1}$ and $\mathcal{P}_{n_2}$, respectively, and $\{p'_1, p'_2, \ldots, p'_{n_1}\}$ and $\{q'_1, q'_2, \ldots, q'_{n_2}\}$ be the sets of vertices of $\mathcal{P}'_{n_1}$ and $\mathcal{P}'_{n_2}$, respectively.

Theorem 2.1. [6] *For an n-vertex connected graph A, we have* $\dim(A) = 1$ *if and only if* $A \cong \mathcal{P}_n$.

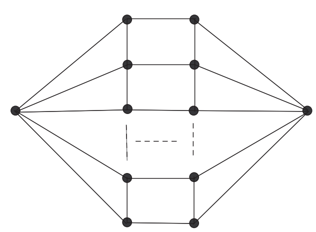

Figure 2. $\mathcal{P}_1\blacktriangledown\mathcal{P}_{n_2}$.

Theorem 2.2. *If $n_2 \geq 1$, then the following holds:*

$$\dim(\mathcal{P}_1\blacktriangledown\mathcal{P}_{n_2}) = \begin{cases} 1 & \text{if } n_2 = 1, \\ 2 & \text{if } n_2 \in \{2, 3, 4\}, \\ 3 & \text{if } n_2 = 5, \\ \lfloor \frac{n_2}{2} \rfloor & \text{if } n_2 \geq 6. \end{cases}$$

Proof. Let $\mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$ (see Figure 2) be the Indu-Bala product of $\mathcal{P}_1$ and $\mathcal{P}_{n_2}$. If $n_2 = 1$, then $\mathcal{P}_1 \blacktriangledown \mathcal{P}_1 \cong \mathcal{P}_4$ and $\dim(\mathcal{P}_1 \blacktriangledown \mathcal{P}_1) = 1$ by Theorem 2.1. If $n_2 \geq 2$, then $\mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$ is not a path. Therefore $\dim(\mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}) \geq 2$ by Theorem 2.1.

Case 1. Let $A = \mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$ and $n_2 \in \{2, 3\}$. We show that the set $A' = \{p_1, q_1\} \subset \mathcal{V}(A)$ is a resolving set of A. The representation of vertices in $\mathcal{V}(A) \setminus A'$ with reference to A' is given by:

$$\mathrm{r}(p'_1|A') = (3, 2), \quad \mathrm{r}(q_l|A') = (1, l-1), \quad \mathrm{r}(q'_l|A') = (2, l),$$

where $1 \leq l \leq n_2$. We see that all vertices of A have different representations. Therefore $A' = \{p_1, q_1\}$ is a resolving set of A and thus $\dim(A) = 2$.

Case 2. Let $A = \mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$, $n_2 = 4$ and $A' = \{q_1, q_4\} \subset \mathcal{V}(A)$. We present that A' is a resolving set for A. For this purpose, we present the representation of vertices in $\mathcal{V}(A) \setminus A'$ with reference to A':

$$\mathrm{r}(p_1|A') = (1, 1), \quad \mathrm{r}(p'_1|A') = (2, 2), \qquad \mathrm{r}(q_2|A') = (1, 2), \qquad \mathrm{r}(q_3|A') = (2, 1),$$
$$\mathrm{r}(q'_l|A') = (l, 3), \quad 1 \leq l \leq n_2 - 2, \quad \mathrm{r}(q'_l|A') = (3, n_2 - l + 1), \quad n_2 - 1 \leq l \leq n_2.$$

From the above representation of vertices, we see that all vertices of A can be resolved by the set of vertices in A'. Therefore, $A' = \{q_1, q_4\}$ is a resolving set of A and thus $\dim(A) = 2$.

Case 3. Let $A = \mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$, $n_2 = 5$ and $A' = \{q_1, q_3, q_5\} \subset \mathcal{V}(A)$. We show that A' is a resolving set of A. For this purpose, we describe the representation of vertices in $\mathcal{V}(A) \setminus A'$ with reference to A':

$$\mathrm{r}(p_1|A') = (1, 1, 1), \quad \mathrm{r}(p'_1|A') = (2, 2, 2), \quad \mathrm{r}(q_2|A') = (1, 1, 2), \quad \mathrm{r}(q_4|A') = (2, 2, 1),$$

$$\mathrm{r}(q'_l|A') = (l, n_2 - l - 1, 3), \qquad 1 \leq l \leq n_2 - 2,$$
$$\mathrm{r}(q'_l|A') = (3, l - 2, n_2 - l + 1), \quad n_2 - 1 \leq l \leq n_2.$$

From the above representation of vertices, we see that all vertices of A can be resolved by the set of vertices in A'. Therefore, $A' = \{q_1, q_3, q_5\}$ is a resolving set of A and thus $\dim(A) = 3$.

Case 4. Let $A = \mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$, $n_2 \equiv 0\ (mod\ 2)$, $n_2 \geq 6$ and $\mathcal{V}(A) = \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}) \cup \{p_1, p'_1\}$. First we show that $\dim(A) \geq \frac{n_2}{2}$ by giving reasoning that there is no resolving set with $\left(\frac{n_2}{2} - 1\right)$ cardinality. Let $\mathcal{V}(\mathcal{P}_{n_2}) = \{q_k \mid 1 \leq k \leq n_2\}$ and $\mathcal{V}(\mathcal{P}'_{n_2}) = \{q'_l \mid 1 \leq l \leq n_2\}$ be the subsets of $\mathcal{V}(A)$. Let A'_1 be a resolving set such that $|A'_1| = \frac{n_2}{2} - 1$. Then, there are the following possibilities:

- If $A'_1 \subset \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2})$, then a vertex $q \in \mathcal{V}(\mathcal{P}_{n_2}) \setminus A'_1$ and p'_1 have the same representation, because q and p'_1 have some equal distances by simple computation. Also, a vertex $q' \in \mathcal{V}(\mathcal{P}'_{n_2}) \setminus A'_1$ and p_1 have the same distance from the vertices in A'_1.
- If A'_1 is a resolving set containing the vertex p_1 (or p'_1) and $\left(\frac{n_2}{2} - 2\right)$ number of vertices from $\mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(P'_{n_2})$, then a pair of vertices of $(\mathcal{V}(\mathcal{P}_{n_2}) \setminus A'_1) \cup (\mathcal{V}(\mathcal{P}'{}_{n_2}) \setminus A'_1)$ have the same distance from the vertices in A'_1 in the structure.

The above cases show that there is no resolving set A'_1 with $|A'_1| = \frac{n_2}{2} - 1$. Thus $\dim(A) \geq \frac{n_2}{2}$. Now, we need to show that $\dim(A) \leq \frac{n_2}{2}$. Let $A' = \{q_2, q_4, \ldots, q_{n_2}\} \subset \mathcal{V}(A)$. We show that A' is the resolving set of A. For this purpose, the representation of vertices in $\mathcal{V}(A) \setminus A'$ with reference to A' is given below:

$\mathrm{r}(p_1|A') = (1, 1, 1, \ldots, 1)$,$\mathrm{r}(p'_1|A') = (2, 2, \ldots, 2)$,

$\mathrm{r}(q_1|A') = (1, 2, 2, \ldots, 2)$,$\mathrm{r}(q_3|A') = (1, 1, 2, \ldots, 2), \ldots, \mathrm{r}(q_{n_2-1}|A') = (2, 2, \ldots, 2, 1, 1)$,

$\mathrm{r}(q'_1|A') = (2, 3, 3, \ldots, 3)$,$\mathrm{r}(q'_3|A') = (2, 2, 3, \ldots, 3), \ldots, \mathrm{r}(q'_{n_2-1}|A') = (3, \ldots, 3, 2, 2)$,

$\mathrm{r}(q'_2|A') = (1, 3, 3, \ldots, 3)$,$\mathrm{r}(q'_4|A') = (3, 1, 3, \ldots, 3), \ldots, \mathrm{r}(q'_{n_2}|A') = (3, \ldots, 3, 1)$.

This implies that all vertices have different representations with reference to A'. Thus A' is a resolving set of A and $\dim(A) \leq \frac{n_2}{2}$. So from above, we conclude that

$$\dim(A) = \frac{n_2}{2}. \tag{2.1}$$

Similarly we can prove that $A = \{q_2, q_4, \ldots, q_{n_2-1}\}$ is a resolving set for $A = \mathcal{P}_1 \blacktriangledown \mathcal{P}_{n_2}$, with $n_2 \equiv 1(mod 2)$, $n_2 \geq 7$ and thus

$$\dim(A) = \frac{n_2 - 1}{2}. \tag{2.2}$$

From equations (2.1) and (2.2), we get $\dim(A) = \left\lfloor \frac{n_2}{2} \right\rfloor$. This gives the desired result. $\square$

Theorem 2.3. *If $n_1 \in \{2, 4\}$ and $n_2 \geq 1$, then we have*

$$\dim(\mathcal{P}_{n_1} \blacktriangledown \mathcal{P}_{n_2}) = \begin{cases} \frac{n_2}{2} + n_1 & \text{if } n_2 \in \{2, 4\}, \\ \left\lfloor \frac{n_2 - 1}{2} \right\rfloor + n_1 & \text{if } n_2 \in \{1, 3, 5, 6, 7 \ldots\}. \end{cases}$$

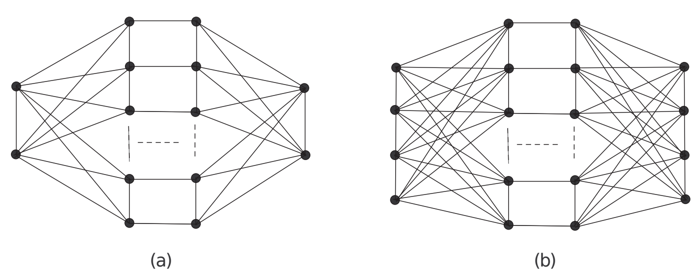

FIGURE 3. (a). $\mathcal{P}_2 \blacktriangledown \mathcal{P}_{n_2}$ (b). $\mathcal{P}_4 \blacktriangledown \mathcal{P}_{n_2}$.

Proof. Let $A = \mathcal{P}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$ (see Figure 3) be the Indu-Bala product of $\mathcal{P}_{n_1}$ and $\mathcal{P}_{n_2}$. Now, we can convert this theorem in two cases.

Case 1. If $n_2 \in \{2, 4\}$, then we take $A' = \{p_1, p_{\frac{n_1}{2}}, q_1, q_{\frac{n_2}{2}}, p'_1, p'_{\frac{n_1}{2}}\} \subset \mathcal{V}(A)$. We show that A' is a resolving set of A. Let $\mathcal{J} = \{p_1, p_2, \cdots, p_{\frac{n_1}{2}}\}$, $\mathcal{K} = \{p'_1, p'_2, \cdots, p'_{\frac{n_1}{2}}\}$, $\mathcal{L} = \{q_1, q_2, \cdots, q_{\frac{n_2}{2}}\}$ and $\mathcal{M} = \{q'_1, q'_2, \cdots, q'_{\frac{n_2}{2}}\}$ be the subsets of $\mathcal{V}(A) = \mathcal{J} \cup \mathcal{K} \cup \mathcal{L} \cup \mathcal{M}$.

From distances (1.1), we see that the vertices $p_1, p_{\frac{n_1}{2}}$ resolve the vertices of set $\mathcal{J}$ and $p'_1, p'_{\frac{n_1}{2}}$ resolve the vertices of set $\mathcal{K}$. From distances (1.1) and (1.6), we see that the vertices $q_1, q_{\frac{n_2}{2}}$ resolve the vertices of sets $\mathcal{L}$ and $\mathcal{M}$. Which implies that a resolving set of A is A' and therefore $\dim(A) \leq \frac{n_2}{2} + n_1$.
On the other side we show that $\dim(A) \geq \frac{n_2}{2} + n_1$ by showing that there is no resolving set with cardinality $\left(\frac{n_2}{2} + n_1 - 1\right)$. On contrary we suppose A'_2 is a resolving set of A with $|A'_2| = \left(\frac{n_2}{2} + n_1 - 1\right)$. Then, we consider the following possibilities:

- If $A'_2 \subset (\mathcal{J} \cup \mathcal{K} \cup \mathcal{L})$ (or $A'_2 \subset (\mathcal{J} \cup \mathcal{L} \cup \mathcal{M})$), then there are the following two possibilities:
  (1) If the set A'_2 contains $\frac{n_2}{2}$ number of vertices of $\mathcal{L}$ (or $\mathcal{M}$) and other vertices of A'_2 belongs to $\mathcal{J} \cup \mathcal{K}$, then a pair of vertices in set $(\mathcal{J} \setminus A'_2) \cup (\mathcal{K} \setminus A'_2)$ have the same representation, by simple computation.
  (2) If the set A'_2 contains $\frac{n_2}{2} - 1$ number of vertices of $\mathcal{L}$ (or $\mathcal{M}$) and other vertices of A'_2 belongs to $\mathcal{J} \cup \mathcal{K}$, then a pair of vertices in $\mathcal{M}$ (or $\mathcal{L}$) have the same representation, by the simple computation.
- If $A'_2 \subset (\mathcal{J} \cup \mathcal{L} \cup \mathcal{M})$, then the vertices in the $\mathcal{K}$ can not be resolved by the vertices in A'_2, because some vertices of $\mathcal{K}$ have equal distances in the structure. Similarly if $A'_2 \subset (\mathcal{K} \cup \mathcal{L} \cup \mathcal{M})$, then the vertices in the $\mathcal{J}$ can not be resolved by the vertices in A'_2, because some vertices of $\mathcal{J}$ have equal distances in the structure.

From above cases we can derive that there is no a resolving set A'_2 with $\left(\frac{n_2}{2} + n_1 - 1\right)$ number of vertices of A. Hence $\dim(A) \geq \frac{n_2}{2} + n_1$. Thus

$$\dim(A) = \frac{n_2}{2} + n_1.$$

Case 2. If $n_2 \equiv 1(mod\ 2)$, then $\mathcal{J} = \{p_r \mid r \in \{1, 2, \ldots, n_1\}\}$, $\mathcal{K} = \{p'_s \mid s \in \{1, 2, \ldots, n_1\}\}$, $\mathcal{L} = \{q_t \mid t \in \{1, 2, \ldots, n_2\}\}$ and $\mathcal{M} = \{q'_l \mid l \in \{1, 2, \ldots, n_2\}\}$ are the subsets of $\mathcal{V}(A) = \mathcal{J} \cup \mathcal{K} \cup \mathcal{L} \cup \mathcal{M}$. Initially we prove that $\dim(A) \geq \frac{n_2 - 1}{2} + n_1$.

Suppose that A'' is the resolving set of A with cardinality $\dfrac{n_2-1}{2}+n_1-1$. Then there are following cases:

- If $A'' \subset (\mathcal{J}\cup\mathcal{K}\cup\mathcal{L})$ (or $A'' \subset (\mathcal{J}\cup\mathcal{K}\cup\mathcal{M})$), then there exists following two cases:
  (1) If the resolving set A'' contains $\frac{n_2-1}{2}$ number of vertices of $\mathcal{L}$ (or $\mathcal{M}$) and other vertices of A'' belongs to $\mathcal{J}\cup\mathcal{K}$, then a pair of vertices in set $(\mathcal{J}\setminus A'')\cup(\mathcal{K}\setminus A'')$ have the same distance from the vertices in A'', by the simple computation.
  (2) If the resolving set A'' contains $\frac{n_2-1}{2}-1$ number of vertices of $\mathcal{L}$ (or $\mathcal{M}$) and other vertices of A'' belongs to $\mathcal{J}\cup\mathcal{K}$, then a pair of vertices in $\mathcal{M}$ (or $\mathcal{L}$)have the same distance from the vertices of A'', by the simple computation.
- If $A'' \subset (\mathcal{K}\cup\mathcal{L}\cup\mathcal{M})$, then the vertices in $\mathcal{J}$ can not be resolved by the vertices in A'', because some vertices of $\mathcal{J}$ have equal distances in the structure. Similarly if $A'' \subset (\mathcal{J}\cup\mathcal{L}\cup\mathcal{M})$, then the vertices in $\mathcal{K}$ can not be resolved by the vertices in A'', because some vertices of $\mathcal{K}$ have equal distances in the structure.

Therefore in every case we get a contradiction. Thus we conclude that there does not exists a resolving set A'' containing $\dfrac{n_2-1}{2}+n_1-1$ vertices of A. Therefore $\dim(A)\geq \dfrac{n_2-1}{2}+n_1$.

Now we find a resolving set A' which contains exactly $\dfrac{n_2-1}{2}+n_1$ number of vertices of A. Let $A'=\{p_i,p'_j,q_k \mid i,j\in\{1,3,\ldots,n_1-1\}\ and\ k\in\{1,3,\ldots,n_2-2\}\}\subset\mathcal{V}(A)$. We prove that A' is a resolving set of A. From distances (1.1) and (1.6), we see that the vertices $q_k\in\mathcal{L}$, $k\in\{1,3,\ldots,n_2-2\}$, resolve the vertices of sets $\mathcal{L}$ and $\mathcal{M}$. The vertices p_i and p'_j have same distance from the vertices of sets $\mathcal{L}$ and $\mathcal{M}$, therefore the vertices p_i and p'_j, for $i,j\in\{1,3,\ldots,n_1-1\}$, resolve the p_i and p'_j, for $i, j \in \{2, 4, \ldots, n_1\}$, respectively. This implies that A' is a resolving set of A and $\dim(A) \leq \frac{n_2 - 1}{2} + n_1$. Therefore we conclude that

$$\dim(A) = \frac{n_2 - 1}{2} + n_1. \tag{2.3}$$

Similarly $A' = \{p_i, p'_j, q_k \mid i, j \in \{1, 3, \ldots, n_1 - 1\}$ *and* $k \in \{2, 4, \ldots, n_2 - 2\}\} \subset \mathcal{V}(A)$ is a resolving set of A for $n_2 \equiv 0(mod\ 2)$, $n_2 \geq 6$ and

$$\dim(A) = \frac{n_2 - 2}{2} + n_1. \tag{2.4}$$

From equations (2.3) and (2.4), we get

$$\dim(A) = \left\lfloor \frac{n_2 - 1}{2} \right\rfloor + n_1.$$

This completes the proof. □

Theorem 2.4. *If $n_2 \geq 1$, then following holds:*

(1) *If $n_1 \equiv 0(mod\ 2)$, $n_1 \geq 6$, then*

$$\dim(\mathcal{P}_{n_1} \blacktriangledown \mathcal{P}_{n_2}) = \begin{cases} \frac{n_2}{2} + n_1 - 2 & \text{if } n_2 \in \{2, 4\}, \\ \left\lfloor \frac{n_2 - 1}{2} \right\rfloor + n_1 - 2 & \text{if } n_2 \in \{1, 3, 5, 6, 7, \ldots\}. \end{cases}$$

(2) *If $n_1 \equiv 1(mod\ 2)$, $n_1 \geq 3$, then*

$$\dim(\mathcal{P}_{n_1} \blacktriangledown \mathcal{P}_{n_2}) = \begin{cases} \frac{n_2}{2} + n_1 - 1 & \text{if } n_2 \in \{2, 4\}, \\ \left\lfloor \frac{n_2 - 1}{2} \right\rfloor + n_1 - 1 & \text{if } n_2 \in \{1, 3, 5, 6, 7, \ldots\}. \end{cases}$$

Proof. Let $A = \mathcal{P}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$ and $n_1 \equiv 0(mod\ 2)$, $n_1 \geq 6$ (see Figure 4). Let $\mathcal{V}(\mathcal{P}_{n_1}) = \{p_r \mid r \in \{1, 2, \ldots, n_1\}\}$, $\mathcal{V}(\mathcal{P}'_{n_1}) = \{p'_s \mid s \in \{1, 2, \ldots, n_1\}\}$, $\mathcal{V}(\mathcal{P}_{n_2}) = \{q_t \mid t \in \{1, 2, \ldots, n_2\}\}$ and $\mathcal{V}(\mathcal{P}'_{n_2}) = \{q'_l \mid l \in \{1, 2, \ldots, n_2\}\}$ be the subsets of $\mathcal{V}(A) = \mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2})$.

Case 1. Let $n_2 \in \{2, 4\}$ and $A' = \{p_2, p_4, \ldots, p_{n_1 - 2}, q_1, q_{\frac{n_2}{2}}, p'_2, p'_4, \ldots, p'_{n_1 - 2}\} \subset$

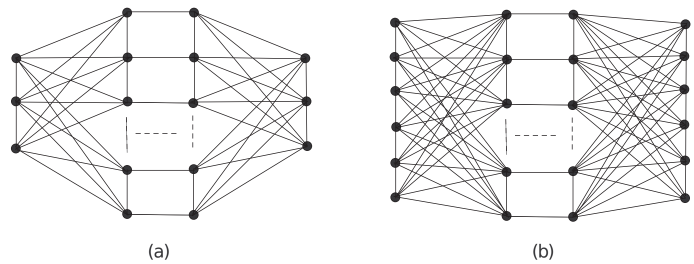

FIGURE 4. (a). $\mathcal{P}_3 \blacktriangledown \mathcal{P}_{n_2}$ (b). $\mathcal{P}_6 \blacktriangledown \mathcal{P}_{n_2}$.

$\mathcal{V}(P_{n_2})$. We show that A' is a resolving set of A. From (1.1) and (1.6), we see that the vertices p_{r_1} and p_{r_2} for each $r_1, r_2 \in \{2, 4, \cdots, n_1 - 2\}$ resolve the vertices in $\mathcal{V}(\mathcal{P}_{n_1})$ and $\mathcal{V}(\mathcal{P}'_{n_1})$, respectively. However, the vertices q_t, $1 \leq t \leq \dfrac{n_2}{2}$, can easily resolve the vertices in sets $\mathcal{V}(\mathcal{P}_{n_2})$ and $\mathcal{V}(\mathcal{P}'_{n_2})$. Thus the set A' is a resolving set of A and

$$\dim(A) \leq \frac{n_2}{2} + n_1 - 2.$$

Now we find that there is no a resolving set with $\left(\dfrac{n_2}{2} + n_1 - 3\right)$ number of vertices. On contrary we suppose that A'_2 is a resolving set of A with $|A'_2| = \dfrac{n_2}{2} + n_1 - 3$. Then there are following cases:

- If $A'_2 \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$, then the vertices p'_s, $1 \leq s \leq n_1$, can not be resolved by the vertices in A'_2. Similarly if $A'_2 \subset (\mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$, then the vertices p_r, $1 \leq r \leq n_1$, can not be resolved by the vertices in A'_2, because these vertices have equal distances in the structure.
- If $A'_2 \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}))$, then there are following two possibilities:
  (1) If the set A'_2 have at least $\frac{n_2}{2}$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in set $(\mathcal{V}(\mathcal{P}_{n_1}) \setminus A'_2) \cup (\mathcal{V}(\mathcal{P}'_{n_1}) \setminus A'_2)$ have the same distance from the vertices in A'_2.
  (2) If the set A'_2 have at least $\frac{n_2}{2} - 1$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in set $\mathcal{V}(\mathcal{P}'_{n_2})$ have the same distance from the vertices in A'_2.

Similarly $A'_2 \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$ is not a resolving set of A.

From above cases, we get that there is no any resolving set A'_2 with $|A'_2| = \frac{n_2}{2}+n_1-3$. Hence $\dim(A) \geq \frac{n_2}{2} + n_1 - 2$. This implies that $\dim(A) = \frac{n_2}{2} + n_1 - 2$ in this case.

Case 2. Let $n_2 = 1$ and $n_2 \equiv 1(mod 2)$. First we drive that $\dim(A) \geq \frac{n_2-1}{2}+n_1-2$. On contrary we suppose that there is a resolving set A'' of A with $|A''| = \frac{n_2-1}{2} + n_1 - 3$. Then there are following cases:

- If $A'' \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$, then the vertices p'_s, $1 \leq s \leq n_1$, can not be resolved by A''. Similarly if $A'' \subset (\mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$, then the vertices p_r, $1 \leq r \leq n_1$, can not be resolved by A'', because these vertices have equal distances in the structure.
- If $A'' \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(P'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}))$, then there are following two possibilities:
  (1) If the set A'' have at least $\frac{n_2-1}{2}$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in set $(\mathcal{V}(\mathcal{P}_{n_1}) \setminus A'') \cup (\mathcal{V}(\mathcal{P}'_{n_1}) \setminus A'')$ have the same distance from the vertices in A'.
  (2) If the set A'' have at least $\frac{n_2-1}{2} - 1$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in set $\mathcal{V}(\mathcal{P}'_{n_2})$ have the same distance from the vertices in A''.

  Similarly we can prove that $A'' \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$ is not a resolving set of A.

Therefore from above discussed cases, we get conclusion that A'' with $\frac{n_2-1}{2}+n_1-2$ vertices is not a resolving set of A. Thus $\dim(A) \geq \frac{n_2-1}{2} + n_1 - 2$.

Next we determine that A' is a resolving set of A having $\left(\frac{n_2-1}{2} + n_1 - 2\right)$ number of vertices. Let $A' = \{p_r, q_t, p'_s \mid r, s \in \{2, 4, \ldots, n_1 - 2\} \ and \ t \in \{1, 3, \ldots, n_2 - 2\}\} \subset \mathcal{V}(P_{n_2})$. We prove that A' is a resolving set of A. We present the representation of distances in $\mathcal{V}(\mathcal{P}_{n_2}) \setminus A'$ with reference to A':

$$\mathrm{r}(q_2|A') = (\overbrace{1,\ldots,1}^{p_r},\overbrace{1,1,2\ldots,2,2}^{q_t},\overbrace{2,\ldots,2}^{p'_s})$$

$$\mathrm{r}(q_4|A') = (\overbrace{1,\ldots,1}^{p_r},\overbrace{2,1,1,2,\ldots,2}^{q_t},\overbrace{2,\ldots,2}^{p'_s})$$

$$\vdots$$

$$\mathrm{r}(q_{n_2-1}|A') = (\overbrace{1,\ldots,1}^{p_r},\overbrace{2\ldots,2,1}^{q_t},\overbrace{2,\ldots,2}^{p'_s})$$

$$\mathrm{r}(q_{n_2}|A') = (\overbrace{1,\ldots,1}^{p_r},\overbrace{2,2,\ldots,2}^{q_t},\overbrace{2,\ldots,2}^{p'_s}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{P}_{n_2}) \setminus A'$ is given below:

$$\mathrm{r}(q'_2|A') = (\overbrace{2,2,\ldots,2,2,2}^{p_r},\overbrace{2,2,3\ldots,3,3}^{q_t},\overbrace{1,1,\ldots,1}^{p'_s})$$

$$\mathrm{r}(q'_4|A') = (\overbrace{2,2,\ldots,2}^{p_r},\overbrace{3,2,2,3\ldots,3}^{q_t},\overbrace{1,1\ldots,1}^{p'_s})$$

$$\vdots$$

$$\mathrm{r}(q'_{n_2-1}|A') = (\overbrace{2,2,\ldots,2}^{p_r},\overbrace{3,3\ldots,3,2}^{q_t},\overbrace{1,1\ldots,1}^{p'_s}).$$

$$\mathrm{r}(q'_1|A') = (\overbrace{2,2,\ldots,2,2,2}^{p_r},\overbrace{1,3\ldots,3,3}^{q_t},\overbrace{1,1,\ldots,1}^{p'_s})$$

$$\mathrm{r}(q'_3|A') = (\overbrace{2,2,\ldots,2}^{p_r},\overbrace{3,1,3\ldots,3}^{q_t},\overbrace{1,1\ldots,1}^{p'_s})$$

$$\vdots$$

$$\mathrm{r}(q'_{n_2-2}|A') = (\overbrace{2,2,\ldots,2}^{p_r},\overbrace{3,3\ldots,3,1}^{q_t},\overbrace{1,1\ldots,1}^{p'_s})$$

$$\mathrm{r}(q'_{n_2}|A') = (\overbrace{2,2,\ldots,2}^{p_r},\overbrace{3,\ldots,3}^{q_t},\overbrace{1,1,\ldots,1,1}^{p'_s}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{P}_{n_1}) \setminus A'$ is given below:

$$\mathrm{r}(p_1|A') = (\overbrace{1,2,2,\ldots,2,2}^{p_r}, \overbrace{1,1,\ldots,1,1}^{q_t}, \overbrace{3,3\ldots,3}^{p'_s})$$

$$\mathrm{r}(p_3|A') = (\overbrace{1,1,2,2,\ldots,2,2}^{p_r}, \overbrace{1,1,\ldots,1,1}^{q_t}, \overbrace{3,3\ldots,3}^{p'_s})$$

$$\mathrm{r}(p_5|A') = (\overbrace{2,1,1,2,\ldots,2,2}^{p_r}, \overbrace{1,1,\ldots,1,1}^{q_t}, \overbrace{3,3\ldots,3}^{p'_s})$$

$$\vdots$$

$$\mathrm{r}(p_{n_1-1}|A') = (\overbrace{2,2,\ldots,2,1}^{p_r}, \overbrace{1,1,\ldots,1,1}^{q_t}, \overbrace{3,3,\ldots,3}^{p'_s})$$

$$\mathrm{r}(p_{n_1}|A') = (\overbrace{2,2,\ldots,2,2}^{p_r}, \overbrace{1,1,\ldots,1,1}^{q_t}, \overbrace{3,3,\ldots,3}^{p'_s}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{P}'_{n_1}) \setminus A'$ is given below:

$$\mathrm{r}(p'_1|A') = (\overbrace{3,3\ldots,3}^{p_r}, \overbrace{2,2,\ldots,2,2}^{q_t}, \overbrace{1,2,2,\ldots,2,2}^{p'_s})$$

$$\mathrm{r}(p'_3|A') = (\overbrace{3,3\ldots,3}^{p_r}, \overbrace{2,2,\ldots,2,2}^{q_t}, \overbrace{1,1,2,2,\ldots,2,2}^{p'_s})$$

$$\mathrm{r}(p'_5|A') = (\overbrace{3,3\ldots,3}^{p_r}, \overbrace{2,2,\ldots,2,2}^{q_t}, \overbrace{2,1,1,2,\ldots,2,2}^{p'_s})$$

$$\vdots$$

$$\mathrm{r}(p'_{n_1-1}|A') = (\overbrace{3,3,\ldots,3}^{p_r}, \overbrace{2,2,\ldots,2,2}^{q_t}, \overbrace{2,2,\ldots,2,1}^{p'_s})$$

$$\mathrm{r}(p'_{n_1}|A') = (\overbrace{3,3,\ldots,3}^{p_r}, \overbrace{2,2,\ldots,2,2}^{q_t}, \overbrace{2,2,\ldots,2}^{p'_s}).$$

One can easy to see that there are no two vertices having the same representations. Thus A' is a resolving set of A and $\dim(A) \leq \frac{n_2-1}{2} + n_1 - 2$. Hence

$$\dim(A) = \frac{n_2-1}{2} + n_1 - 2. \tag{2.5}$$

Case 3. Let $n_2 \equiv 0(mod\ 2)$. First we prove that $\dim(A) \geq \frac{n_2}{2} + n_1 - 2$. On contrary we suppose that there is a resolving set A'_2 of A with $|A'_2| = \frac{n_2}{2} + n_1 - 3$. Then there are following cases:

- If $A'_2 \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$, then the vertices p'_s, $1 \leq s \leq n_1$, can not be resolved by the vertices in A'_2. Similarly if $A'_2 \subset (\mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(P'_{n_2}))$, then the vertices p_r, $1 \leq r \leq n_1$, can not be resolved by the vertices in A'_2, because these vertices have equal distances in the structure.
- If $A'_2 \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}))$, then there are following two possibilities:
  (1) If the set A'_2 have at least $\frac{n_2}{2}$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in set $\mathcal{V}(\mathcal{P}_{n_1}) \setminus A'_2$ and $\mathcal{V}(\mathcal{P}'_{n_1}) \setminus A'_2$ have the same distance from the vertices in A'_2.
  (2) If the set A'_2 have at least $\frac{n_2}{2} - 1$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in set $\mathcal{V}(\mathcal{P}'_{n_2})$ have the same distance from the vertices in A'_2.

  Similarly we can prove that $A'_2 \subset (\mathcal{V}(\mathcal{P}_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$ is not a resolving set of A.

Thus, in every case we obtain a contradiction. Therefore any resolving set of A contains at least $\frac{n_2}{2} + n_1 - 2$ vertices and we get $\dim(A) \geq \frac{n_2}{2} + n_1 - 2$.
Now we find the resolving set of A having exactly $\left(\frac{n_2}{2} + n_1 - 2\right)$ number of vertices of A. Let $A' = \{p_r, q_t, p'_s \mid r, s \in \{2, 4, \ldots, n_1 - 2\}\ and\ t \in \{2, 4, \ldots, n_2 - 2\}\} \subset \mathcal{V}(A)$. Now we derive that A' is a set of vertices to resolve the vertices in $\mathcal{V}(A) \setminus A'$. Let $A'_1 = \{p_r, p'_s \mid r, s \in \{2, 4, \ldots, n_1 - 2\}\} \subset A'$. We describe the representation of vertices $\mathcal{V}(A) \setminus A'_1$ with reference to A'_1:

$$\mathrm{r}(q_t|A'_1) = (\overbrace{1, 1, \ldots, 1, 1}^{p_r}, \overbrace{2, 2, \ldots, 2}^{p'_s}), \quad 1 \leq t \leq n_2,$$

$$\mathrm{r}(q'_l|A'_1) = (\overbrace{2, 2, \ldots, 2, 2}^{p_r}, \overbrace{1, 1, \ldots, 1}^{p'_s}), \quad 1 \leq l \leq n_2,$$

$$\mathrm{r}(p_{n_1}|A'_1) = (\overbrace{2,2,\ldots,2,2}^{p_r},\overbrace{1,1,\ldots,1,1}^{p'_s}), \quad \mathrm{r}(p'_{n_1}|A'_1) = (\overbrace{2,2,\ldots,2,2}^{p_r},\overbrace{1,1,\ldots,1}^{p'_s}).$$

The representation of vertices p_i and p'_j, for $i,j \in \{1,3,\ldots,n_1-1\}$ are

$$\mathrm{r}(p_1|A'_1) = (\overbrace{1,2,2,\ldots,2,2}^{p_r},\overbrace{3,3\ldots,3}^{p'_s})$$
$$\mathrm{r}(p_3|A'_1) = (\overbrace{1,1,2,2,\ldots,2,2}^{p_r},\overbrace{3,3\ldots,3}^{p'_s})$$
$$\mathrm{r}(p_5|A'_1) = (\overbrace{2,1,1,2,\ldots,2,2}^{p_r},\overbrace{3,3\ldots,3}^{p'_s})$$
$$\vdots$$
$$\mathrm{r}(p_{n_1-1}|A'_1) = (\overbrace{2,2,\ldots,2,1}^{p_r},\overbrace{3,3,\ldots,3}^{p'_s}).$$
$$\mathrm{r}(p'_1|A'_1) = (\overbrace{3,3\ldots,3}^{p_r},\overbrace{1,2,2,\ldots,2,2}^{p'_s})$$
$$\mathrm{r}(p'_3|A'_1) = (\overbrace{3,3\ldots,3}^{p_r},\overbrace{1,1,2,2,\ldots,2,2}^{p'_s})$$
$$\mathrm{r}(p'_5|A'_1) = (\overbrace{3,3\ldots,3}^{p_r},\overbrace{2,1,1,2,\ldots,2,2}^{p'_s})$$
$$\vdots$$
$$\mathrm{r}(p'_{n_1-1}|A'_1) = (\overbrace{3,3,\ldots,3}^{p_r},\overbrace{2,2,\ldots,2,1}^{p'_s}).$$

From above representations of vertices we see that all vertices in $\mathcal{V}(A) \setminus A'$ have distinct representations except the vertices $p_{n_1}, q'_l, p'_{n_1}, q_t,$, $t,l \in \{1,2,\ldots,n_2\}$. The vertices q_t for each $t \in \{2,4,\ldots,n_2-2\}$ resolve the vertices $p_{n_1}, q'_l, p'_{n_1}, q_t$, where $t,l \in \{1,2,\ldots,n_2\}$. This implies that $A' = A'_1 \cup \{q_2,q_4,\ldots,q_{n_2-2}\}$ is a resolving set of A and $\dim(A) \le \dfrac{n_2}{2} + n_1 - 2$. Therefore

$$\dim(A) = \frac{n_2}{2} + n_1 - 2. \tag{2.6}$$

From equations (2.5) and (2.6), we have

$$\dim(A) = \left\lfloor \frac{n_2 - 1}{2} \right\rfloor + n_1 - 2.$$

Similarly we can find the resolving set and the matric dimension of $\mathcal{P}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$ for $n_1 \equiv 1(mod\ 2)$, $n_2 \geq 1$, (see Figure 4). This finishes the proof. □

3. Metric dimension of $\mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$

Khuller et al. [16] and Chartrand et al. [6] showed that $\dim(\mathcal{C}_n) = 2$. In this section, we compute the metric dimension of Indu-Bala product of $\mathcal{C}_{n_1}$ and $\mathcal{P}_{n_2}$. Let $\{c_1, c_2, \ldots, c_{n_1}\}$ and $\{p_1, p_2, \ldots, p_{n_2}\}$ be the set of vertices of $\mathcal{C}_{n_1}$ and $\mathcal{P}_{n_2}$, respectively. Let $\mathcal{C}'_{n_1}$ and $\mathcal{P}'_{n_2}$ be the copies of $\mathcal{C}_{n_1}$ and $\mathcal{P}_{n_2}$, respectively and $\mathcal{V}(\mathcal{C}'_{n_1}) = \{c'_1, c'_2, \ldots, c'_{n_1}\}$ and $\mathcal{V}(\mathcal{P}_{n_2}) = \{p'_1, p'_2, \ldots, p'_{n_2}\}$. Let $A = \mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$ be a Indu-Bala product of $\mathcal{C}_{n_1}$ and $\mathcal{P}_{n_2}$ with vertex set $\mathcal{V}(A) = \mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{C}'_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_2})$.

Theorem 3.1. *If $n_1 \in \{3, 4\}$ and $n_2 \geq 2$, then we have* $\dim(\mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}) = \left\lfloor \frac{n_2}{2} \right\rfloor + 4$.

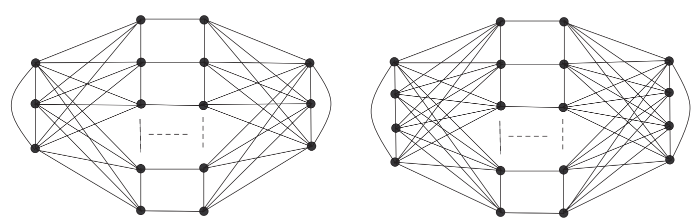

Figure 5. (a). $\mathcal{C}_3 \blacktriangledown \mathcal{P}_{n_2}$ (b). $\mathcal{C}_4 \blacktriangledown \mathcal{P}_{n_2}$.

Proof. Let $A = \mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$, $n_2 \equiv 1(mod\ 2)$ (see Figure 5) and $A' = \{c_1, c_2, p_1, p_3, \ldots, p_{n_2-2}, c'_1, c'_2\} \subset \mathcal{V}(A)$. We need to derive that A' is a resolving set of A. Let $A'' = \{p_t \mid t \in \{1, 3, \ldots, n_2 - 2\}\} \subset A'$. We present the representation of vertices of $\mathcal{V}(A) \setminus A''$

with reference to A'':

$$\mathrm{r}(p_2|A'') = (1,1,2,2,\ldots,2)$$

$$\mathrm{r}(p_4|A'') = (2,1,1,2,\ldots,2,2)$$

$$\vdots$$

$$\mathrm{r}(p_{n_2-1}|A'') = (2,2,\ldots,2,1)$$

$$\mathrm{r}(p_{n_2}|A'') = (2,2,\ldots,2).$$

The representation of vertices $p'_l \in \mathcal{V}(\mathcal{P}'_{n_2})$, $l \in \{2,4,\ldots,n_2-1\}$ is given below:

$$\mathrm{r}(p'_2|A'') = (2,2,3,\ldots,3,3), \mathrm{r}(p'_4|A'') = (3,2,2,3,\ldots,3), \ldots, \mathrm{r}(p'_{n_2-1}|A'') = (3,3,\ldots,3,2).$$

The representation of vertices $p'_l \in \mathcal{V}(\mathcal{P}'_{n_2})$, $l \in \{1,3,\ldots,n_2\}$ is given below:

$$\mathrm{r}(p'_1|A'') = (1,3,\ldots,3,3)$$

$$\mathrm{r}(p'_3|A'') = (3,1,3,\ldots,3)$$

$$\vdots$$

$$\mathrm{r}(p'_{n_2-2}|A'') = (3,3,\ldots,3,1)$$

$$\mathrm{r}(p'_{n_2}|A'') = (3,3,\ldots,3,3).$$

The representations of vertices of set $\mathcal{V}(\mathcal{C}_{n_1})$ and $\mathcal{V}(\mathcal{C}'_{n_1})$ are given below:

$$\mathrm{r}(c_s|A'') = (1,1,\ldots,1,1), \qquad \mathrm{r}(c'_t|A'') = (2,2,\ldots,2,2).$$

where $1 \le s,t \le n_1$. From above we conclude that all vertices in $\mathcal{V}(A) \setminus A''$ have distinct representation except the vertices in sets $\mathcal{V}(\mathcal{C}_{n_1})$ and $\mathcal{V}(\mathcal{C}'_{n_1})$. The vertices c_1, c_2, c'_1, c'_2 resolve the vertices of sets $\mathcal{V}(\mathcal{C}_{n_1})$ and $\mathcal{V}(\mathcal{C}'_{n_1})$. Which implies that $A' = A'' \cup \{c_1, c_2, c'_1, c'_2\}$ is a resolving set of A and $\dim(A) \le \dfrac{n_2-1}{2} + 4$.

Also, we need to find that $\dim(A) \ge \dfrac{n_2-1}{2} + 4$. On contrary, we suppose that there exists a resolving set A'_1 with $|A'_1| = \dfrac{n_2-1}{2} + 3$. Then there are following possibilities:

- If $A'_1 \subset (\mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$, then the vertices of set $\mathcal{C}'_{n_1}$ have same distance from the vertices in A'_1. Similarly, we can prove that $A'_1 \subset (\mathcal{V}(\mathcal{C}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$ is not a resolving set of A, because some vertices have equal distances in the structure.
- If $A'_1 \subset (\mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{C}'_{n_1}) \cup \mathcal{V}(\mathcal{P}_{n_2}))$, then there are two possibilities:
  (1) If the set A'_1 have exactly two vertices of $\mathcal{C}_{n_1}$ and two vertices of $\mathbb{C}'_{n_1}$, then a pair of vertices of $\mathcal{V}(\mathcal{P}'_{n_2}) \setminus A'_1$ have the same distance from the vertices in A'_1.
  (2) If the set A'_1 have at least $\frac{n_2-1}{2}$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices in $\mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{C}'_{n_1})$ have the same distance from the vertices in A'_1.

  Similarly we can prove that $A'_1 \subset (\mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{C}'_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$ does not resolved the elements of $\mathcal{V}(A)$.

Therefore from above cases, we see that any resolving set of A have at least $\dfrac{n_2-1}{2}+4$ vertices and we get $\dim(A) \geq \dfrac{n_2-1}{2} + 4$. Thus $\dim(A) = \dfrac{n_2-1}{2} + 4$.
Similarly we can prove that $A' = \{c_1, c_2, p_2, p_4, \ldots, p_{n_2-2}, c'_1, c'_2\}$ is a resolving set for $A = \mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$, $n_2 \equiv 0(mod\ 2)$ and $\dim(A) = \dfrac{n_2}{2} + 4$. This finishes the proof. □

Theorem 3.2. *If $n_1 \geq 5$ and $n_2 \geq 1$, then following holds:*

$$\dim(\mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}) = \begin{cases} \left\lfloor \dfrac{n_2}{2} \right\rfloor + n_1 - 2 & \textit{if } n_1 \geq 6 \textit{ even}, \\ \\ \left\lfloor \dfrac{n_2}{2} \right\rfloor + n_1 - 1 & \textit{if } n_1 \geq 5 \textit{ odd}. \end{cases}$$

Proof. Let $A = \mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$, $n_1 \equiv 0(mod\ 2)$, $n_1 \geq 6$ (see Figure 6).
Case 1. Let $n_2 \equiv 0(mod\ 2)$. First we show that $\dim(A) \geq \dfrac{n_2}{2} + n_1 - 2$. We suppose contrary that A'_1 with $|A'_1| = \dfrac{n_2}{2} + n_1 - 3$ is a resolving set. Then there are following possibilities:

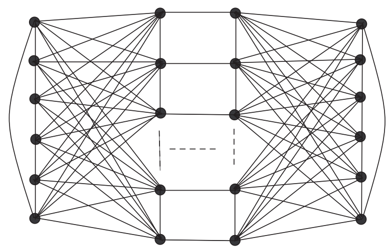

FIGURE 6. $\mathcal{C}_6 \blacktriangledown \mathcal{P}_{n_2}$.

- If A'_1 can not contain any vertex of $\mathcal{C}'_{n_1}$, then all vertices of set $\mathcal{C}'_{n_1}$ have same representation with respect to A'_1. Similarly we can prove that if A'_1 can not contain any vertex of $\mathcal{C}_{n_1}$, then A'_1 is not a resolving set of A, because elements of A'_1 have same representations.
- If A'_1 can not contain any vertex of $\mathcal{V}(\mathcal{P}'_{n_2})$, then there are two possibilities:
  (1) If the set A'_1 have $\dfrac{n_2}{2}$ vertices of set $\mathcal{V}(\mathcal{P}_{n_2})$, then pair of vertices in set $\mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{C}'_{n_1})$ have the same distance from the vertices in A'_1.
  (2) If the set A'_1 have $\dfrac{n_2}{2} - 1$ number of vertices from set $\mathcal{V}(\mathcal{P}_{n_2})$, then a pair of vertices of $\mathcal{V}(\mathcal{P}'_{n_2})$ have the same distance from the vertices in A'_1.

  Similarly we can prove that $A'_1 \subset (\mathcal{V}(\mathcal{C}_{n_1}) \cup \mathcal{V}(\mathcal{C}'_{n_1}) \cup \mathcal{V}(\mathcal{P}'_{n_2}))$ is not a resolving set of A.

Therefore from above cases, we see that any resolving set of A have at least $\dfrac{n_2}{2} + n_1 - 2$ vertices and we get $\dim(A) \geq \dfrac{n_2}{2} + n_1 - 2$. Thus $\dim(A) = \dfrac{n_2}{2} + n_1 - 2$.
Now we determine a resolving set A' which have exactly $\left(\dfrac{n_2}{2} + n_1 - 2\right)$ number of vertices. Let $A' = \{c_s, p_k, c'_t \mid s, t \in \{2, 4, \ldots, n_1 - 2\}$ *and* $k \in \{2, 4, \ldots, n_2 - 2\}\}$. Let $A'' = \{c_s, c'_t \mid s, t \in \{2, 4, \ldots, n_1 - 2\}\} \subset A'$. We determine that the elements of A' resolve all the vertices of A. For this we see the representation of elements of $\mathcal{V}(\mathcal{C}_{n_1}) \setminus A''$ with reference to A'':

$$\mathrm{r}(c_1|A'') = (\overbrace{1, 2, \ldots, 2}^{c_s}, \overbrace{3, \ldots, 3}^{c'_t})$$

$$\mathrm{r}(c_3|A'') = (\overbrace{1,1,2,\ldots,2}^{c_s}, \overbrace{3,\ldots,3}^{c'_t})$$

$$\vdots$$

$$\mathrm{r}(c_{n_2-1}|A'') = (\overbrace{2,\ldots,2,1}^{c_s}, \overbrace{3,\ldots,3}^{c'_t})$$

$$\mathrm{r}(c_{n_2}|A'') = (\overbrace{2,\ldots,2}^{c_s}, \overbrace{3,\ldots,3}^{c'_t}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{C}'_{n_1}) \setminus A''$ is given below:

$$\mathrm{r}(c'_1|A'') = (\overbrace{3,\ldots,3}^{c_s}, \overbrace{1,2,\ldots,2}^{c'_t})$$

$$\mathrm{r}(c'_3|A'') = (\overbrace{3,\ldots,3}^{c_s}, \overbrace{1,1,2,\ldots,2}^{c'_t})$$

$$\vdots$$

$$\mathrm{r}(c'_{n_2-1}|A'') = (\overbrace{3,\ldots,3}^{c_s}, \overbrace{2,\ldots,2,1}^{c'_t})$$

$$\mathrm{r}(c'_{n_2}|A'') = (\overbrace{3,\ldots,3}^{c_s}, \overbrace{2,\ldots,2}^{c'_t}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{P}_{n_2})$ and $\mathcal{V}(\mathcal{P}'_{n_2})$ are given below:

$$\mathrm{r}(p_k|A'') = (\overbrace{1,1,\ldots,1,1}^{c_s}, \overbrace{2,2,\ldots,2,2}^{p'_t}), \quad \mathrm{r}(p'_l|A'') = (\overbrace{2,2,\ldots,2,2}^{c_s}, \overbrace{1,1,\ldots,1,1}^{p'_t}).$$

where $1 \leq k, l \leq n_2$. It can be seen that except the vertices in sets $\mathcal{V}(\mathcal{C}_{n_1})$ and $\mathcal{V}(\mathcal{C}'_{n_1})$, all other elements of $\mathcal{V}(A)$ have same representations with reference to A''. The vertices p_k for $k \in \{2, 4, \ldots, n_2 - 2\}$ resolve the vertices of $\mathcal{V}(\mathcal{P}_{n_2})$ and $\mathcal{V}(\mathcal{P}_{n_2})$. Hence A' is a resolving set of A and $\dim(A) \leq \dfrac{n_2}{2} + n_1 - 2$.

Case 2. Let $n_2 = 1$ and $n_2 \equiv 1(mod\, 2)$. First we show that $\dim(A) \geq \dfrac{n_2 - 1}{2} + n_1 - 2$. There are following possibilities:

- The vertices p_k for $k \in \{1, 3, \ldots, n_2 - 2\}$ resolve the vertices of $\mathcal{C}_{n_1}$ and $\mathcal{C}'_{n_1}$.

- The vertices p'_l for $l \in \{1, 3, \dots, n_2 - 2\}$ resolve the vertices of $\mathcal{C}_{n_1}$ and $\mathcal{C}'_{n_1}$.
- The vertices c_s and c'_t for $s, t \in \{2, 4, \dots, n_1 - 2\}$ resolve the vertices c_s and c'_t for $s, t \in \{1, 3, \dots, n_1 - 1, n_1\}$, respectively.

Hence $\dim(A) \geq \dfrac{n_2 - 1}{2} + n_1 - 2$. Now we find a resolving set A' which consist of exactly $\dfrac{n_2 - 1}{2} + n_1 - 2$ vertices. Let $A' = \{c_s, p_k, c'_t \mid s, t \in \{2, 4, \dots, n_1 - 2\} \textit{ and } k \in \{1, 3, \dots, n_2 - 2\}\}$. We derive that A' is a resolving set of A. Let $A'' = \{c_s, c'_t \mid s, t \in \{2, 4, \dots, n_1 - 2\}\} \subset A'$. Then we describe the representation of vertices of $\mathcal{V}(\mathcal{C}_{n_1}) \setminus A''$ with reference to A'':

$$\mathrm{r}(c_1|A'') = (\overbrace{1, 2, \dots, 2}^{c_s}, \overbrace{3, \dots, 3}^{c'_t})$$

$$\mathrm{r}(c_3|A'') = (\overbrace{1, 1, 2, \dots, 2}^{c_s}, \overbrace{3, \dots, 3}^{c'_t})$$

$$\vdots$$

$$\mathrm{r}(c_{n_2-1}|A'') = (\overbrace{2, \dots, 2, 1}^{c_s}, \overbrace{3, \dots, 3}^{c'_t})$$

$$\mathrm{r}(c_{n_2}|A'') = (\overbrace{2, \dots, 2}^{c_s}, \overbrace{3, \dots, 3}^{c'_t}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{C}'_{n_2}) \setminus A''$ is given below:

$$\mathrm{r}(c'_1|A'') = (\overbrace{3, \dots, 3}^{v_i}, \overbrace{1, 2, \dots, 2}^{v'_j})$$

$$\mathrm{r}(c'_3|A'') = (\overbrace{3, \dots, 3}^{v_i}, \overbrace{1, 1, 2, \dots, 2}^{v'_j})$$

$$\vdots$$

$$\mathrm{r}(c'_{n_2-1}|A'') = (\overbrace{3, \dots, 3}^{v_i}, \overbrace{2, \dots, 2, 1}^{v'_j})$$

$$\mathrm{r}(c'_{n_2}|A'') = (\overbrace{3, \dots, 3}^{v_i}, \overbrace{2, \dots, 2}^{v'_j}).$$

The representation of vertices of set $\mathcal{V}(\mathcal{P}_{n_2})$ and $\mathcal{V}(\mathcal{P}'_{n_2})$ are given below:

$$\mathrm{r}(p_k|A') = (\overbrace{1,1,\ldots,1,1}^{c_s},\overbrace{2,2,\ldots,2,2}^{c'_t}), \quad \mathrm{r}(p'_l|A') = (\overbrace{2,2,\ldots,2,2}^{c_s},\overbrace{1,1,\ldots,1,1}^{c'_t}).$$

It can be seen that the vertices p_k and p'_l have same representations with respect to A''. The vertices p_k for $k \in \{1,3,\ldots,n_2-2\}$ resolve the vertices of $\mathcal{V}(\mathcal{P}_{n_2}) \setminus A'$ and $\mathcal{V}(\mathcal{P}'_{n_2})$. Hence A' is a resolving set of A and $\dim(A) \leq \dfrac{n_2-1}{2} + n_1 - 2$. Thus $\dim(A) = \dfrac{n_2-1}{2} + n_1 - 2$. Similarly we can find the metric dimension of $\mathcal{C}_{n_1} \blacktriangledown \mathcal{P}_{n_2}$, $n_1 \equiv 1(mod\ 2)$, $n_1 \geq 5$. This completes the proof. $\square$

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