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A STABLE METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS WITH TWO CONSTANT DELAYS

R. KATANI(1) AND S. SHAHMORAD(2)

Abstract. The main purpose of this paper is to propose a block by block method for a class of the Volterra integral equations (VIEs) with double constant delays. The convergence analysis is established and the fifth order of convergence is obtained. Then the stability analysis of the presented method is carried out with respect to the basic test equation

$$y(t) = 1 + \lambda \int_{t-\tau_2}^{t-\tau_1} y(s)ds, \qquad t > 0.$$

The analytical behavior of the solution of test equation is investigated and the properties of the numerical solution are derived. Numerical examples are presented to illustrate the capability and efficiency of the proposed method.

1. Introduction

Consider the Volterra integral equation

$$(1.1) \qquad y(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t, s, y(s))ds, \qquad t \in [0, T] =: I,$$

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with the constant delays τ_1, τ_2 $(\tau_2 > \tau_1 > 0)$ and $y(t) = \phi(t)$, $t \in [-\tau_2, 0]$, where $\phi(t)$ is a given function such that

$$\phi(0) = g(0) + \int_{-\tau_2}^{-\tau_1} k(0, s, \phi(s))ds. \tag{1.2}$$

By virtue of the above, we have $y(0^+) = \phi(0)$ for every solution of Eq.(1.1).

We assume that the given functions $\phi : [-\tau_2, 0] \to \mathbb{R}$, $g : I \to \mathbb{R}$ and $k : S \times \mathbb{R} \to \mathbb{R}$ (with $S := \{(t, s) : t \in I, t - \tau_2 \leq s \leq t - \tau_1\}$) are at least continuous on their respective domains and $k(t, s, y)$ satisfies the Lipschitz condition with respect to y and with the Lipschitz constant L:

$$|k(t, s, y) - k(t, s, z)| \leq L|y - z|, \qquad \forall y, z \in \mathbb{R}.$$

By these assumptions, we can obtain the existence and uniqueness results for Eq.(1.1) by considering the theory of VIEs [23, 31, 32].

Many physical phenomena are modeled by using delay integral equations. These integral equations are used for modeling of systems with history, such as electric circuits, dynamical and mechanical systems. These equations also play a major role in population dynamics, e.g., as models for growth of a population structured by age with a finite life span [11, 25, 26] and models for the spread of certain infectious diseases [21, 22, 38].

In recent years, various aspects of numerical methods have been studied for delay integral equations. For example, in [10, 12, 13, 14, 15, 16, 30, 40] the collocation methods are used to solve VIEs with delay. In [1, 2, 3] the spectral collocation methods are proposed to solve Volterra type integral and related differential equations with proportional delay. Bai [7] used collocation methods to solve the VIEs with vanishing delay. In [28] a Nystrom method proposed for nonlinear Urysohn type VIEs with proportional delay and Gu et al. [24] introduced a Chebyshev spectral collocation method for weakly singular VIEs with proportional delay. Mosleh and

Otadi [35] utilized last squares technique to solve delay VIEs and in [36] a single-term Walsh series approach used for solving these equations. Other methods employed in the litrature to solve such equations and similar ones, include Haar wavelet method [5, 6], Chebyshev Galerkin method [27], TDQ (Trapezoidal Direct Quadrature) and DQ (Direct Quadrature) methods [32, 33], Gaussian direct quadrature methods [20], sinc method [37] and so on.

In this paper, we are interested to proposing a block by block method for solving delay Volterra integral equations. The concept of block by block methods for integral equations seems to be described for the first time by Young [41]. A similar technique for differential equations was given by Milne [34]. In each step, this method compute several values of the unknown function at the same time, therefore it's called block by block method. Most of the available methods for the solution of VIEs are based on the expansions of solutions, e.g., the Taylor and Chebyshev expansion methods, the Tau method, the Adomian and homotopy methods and etc. These methods are efficient only for intervals of small length (say, $[0, 1]$ or $[-1, 1]$) and are useless for large intervals. The method used in the present paper is one of the most suitable methods for large intervals [29]. In addition, the computation time for this method is shorter than some well-known methods.

Concerning the stability of the method, we consider the test equation

$$y(t) = 1 + \lambda \int_{t-\tau_2}^{t-\tau_1} y(s)ds, \qquad t > 0, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ and we refer to [9], [17](chapter 7) and [31] (chapter 7) for a discussion on the use of this test equations in the study of stability for VIEs. We study stability of the presented method by using this test equation. For this order, we will show that the presented method preserve the bound, limiting value and oscillating character of the exact solution of (1.3). See [4, 8, 19, 32, 33, 39] for similar approaches.

The rest of the paper is organized as follows. In Section 2, we introduce the block by block method and in Section 3, we prove its convergence. In Section 4, we study properties of the test equation. Then we carry out analogous studies on the presented method and characterize the values of the step size h which lead to a numerical solution that catch the properties of the exact one. Finally, we give some numerical illustrations in Section 5 and we end the paper with some conclusions.

2. Description of the method

Let $t_n := nh$, $n = 0, ..., N-1$, $t_N = T$, define a uniform partition for $I = [0, T]$ and set $X_N := \{t_0, t_1, ..., t_N\}$, $I_n := [t_n, t_{n+1}]$, $n \geq 0$. The mesh X_N is assumed to be constrained, such that $h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$, for some $r_1, r_2 \in \mathbb{N}$.
For given real numbers $\{c_j\}$ with $0 = c_0 < c_1 < ... < c_4 = 1$, define the set $\Pi_N := \{t_{n,j}\}$ of collocation points by $t_{n,j} := t_n + c_j h$, $j = 0, 1, ..., 4$, $n = 0, ..., N-1$, where $t_n \in X_N$ and $c_j = \frac{j}{4}$. Set $t = t_{n,j}$ in (1.1) then

$$y(t_{n,j}) = g(t_{n,j}) + \int_{t_{n,j}-\tau_2}^{t_{n,j}-\tau_1} k(t_{n,j}, s, y(s))ds, \qquad n = 0, 1, ..., N-1, j = 0, 1, ..., 4. \tag{2.1}$$

If $t = t_{n,j}$ is such that $t_{n,j} - \tau_2 (= t_{n-r_2,j}) < 0$, then we have

$$y(t_{n,j}) = g(t_{n,j}) + \int_{t_{n,j}-\tau_2}^{0} k(t_{n,j}, s, \phi(s))ds + \int_{0}^{t_{n,j}-\tau_1} k(t_{n,j}, s, y(s))ds, \tag{2.2}$$

for $j = 0, 1, ..., 4$ and $n = 0, ..., N-1$; we define

$$\Phi(t) := \int_{t-\tau_2}^{0} k(t, s, \phi(s))ds. \tag{2.3}$$

This last expression represents a further potential source of error, since in applications, one will not be often able to evaluate the integral $\Phi(t)$ exactly; instead, one will have to resort suitable numerical integration formula to approximate it.

In order to put (2.1) into an amenable form to numerical computation, define

$$F(t_{n,j}) := \int_{t_{n,j}-\tau_2}^{t_n-\tau_1} k(t_{n,j}, s, y(s))ds. \tag{2.4}$$

Then for $t_{n,j}$ with $0 \le n < r_2$, we have

$$F(t_{n,j}) = \Phi(t_{n,j}) + \int_{t_0}^{t_1} k(t_{n,j}, s, y(s))ds + ... + \int_{t_{n-r_1-1}}^{t_{n-r_1}} k(t_{n,j}, s, y(s))ds, \tag{2.5}$$

if $t_{n,j}$ is such that $r_2 \le n < N-1$, then

$$\begin{aligned} F(t_{n,j}) &= \int_{t_{n,j}-\tau_2}^{t_{n+1}-\tau_2} k(t_{n,j}, s, y(s))ds + \int_{t_{n+1}-\tau_2}^{t_n-\tau_1} k(t_{n,j}, s, y(s))ds \\ &= h\int_{c_j}^{1} k(t_{n,j}, t_{n-r_2}+\nu h, y(t_{n-r_2}+\nu h))d\nu + \int_{t_{n-r_2+1}}^{t_{n-r_2+2}} k(t_{n,j}, s, y(s))ds \\ &+ ... + \int_{t_{n-r_1-1}}^{t_{n-r_1}} k(t_{n,j}, s, y(s))ds, \end{aligned} \tag{2.6}$$

where for the first integral, we used the change of variable $s = t_{n-r_2} + \nu h$.

Thus, the equation (2.1) may be written in the form

$$\begin{aligned} y(t_{n,j}) &= g(t_{n,j}) + F(t_{n,j}) + \int_{t_n-\tau_1}^{t_{n,j}-\tau_1} k(t_{n,j}, s, y(s))ds \\ &= g(t_{n,j}) + F(t_{n,j}) + h\int_0^{c_j} k(t_{n,j}, t_{n-r_1}+\nu h, y(t_{n-r_1}+\nu h))d\nu, \quad j = 0, 1, ..., 4. \end{aligned} \tag{2.7}$$

2.1. **Romberg quadrature rule.** By using the trapezoidal rule for $\int_\beta^\alpha k(t_{n,j}, s, y(s))ds$ with $\alpha, \beta \in \Pi_N$ we define

$$\begin{aligned} T^{00} &:= \frac{\alpha-\beta}{2}\left[k(t_{n,j}, \alpha, y(\alpha)) + k(t_{n,j}, \beta, y(\beta))\right], \\ T^{01} &:= \frac{1}{2}T^{00} + \frac{\alpha-\beta}{2}k(t_{n,j}, \frac{\alpha+\beta}{2}, y(\frac{\alpha+\beta}{2})), \\ T^{02} &:= \frac{1}{2}T^{01} + \frac{\alpha-\beta}{4}\left[k(t_{n,j}, \frac{\alpha+3\beta}{4}, y(\frac{\alpha+3\beta}{4})) + k(t_{n,j}, \frac{3\alpha+\beta}{4}, y(\frac{3\alpha+\beta}{4}))\right], \end{aligned}$$

then by using Romberg quadrature rule with two steps we obtain

$$\int_\beta^\alpha k(t_{n,j},s,y(s))ds \approx \frac{64T^{02}-20T^{01}+T^{00}}{45}$$
$$= \frac{\alpha-\beta}{90}\Big[7\Big(\,k(t_{n,j},\alpha,y(\alpha))+k(t_{n,j},\beta,y(\beta))\Big)$$
$$+\,12k(t_{n,j},\frac{\alpha+\beta}{2},y(\frac{\alpha+\beta}{2}))$$
$$+\,32\left(k(t_{n,j},\frac{3\alpha+\beta}{4},y(\frac{3\alpha+\beta}{4}))+k(t_{n,j},\frac{\alpha+3\beta}{4},y(\frac{\alpha+3\beta}{4}))\right)\Big]. \tag{2.8}$$

2.2. **The block by block method.** Assume that y_{n,c_j} be an approximation for the exact solution $y(t)$ in the point $t_{n,j}$, a.e. $y_{n,c_j} \approx y(t_n+c_jh)$ for $j=0,1,...,4$ and $n=0,1,...,N-1$, then by using (2.8), we can approximate $F(t_{n,j})$ from (2.5)

$$F(t_{n,j}) \approx \Phi(t_{n,j})+h\sum_{i=0}^{4} w_i k(t_{n,j},t_{0,i},y_{0,c_i})+...+h\sum_{i=0}^{4} w_i k(t_{n,j},t_{n-r_1-1,i},y_{n-r_1-1,c_i})$$
$$=: \Phi(t_{n,j})+A(t_{n,j}), \tag{2.9}$$

for $0 \le n < r_2$ where $w_0=w_4=7/90$, $w_2=2/15$ and $w_1=w_3=16/45$. Similarly from (2.6) we have

$$F(t_{n,j}) \approx h(1-c_j)\Big[\,w_0k(t_{n,j},t_{n-r_2,j},y_{n-r_2,c_j})$$
$$+\,w_1k(t_{n,j},t_{n-r_2}+\frac{h}{4}(1+3c_j),y_{n-r_2,\frac{1+3c_j}{4}})$$
$$+\,w_3k(t_{n,j},t_{n-r_2}+\frac{h}{4}(3+c_j),y_{n-r_2,\frac{3+c_j}{4}})$$
$$+w_2k(t_{n,j},t_{n-r_2}+h(1+c_j)/2,y_{n-r_2,\frac{1+c_j}{2}})+w_4k(t_{n,j},t_{n-r_2+1},y_{n-r_2,1})\Big]$$
$$+\,h\sum_{i=0}^{4} w_i\left[k(t_{n,j},t_{n-r_2+1,i},y_{n-r_2+1,c_i})+...+k(t_{n,j},t_{n-r_1-1,i},y_{n-r_1-1,c_i})\right], \tag{2.10}$$

for $r_2 \le n < N-1$. The points $t_{n-r_2}+\frac{h(1+3c_j)}{4}$ and $t_{n-r_2}+\frac{h(3+c_j)}{4}$ for $j=1,2,3$ and points $t_{n-r_2}+\frac{h(1+c_j)}{2}$ for $j=1,3$ do not belong to the collocation points (mesh Π_N). In order to overcome this problem, we adopt an interpolation technique. By

Lagrange interpolation, we can write

$$y(t_n+\nu h)\approx P(t_n+\nu h)=\sum_{i=0}^{4} l_i(\nu)y_{n,c_i},\qquad t_n+\nu h\in I_n,$$

where $l_i(\nu):=\prod_{\substack{j=0\\ j\neq i}}^{4}\frac{\nu-c_j}{c_i-c_j}$. Then for $j=2$ we have from (2.10)

$$\begin{aligned}
F(t_{n,j}) \approx\ & h(1-c_j)\Big[\, w_0k(t_{n,j},t_{n-r_2,j},y_{n-r_2,c_j})\\
&+w_1k(t_{n,j},t_{n-r_2}+h(1+3c_j)/4,\sum_{i=0}^{4} l_i((1+3c_j)/4)y_{n-r_2,c_i})\\
&+w_3k(t_{n,j},t_{n-r_2}+h(3+c_j)/4,\sum_{i=0}^{4} l_i((3+c_j)/4)y_{n-r_2,c_i})\\
&+w_2k(t_{n,j},t_{n-r_2}+h(1+c_j)/2,y_{n-r_2,\frac{1+c_j}{2}})+w_4k(t_{n,j},t_{n-r_2+1},y_{n-r_2,1})\Big]\\
&+h\sum_{i=0}^{4} w_ik(t_{n,j},t_{n-r_2+1,i},y_{n-r_2+1,c_i})+\ldots\\
&+h\sum_{i=0}^{4} w_ik(t_{n,j},t_{n-r_1-1,i},y_{n-r_1-1,c_i})=:B(t_{n,j}),\qquad r_2\le n<N-1,
\end{aligned}\tag{2.11}$$

and for $j=1$ and $j=3$ we use interpolation technique for $y_{n-r_2,\frac{1+c_j}{2}}$, too. Similarly, for approximate the end integral in (2.7), we can write from (2.8)

$$\begin{aligned}
\int_0^{c_j} k(t_{n,j},t_{n-r_1}+\nu h,y(t_{n-r_1}+\nu h))d\nu \approx\ & c_jh\Big[\, w_0k(t_{n,j},t_{n-r_1},y_{n-r_1,0})\\
&+w_1k(t_{n,j},t_{n-r_1}+c_j/4h,y_{n-r_1,c_j/4})+w_3k(t_{n,j},t_{n-r_1}+3c_j/4h,y_{n-r_1,3c_j/4})\\
&+w_2k(t_{n,j},t_{n-r_1}+c_j/2h,y_{n-r_1,c_j/2})+w_4k(t_{n,j},t_{n-r_1,j},y_{n-r_1,c_j})\Big],\\
&\qquad j=0,1,\ldots,4,\ n=0,1,\ldots,N-1,
\end{aligned}$$

where $t_{n-r_1}+c_j/4h$ and $t_{n-r_1}+3c_j/4h$ do not belong to the collocation points for $j=1,2,3$, also, $t_{n-r_1}+c_j/2h\notin\Pi_N$ for $j=1,3$. Then we use again the Lagrange interpolation and for $j = 1$ and $j = 3$ we obtain

$$\begin{aligned}\int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, y(t_{n-r_1} + \nu h))d\nu &\approx c_j h \Big[w_0 k(t_{n,j}, t_{n-r_1}, y_{n-r_1,0}) \\ &+ w_1 k(t_{n,j}, t_{n-r_1} + c_j/4h, \sum_{i=0}^{4} l_i(\frac{c_j}{4}) y_{n-r_1,c_i}) \\ &+ w_3 k(t_{n,j}, t_{n-r_1} + 3c_j/4h, \sum_{i=0}^{4} l_i(\frac{3c_j}{4}) y_{n-r_1,c_i}) \\ &+ w_2 k(t_{n,j}, t_{n-r_1} + c_j/2h, \sum_{i=0}^{4} l_i(\frac{c_j}{2}) y_{n-r_1,c_i}) \\ &+ w_4 k(t_{n,j}, t_{n-r_1,j}, y_{n-r_1,c_j}) \Big] =: C(t_{n,j}), \end{aligned} \tag{2.12}$$

for each $n = 0, ..., N-1$. Similar to (2.11), for $j = 2$ does not need use of Lagrange interpolation for $y_{n-r_2,\frac{c_j}{2}}$.

Finally, for $r_2 \leq n < N-1$, by substituting the approximations (2.10) and (2.12) in (2.7) we will have a system of equations. But for $0 \leq n < r_2$, if $\Phi(t)$ is not computable exactly, we need to approximate $\Phi(t_{n,j})$, too. Then we can write

$$\begin{aligned}\Phi(t_{n,j}) &= \int_{t_{n,j}-\tau_2}^{0} k(t_{n,j}, s, \phi(s))ds \\ &= \int_{t_{n,j}-\tau_2}^{t_{n+1}-\tau_2} k(t_{n,j}, s, \phi(s))ds + \int_{t_{n+1}-\tau_2}^{0} k(t_{n,j}, s, \phi(s))ds,\end{aligned}$$

for the first integral we use the change of variable $s = t_{n-r_2} + \nu h$ and obtain

$$\begin{aligned}\Phi(t_{n,j}) &= h \int_{c_j}^{1} k(t_{n,j}, t_{n-r_2} + \nu h, \phi(t_{n-r_2} + \nu h))d\nu \\ &+ \int_{t_{n-r_2+1}}^{t_{n-r_2+2}} k(t_{n,j}, s, \phi(s))ds + ... + \int_{t_1}^{t_0} k(t_{n,j}, s, \phi(s))ds.\end{aligned} \tag{2.13}$$

Using (2.8) leads to

$$\begin{aligned}
\Phi(t_{n,j}) \approx\ & h(1-c_j)[w_0 k(t_{n,j}, t_{n-r_2,j}, \phi(t_{n-r_2,j})) \\
& + w_1 k(t_{n,j}, t_{n-r_2} + h(1+3c_j)/4, \phi(t_{n-r_2} + h(1+3c_j)/4)) \\
& + w_3 k(t_{n,j}, t_{n-r_2} + h(3+c_j)/4, \phi(t_{n-r_2} + h(3+c_j)/4)) \\
& + w_2 k(t_{n,j}, t_{n-r_2} + h(1+c_j)/2, \phi(t_{n-r_2} + h(1+c_j)/2)) \\
& + w_4 k(t_{n,j}, t_{n-r_2+1}, \phi(t_{n-r_2+1}))] + h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_2+1,i}, \phi(t_{n-r_2+1,i})) \\
& + \ldots + h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{-1,i}, \phi(t_{-1,i})) =: \hat{\Phi}(t_{n,j}), \qquad (2.14)
\end{aligned}$$

where $w_0 = w_4 = 7/90$, $w_2 = 6/45$, $w_1 = w_3 = 16/45$ and $t_{-1,i} = (c_i - 1)h$. Therefore by substituting these approximations in relation (2.7), we will have

$$y_{n,c_j} = \begin{cases} g(t_{n,j}) + \hat{\Phi}(t_{n,j}) + A(t_{n,j}) + C(t_{n,j}), & 0 \le n \le r_2, \\ \\ g(t_{n,j}) + B(t_{n,j}) + C(t_{n,j}), & r_2 \le n \le N-1. \end{cases}$$

For $j = 1, ..., 4$ we will obtain a system of algebraic equations in each step ($n = 0, 1, .., N-1$). By solving this system we obtain a block of unknowns $U := (u_{n,1}, ..., u_{n,4})$. Of course, it is easy to generalize this method by increasing j and obtain a method with higher order of convergence that give more unknowns in each step.

3. Convergence of the numerical method

Theorem 3.1. *Assume that the integral*

$$\Phi(t) := \int_{t-\tau_2}^{0} k(t, s, \phi(s))ds, \qquad t \in [-\tau_2, 0],$$

is given exactly. Then for all sufficiently small h

$$h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}, \qquad r_1, r_2 \in \mathbb{N}, \qquad (3.1)$$

the numerical solution given in previous section is convergent to $y(t)$. *In addition, let us assume that* $k \in C^6(S \times \mathbb{R})$, $g \in C^6[0,T]$ *and* $\phi \in C^6[-\tau_2, 0]$. *Then, for all sufficiently small* h *satisfying in condition (3.1), the numerical solution* y_{n,c_j} *satisfies*

$$\max_{0 \le n \le N, 0 \le j \le 4} |y(t_{n,j}) - y_{n,c_j})| \le Ch^5,$$

for some constant C *which does not depend on* h. *This estimate holds for all collocation parameters* $\{c_j\}$ *with* $0 = c_0 < c_1 < ... < c_4 = 1$.

Remark 1. *Provided that* k, g *and* ϕ *in Eq.(1.1) are sufficiently smooth, condition (1.2) assures the continuity of* $y(t)$ *for* $t \ge 0$ *and by successively differentiating (1.1) it is easy to verify that* $y^{(l)}(t)$, $l = 1, 2, ...$ *present some points,* θ_1, θ_2,..., θ_Z, *of primary discontinuities (*$\theta_1 = 0$ *for* y', $\theta_1 = 0$, $\theta_2 = \tau_1$, $\theta_3 = \tau_2$ *for* y'',...*) and it is continuous for* $t > (l-1)\tau_2$.

Remark 2. *We have* $t_{n,j} \in [t_n, t_{n+1}]$ *for* $j = 0, 1, ..., 4$ *and assume that* h *satisfies the condition (3.1), which implies that* $t_{n,0}, ..., t_{n,4} \in [\theta_z, \theta_{z+1}]$ *or* $t_{n,0} \ge \theta_Z$ *for the discontinuity points* $\theta_1, ..., \theta_Z$. *Then from the smoothness hypotheses on* ϕ, g *and* k, *the exact solution* $y(t)$ *of (1.1) is at least 5 times continuously differentiable on* $[\theta_z, \theta_{z+1}]$, $z = 1, ..., Z-1$, *and on* $[\theta_Z, T]$. *From the expression for* $y^{(v)}(t)$, $v = 0, 1, ..., 5$, *obtained by successively differentiating (1.1) with respect to* t, *it is obvious that both the left and right limits of* $y^{(v)}(t)$ *as* $t \to \theta_z$ *exist and are finite.*

Proof of Theorem 3.1. Let $e(t_{n,j}) = y(t_{n,j}) - y_{n,j}$ and assume that $j = 1$ or $j = 3$ (for other values of j do similarly). If $t_n < \tau_2$, then $t_{n-r_2,j} = t_n + c_j h - \tau_2 \le 0$. From

(1.1) we have

$$
\begin{aligned}
y(t_{n,j}) &= g(t_{n,j}) + \int_{t_{n,j}-\tau_2}^{t_{n,j}-\tau_1} k(t_{n,j}, s, y(s))ds \\
&= g(t_{n,j}) + \Phi(t_{n,j}) + \int_0^{t_1} k(t_{n,j}, s, y(s))ds + ... \\
&+ \int_{t_{n-r_1-1}}^{t_{n-r_1}} k(t_{n,j}, s, y(s))ds + h\int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, y(t_{n-r_1} + \nu h))d\nu,
\end{aligned}
$$

and by substituting the approximations (2.9) and (2.12) in (2.7) we obtain $y(t_{n,j})$. Since $y(t) = \phi(t)$ on $[-\tau_2, 0]$ and $\Phi(t)$ is given exactly, then

$$
\begin{aligned}
e_{n,j} := e(t_{n,j}) &= \int_0^{t_1} k(t_{n,j}, s, y(s))ds + ... + \int_{t_{n-r_1-1}}^{t_{n-r_1}} k(t_{n,j}, s, y(s))ds \\
&+ h\int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, y(t_{n-r_1} + \nu h))ds - h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{0,i}, y_{0,c_i}) - ... \\
&- h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1-1,i}, y_{n-r_1-1,c_i}) - hc_j\Big[w_0 k(t_{n,j}, t_{n-r_1}, y_{n-r_1,0}) \\
&+ w_1 k(t_{n,j}, t_{n-r_1} + \frac{c_j}{4}h, P(t_{n-r_1} + \frac{c_j}{4}h)) + ... + w_4 k(t_{n,j}, t_{n-r_1,j}, y_{n-r_1,c_j})\Big].
\end{aligned}
$$

By adding and diminishing terms

$$
\begin{aligned}
&h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{0,i}, y(t_{0,i})),\ ...,\ h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1-1,i}, y(t_{n-r_1-1,i})) \\
&,\ hc_j\Big[w_0 k(t_{n,j}, t_{n-r_1}, y(t_{n-r_1})), w_1 k(t_{n,j}, t_{n-r_1} + \frac{c_j}{4}h, \sum_{i=0}^{4} l_i(\frac{c_j}{4}) y(t_{n-r_1,i})), \\
&..., w_4 k(t_{n,j}, t_{n-r_1,j}, y(t_{n-r_1,j}))\Big],
\end{aligned}
$$

and by setting

$$
\begin{aligned}
&w_0 k(t_{n,j}, t_{n-r_1}, y(t_{n-r_1})) + w_1 k(t_{n,j}, t_{n-r_1} + \frac{c_j}{4}h, \sum_{i=0}^{4} l_i(\frac{c_j}{4}) y(t_{n-r_1,i})) + ... \\
&+ w_4 k(t_{n,j}, t_{n-r_1,j}, y(t_{n-r_1,j})) := \sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1} + c_i c_j h, P(t_{n-r_1} + c_i c_j h)),
\end{aligned}
$$

we will have

$$
\begin{aligned}
|e_{n,j}| \leq & \left| \int_0^{t_1} k(t_{n,j}, s, y(s))ds - h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{0,i}, y(t_{0,i})\right| + ... \\
& + \left| \int_{t_{n-r_1-1}}^{t_{n-r_1}} k(t_{n,j}, s, y(s))ds - h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1-1,i}, y(t_{n-r_1-1,i})\right| \\
& + h\left| \int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, y(t_{n-r_1} + \nu h))d\nu \right. \\
& \pm \int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, P(t_{n-r_1} + \nu h))d\nu \\
& \left. - hc_j \sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1} + c_i c_j h, P(t_{n-r_1} + c_i c_j h)\right| \\
& + \left|h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{0,i}, y_{0,i}) - h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{0,i}, y(t_{0,i})\right| + ... \\
& + \left|h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1-1,c_i}, y_{n-r_1-1,c_i}) - h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1-1,i}, y(t_{n-r_1-1,i})\right| \\
& + hc_j \left[w_0|k(t_{n,j}, t_{n-r_1}, y(t_n - r_1)) - k(t_{n,j}, t_{n-r_1,0}, y_{n-r_1,0})| \right. \\
& + w_1|k(t_{n,j}, t_{n-r_1} + \frac{c_j}{4}h, y(t_{n-r_1} + \frac{c_j}{4}h)) - k(t_{n,j}, t_{n-r_1} + \frac{c_j}{4}h, P(t_{n-r_1, \frac{c_j}{4}h})| + \\
& \left. ... + w_4|k(t_{n,j}, t_{n-r_1,j}, y(t_{n-r_1}, j)) - k(t_{n,j}, t_{n-r_1,j}, y_{n-r_1,c_j})|\right].
\end{aligned}
$$

By using the Lipschitz condition for k with the Lipschitz constant L, we obtain

$$
\begin{aligned}
|e_{n,j}| \leq & |R_0| + ... + |R_{n-r_1-1}| + h\left| \int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, y(t_{n-r_1} + \nu h))d\nu \right. \\
& \left. - \int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, P(t_{n-r_1} + \nu h))d\nu \right| \\
& + h\left| \int_0^{c_j} k(t_{n,j}, t_{n-r_1} + \nu h, P(t_{n-r_1} + \nu h))d\nu \right. \\
& \left. - c_j \sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1} + c_i c_j h, P(t_{n-r_1} + c_i c_j h)) \right| \\
& + hL\sum_{i=0}^{4} w_i |e_{0,i}| + ... + hL\sum_{i=0}^{4} w_i |e_{n-r_1-1,i}| \\
& + hLc_j \left[w_0|e_{n-r_1,0}| + w_1 \sum_{i=0}^{4} |l_i(\frac{c_j}{4})||e_{n-r_1,i}| \right. \\
& \left. + w_3 \sum_{i=0}^{4} |l_i(\frac{3c_j}{4})||e_{n-r_1,i}| + w_2 \sum_{i=0}^{4} |l_i(\frac{c_j}{2})||e_{n-r_1,i}| + w_4|e_{n-r_1,j}| \right],
\end{aligned}
$$

where $|R_i|$, $i=0,...,n-r_1-1$ are the errors of numerical integrations, also,

$$\left|\int_0^{c_j} k(t_{n,j},t_{n-r_1}+\nu h,P(t_{n-r_1}+\nu h))d\nu - c_j\sum_{i=0}^{4} w_i k(t_{n,j},t_{n-r_1}+c_ic_jh,P(t_{n-r_1}+c_ic_jh))\right|,$$

is error of the numerical integration that we show by $|R_{n-r_1}|$. Again, by adding and diminishing the term $\int_0^{c_j} k(t_{n,j},t_{n-r_1}+\nu h,\sum_{i=0}^{4} l_i(\nu)y(t_{n-r_1,i}))d\nu$ and using the Lipschitz condition we have

$$\begin{aligned}
|e_{n,j}| &\le |R_0|+...+|R_{n-r_1-1}|+h|R_{n-r_1}| \\
&+hL\int_0^{c_j}\left|y(t_{n-r_1}+\nu h)-\sum_{i=0}^{4} l_i(\nu)y(t_{n-r_1,i})\right|d\nu \\
&+hL\int_0^{c_j}\left|\sum_{i=0}^{4} l_i(\nu)\left(y(t_{n-r_1,i})-y_{n-r_1,c_i}\right)\right|d\nu + hL\sum_{m=0}^{n-r_1-1}\sum_{i=0}^{4} w_i|e_{m,i}| \\
&+hLc_j\Bigg[w_0|e_{n-r_1,0}|+w_1\max_i\{l_i(\frac{c_j}{4})\}\sum_{i=0}^{4}|e_{n-r_1,i}| \\
&+w_3\max_i\{l_i(\frac{3c_j}{4})\}\sum_{i=0}^{4}|e_{n-r_1,i}| + w_2\max_i\{l_i(\frac{c_j}{2})\}\sum_{i=0}^{4}|e_{n-r_1,i}|+w_4|e_{n-r_1,j}|\Bigg],
\end{aligned}$$

by define $c_1' := Lc_jw_1\max_i\{l_i(\frac{c_j}{4})\}$, $c_2' := Lc_jw_3\max_i\{l_i(\frac{3c_j}{4})\}$ and $c_3' := Lc_jw_2\max_i\{l_i(\frac{c_j}{2})\}$ we obtain

$$\begin{aligned}
|e_{n,j}| &\le |R_0|+...+|R_{n-r_1-1}|+h|R_{n-r_1}|+hL\int_0^{c_j} I_{n-r_1}d\nu + hL\int_0^{c_j}\left|\sum_{i=0}^{4} l_i(\nu)e_{n-r_1,i}\right|d\nu \\
&+hLw'\sum_{m=0}^{n-r_1-1}\sum_{i=0}^{4}|e_{m,i}|+hLc_jw_0|e_{n-r_1,0}|+hc_1'\sum_{i=0}^{4}|e_{n-r_1,i}| \\
&+hc_2'\sum_{i=0}^{4}|e_{n-r_1,i}|+hc_3'\sum_{i=0}^{4}|e_{n-r_1,i}|+hLc_jw_4|e_{n-r_1,j}|,
\end{aligned}$$

where I_{n-r_1} is the interpolation error at $t_{n-r_1}+\nu h$ and c' is constant, then

$$|e_{n,j}| \leq |R_0| + ... + |R_{n-r_1-1}| + h|R_{n-r_1}| + hLc_j I_{n-r_1} + hc\sum_{i=0}^{4}|e_{n-r_1,i}|$$
$$+ hLw' \sum_{m=0}^{n-r_1-1}\sum_{i=0}^{4}|e_{m,i}|.$$

We can obtain the same inequalities for $j = 2$ and $j = 4$. Without diminishing universality, we assume that $|\widetilde{e}_{n,1}| = \max_{j=1,2,3,4} |e_{n,j}|$, therefore

$$|\widetilde{e}_{n,1}| \leq |R_0| + ... + |R_{n-r_1-1}| + h|R_{n-r_1}|$$
$$+ hLc_j I_{n-r_1} + 4hc\,|\widetilde{e}_{n-r_1,1}| + 4hLw' \sum_{m=0}^{n-r_1-1} |\widetilde{e}_{m,1}|$$
$$\leq |R_0| + ... + |R_{n-r_1-1}| + h|R_{n-r_1}| + hLc_j I_{n-r_1} + hc'' \sum_{m=0}^{n-r_1} |\widetilde{e}_{m,1}|,$$

where R_i, $i = 0, ..., n-r_1$ are the error of the numerical integrations and I_{n-r_1} is the interpolation error. So from the smoothness hypotheses on ϕ, g and k and Remark 2, there exists $C_1 > 0$ and $C_2 > 0$ such that $|R_i| \leq C_1 h^6$ and $|I_{n-r_1}| \leq C_2 h^5$ [18], therefore

$$|\widetilde{e}_{n,1}| \leq (n - r_1 + h)C_1 h^6 + C_2 Lc_j h^6 + hc'' \sum_{m=0}^{n-1} |\widetilde{e}_{m,1}|$$
$$\leq TC_1 h^5 + C_2 Lc_j h^6 + hc'' \sum_{m=0}^{n-1} |\widetilde{e}_{m,1}| \leq \tilde{C} h^5 + hc'' \sum_{m=0}^{n-1} |\widetilde{e}_{m,1}|,$$

by using the discrete Gronwall inequality ([17]. p.41) we get the error bound

$$|\widetilde{e}_{n,1}| \leq \tilde{C} h^5 e^{c''T}.$$

For $t_n \geq t_{r_2} (= \tau_2)$, a similar process implies

$$|\widetilde{e}_{n,1}| \leq \tilde{C} h^5 e^{c''T},$$

thus $|\widetilde{e}_{n,1}| \longrightarrow 0$ as $h \longrightarrow 0$ and $|\widetilde{e}_{n,1}| = O(h^5)$. $\square$

Now we turn to the case where we approximate integral in (2.3) by the quadrature rule given by (2.14).

Theorem 3.2. *Let assumptions of Theorem 3.1 hold, except for* $t = t_{n,j}$ $(j = 0, ..., 4$ *and* $0 \le n \le r_2 - 1)$ *the integral*

$$\Phi(t) = \int_{t-\tau_2}^{0} k(t, s, \phi(s))ds,$$

is approximated by the quadrature formula given by (2.14). Define the quadrature error by $E_0(t) := \Phi(t) - \hat{\Phi}(t)$ *such that*

$$|E_0(t)| \le Q_0(t)h^q, \qquad t = t_{n,j}, \qquad 0 \le n < r_2,$$

for some $q > 0$. *Then for the approximate solution* u *we have the error bound*

$$\max_{0 \le n \le N, 0 \le j \le 4} |y(t_{n,j}) - y_{n,j})| \le Ch^m,$$

for sufficiently small h *and* $m := min\{5, q\}$.

Proof. Assume that $j = 1$ or $j = 3$ (for other values of j do similarly) and $0 \le n < r_2$. Subtracting (2.9) and (2.12) from (2.2), implies

$$\begin{aligned} e(t_{n,j}) &= (\Phi(t_{n,j}) - \hat{\Phi}(t_{n,j})) + \int_0^{t_{n,j}-\tau_1} k(t_{n,j}, s, y(s))ds \\ &- h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{0,i}, y_{0,c_i}) - ... - h\sum_{i=0}^{4} w_i k(t_{n,j}, t_{n-r_1-1,i}, y_{n-r_1-1,c_i}) \\ &- hc_j\Big[w_0 k(t_{n,j}, t_{n-r_1}, y_{n-r_1,0}) + w_1 k(t_{n,j}, t_{n-r_1} + \frac{c_j}{4}, P(t_{n-r_1} + \frac{c_j}{4}h) + \\ &\quad ... + w_4 k(t_{n,j}, t_{n-r_1,j}, y_{n-r_1,c_j})\Big], \end{aligned}$$

then similar to the proof of Theorem 3.1, we obtain

$$|\widetilde{e}_{n,1}| \le |\Phi - \hat{\Phi}| + \tilde{C}h^5 + hc' \sum_{m=0}^{n-1} |\widetilde{e}_{m,1}|, \qquad 0 \le n < r_2,$$

for some constants C_1 and C_2. Therefore using the discrete Gronwall inequality implies

$$|\widetilde{e}_{n,1}| \le \{|\Phi - \hat{\Phi}| + \tilde{C}h^5\}e^{c'T}, \qquad 0 \le n < r_2.$$

Hence $|\widetilde{e}_{n,1}| \longrightarrow 0$, as $h \longrightarrow 0$, $T = Nh$ and $|\widetilde{e}_{n,1}| = O(h^m)$ if and only if $m := min\{5, q\}$.□

4. Numerical stability

4.1. **Properties of the test equation.** In this section we investigate the analytical behavior of the solution of test equation

$$y(t) = 1 + \lambda \int_{t-\tau_2}^{t-\tau_1} y(s)ds, \qquad t \in [0, T], \tag{4.1}$$

that it is the test equation introduced in [33].

Firstly, we prove the following lemma.

Lemma 4.1. *Assume that $\lambda > 0$, then if the solution $y(t)$ of (4.1) is a monotone increasing (decreasing) function in $[\overline{t} - \tau_2, \overline{t}]$ for some $\overline{t} \ge 0$, then it is ultimately increasing (decreasing) for all $t \ge \overline{t}$.*

Proof. Since $y(s)$ is continuous for $t - \tau_2 \le s \le t - \tau_1$ and $\tau_2 \le t \le T$ we obtain from (4.1)

$$y'(t) = \lambda[y(t - \tau_1) - y(t - \tau_2)].$$

For $t \in [\overline{t}, \overline{t} + \tau_1]$ we have $t - \tau_2$, $t - \tau_1 \in [\overline{t} - \tau_2, \overline{t}]$, so if y is increasing in $[\overline{t} - \tau_2, \overline{t}]$, then $y(t - \tau_1) \ge y(t - \tau_2)$ and hence $y'(t) \ge 0$. Continuing step-by-step through the adjacent intervals $[\overline{t} + \tau_1, \overline{t} + 2\tau_1]$,... we conclude that $y'(t) \ge 0$, $\forall t \in [\overline{t}, T]$.

The same procedure can be applied if we assume that $y' < 0$ in $[\overline{t} - \tau_2, \overline{t}]$. In this case $y'(t) \le 0$ for all $t \in [\overline{t}, T]$ and this yields the result stated in this lemma. □

We recall the following lemma and theorems from [33].

Lemma 4.2. *Let $y(t)$ be the solution of (4.1). If $\lambda \leq 0$ then $y'(t)$ is neither ultimately positive nor negative.*

Theorem 4.1. *Let $\phi(t) \geq 0$ for all $t \in [-\tau_2, 0]$. Then the solution $y(t)$ of (4.1) is positive for $\lambda > 0$. Furthermore, if $\phi(t) \leq 1$ and*

$$|\lambda|(\tau_2 - \tau_1) < 1, \tag{4.2}$$

then $y(t)$ is positive and bounded for all $t \geq 0$ and any λ.

Theorem 4.2. *Assume that the hypotheses of Theorem 4.1 hold, then the solution $y(t)$ of (4.1) is convergent and*

$$\lim_{t \longrightarrow \infty} y(t) = y^* = \frac{1}{1 - \lambda(\tau_2 - \tau_1)}. \tag{4.3}$$

Thus $y(t)$ is positive and bounded for all $t \geq 0$ and for any λ.

4.2. **Properties of the numerical solution.** In this section we investigate numerical stability of the block by block method for solving (4.1).

By applying the block by block method to the test equation (4.1), we obtain

$$\begin{aligned} y(t_{n,j}) &= 1 + \lambda h \int_{c_j}^{1} y(t_n - \tau_2 + \nu h))d\nu + \lambda \int_{t_{n-r_2+1}}^{t_{n-r_2+2}} y(s)ds \\ &+ \ldots + \lambda \int_{t_{n-r_1-1}}^{t_{n-r_1}} y(s)ds + \lambda h \int_0^{c_j} y(t_n - \tau_1 + \nu h)d\nu \\ &\approx 1 + \lambda h(1 - c_j)[w_0 y_{n-r_2,c_j} \\ &+ \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(i + (4-i)c_j/4) y_{n-r_2,c_{ii}} + w_4 y_{n-r_2+1}] \\ &+ \lambda h \sum_{i=0}^{4} w_i y_{n-r_2+1,c_i} + \ldots + \lambda h \sum_{i=0}^{4} w_i y_{n-r_1-1,c_i} \\ &+ \lambda h c_j [w_0 y_{n-r_1} + \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(ic_j/4) y_{n-r_1,c_{ii}} + w_4 y_{n-r_1,c_j}], \\ & \qquad j = 0, \ldots, 4, \quad r_2 \leq n \leq N-1, \end{aligned} \tag{4.4}$$

where $l_{ii}(\nu) = \prod_{\substack{i=0 \\ i\neq ii}}^{4} \frac{\nu - c_i}{c_{ii} - c_i}$. Similarly for $r_2 > n \geq 0$ we obtain

$$\begin{aligned} y(t_{n,j}) &\approx 1 + \lambda h(1-c_j)\sum_{i=0}^{4} w_i\phi(t_{n-r_2} + h(i+(4-i)c_j)/4) \\ &+ \lambda h \sum_{i=0}^{4} w_i \phi(t_{n-r_2+1,i}) + ... + \lambda h \sum_{i=0}^{4} w_i\phi(t_{-1,i}) + \lambda h \sum_{i=0}^{4} w_i y_{0,c_i} + \\ & ... + \lambda h \sum_{i=0}^{4} w_i y_{n-r_1-1,c_i} + \lambda h c_j [w_0 y_{n-r_1} \\ &+ \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(ic_j/4) y_{n-r_1,c_{ii}} + w_4 y_{n-r_1,c_j}], \qquad j = 0, ..., 4, \end{aligned} \tag{4.5}$$

with $h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$, $y_{-r_2,c_j} = \phi(t_{-r_2,j})$ $(t_{-r_2,j} = (c_j - r_2)h)$, . . . , $y_{0,c_j} = \phi(t_{0,j})$ and $y_0 = 1$.

Hence, we look for conditions on the parameters of (4.4), (4.5) that lead to a numerical solution y_n which replicates the global properties obtained in the previous subsection for the analytical solution $y(t)$.

In the following theorems which are the discrete analogues of Lemmas 4.1 and 4.2, we show the monotonicity and oscillatory behavior of y_n for $n > 0$.

Theorem 4.3. *Assume that $\lambda > 0$, then if there exists $\bar{n} \in \mathbb{N}$ such that y_{n,c_j} is monotone increasing (decreasing) for $n = \bar{n} - r_2, ...\bar{n}$, $j = 0, ..., 4$; then y_{n,c_j} is increasing (decreasing) for all $n \geq \bar{n}$ and $j = 0, ..., 4$.*

Proof. Applying the block by block method on the test equation, we obtain

$$\begin{aligned} y_{n+1,c_j} - y_{n,c_j} &= \lambda h(1-c_j)[w_0(y_{n-r_2+1,c_j} - y_{n-r_2,c_j}) + w_4(y_{n-r_2+2} - y_{n-r_2+1}) \\ &+ \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(i+(4-i)c_j)/4)(y_{n-r_2+1,c_{ii}} - y_{n-r_2,c_{ii}})] \end{aligned} \tag{4.6}$$

$$+ \lambda h \sum_{i=0}^{4} w_i (y_{n-r_2+2,c_i} - y_{n-r_2+1,c_i}) + ... + \lambda h \sum_{i=0}^{4} w_i (y_{n-r_1,c_i} - y_{n-r_1-1,c_i})$$
$$+ \lambda h c_j [w_0 (y_{n-r_1+1} - y_{n-r_1}) + w_4 (y_{n-r_1+1,c_j} - y_{n-r_1,c_j})$$
$$+ \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(ic_j/4)(y_{n-r_1+1,c_{ii}} - y_{n-r_1,c_{ii}})], \quad r_2 \le n \le N-1, \ j = 0, ..., 4.$$

For $n = \bar{n}$,...,$\bar{n} + r_1$, $j = 0, ..., 4$, the values of y_{n,c_j} (in the parentheses involved in the previous expression) are equal to $\bar{n} - r_2$,...,$\bar{n}$. If for these values of n, $y_{n+1,c_j} \ge y_{n,c_j}$, then all the parentheses involved in the previous expression are positive. Since $w_i > 0$ and $\lambda > 0$, then $y_{n+1,c_j} \ge y_{n,c_j}$ for $n = \bar{n} + 1 ....\bar{n} + r_1$. Proceeding step-by-step and applying the same procedure for $n = \bar{n} + r_1 + 1, ..., \bar{n} + 2r_1$, $n = \bar{n} + 2r_1 + 1, ..., \bar{n} + 3r_1$ and so on, we obtain $y_{n+1,c_j} \ge y_{n,c_j}$ for all $n \ge \bar{n}$ and $j = 0, ..., 4$. The same proof can be carried out if we assume that $y_{n+1,c_j} \le y_{n,c_j}$ for $n = \bar{n} - r_1$,...,$\bar{n}$, then it comes out that $y_{n+1,c_j} \le y_{n,c_j}$ for all $n \ge \bar{n}$, $j = 0, ..., 4$. This yields the result stated in the theorem. □

Theorem 4.4. *Assume that $\lambda < 0$, then, for $n \ge 0$ and $j = 0, ..., 4$, y_{n,c_j} is an oscillatory sequence in the sense that it is not ultimately increasing nor decreasing.*

Proof. Assume that y_{n,c_j} is increasing for all n and $j = 0, ..., 4$, then the right-hand side of (4.6) is negative, which contrasts our assumption. The proof is similar, when y_{n,c_j} is decreasing. □

A simple procedure based on the method of steps helps us to prove the following theorem which is the discrete analogue of Theorem 4.1 and gives conditions for the positiveness and boundedness of y_{n,c_j} for $n > 0$.

Theorem 4.5. *Assume that $\phi_{l,j} \ge 0$ for $l = -r_2, ..., 0$, $j = 0, ..., 4$ and $h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$, then the solution y_{n,c_j} of (4.4) is positive when $\lambda > 0$. Furthermore, if $\phi_{l,j} \le 1$ for*

$l=-r_2,...,-1,\ j=0,...,4$ and if

$$h|\lambda|(r_2-r_1)<1, \tag{4.7}$$

then y_{n,c_j} is positive and bounded for all $n\geq 0,\ j=0,...,4$ and for each λ.

Proof. We prove the theorem for the positive and negative values of λ separately. Let us assume $\lambda>0$. If $n=0,1,...,r_1-1$, then $n-r_2+1,...,n-r_1$ are equal to $-r_2,...,-1$, so $y_{n-r_2+1,c_j}=\phi_{n-r_2+1,j}$,..., $y_{n-r_1,c_j}=\phi_{n-r_1,j}$. Since $\phi_{l,j}$ is positive and bounded by 1, it is easy to show that

$$0\leq y_{n,c_j}\leq 1+\lambda h(r_2-r_1),\qquad \forall n=0,1,...,r_1-1,\ j=0,1,...,4.$$

For $n=r_1,r_1+1,...,2r_1-1$, we have $0\leq n-r_1\leq r_1-1$, thus $0\leq y_{n-r_1,c_j}\leq 1+\lambda h(r_2-r_1)$ and so

$$0\leq y_{n,c_j}\leq 1+\lambda h(r_2-r_1)+\lambda^2h^2(r_2-r_1)^2,\qquad n=r_1,r_1+1,...,2r_1-1,\ j=0,1,...,4.$$

Continuing this procedure for the next values of n, we obtain

$$0\leq y_{n,c_j}\leq \sum_{j=0}^{k}\lambda^j h^j(r_2-r_1)^j. \tag{4.8}$$

Since the series on the right side of (4.8) converges for $\lambda h(r_2-r_1)<1$, the theorem is proved for $\lambda>0$.

If $\lambda<0$, then for $n=0,1,...,r_1-1$, it is easy to show that

$$1+\lambda h(r_2-r_1)<y_{n,c_j}<1. \tag{4.9}$$

Thus, y_{n,c_j} is bounded and it is positive for $h|\lambda|(r_2-r_1)<1$. Repeating the process of previous case implies that (4.9) satisfies for each value of n and this yields the result stated in the theorem. □

Since conditions (4.7) and (4.2) coincide, Theorem 4.5 states that the numerical solution of (4.1), obtained by the block by block method, is positive and bounded as the analytical solution. The same correspondence holds on the asymptotic behavior, as shown in the following theorem.

Theorem 4.6. *Assume that (4.7) holds, then* y_{n,c_j} *converges and its limit is*

$$\lim_{n\longrightarrow\infty} y_{n,c_j} = y^* = \frac{1}{1-h\lambda(r_2-r_1)}. \tag{4.10}$$

Proof. Let us suppose that y_{n,c_j} is not regular, hence there exist $\{k'_n\}$ and $\{k''_n\}$ such that $l' = \liminf_n \ y_{n,c_j} = \lim y_{k'_n,c_j} < \lim y_{k''_n,c_j} = \limsup_n \ y_{n,c_j} = l''$. Set

$$\begin{aligned} z_{n,j} := (1-c_j)[w_0 y_{n-r_2,c_j} + \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(i(1-c_j)h/4) y_{n-r_2,c_{ii}} + w_4 y_{n-r_2+1}] \\ + \sum_{i=0}^{4} w_i y_{n-r_2+1,c_i} + ... + \sum_{i=0}^{4} w_i y_{n-r_1-1,c_i} \\ + c_j[w_0 y_{n-r_1} + \sum_{i=1}^{3} w_i \sum_{ii=0}^{4} l_{ii}(ic_j h/4) y_{n-r_1,c_{ii}} + w_4 y_{n-r_1,c_j}], \ j = 0, ..., 4, \end{aligned}$$

then $y_{n,c_j} = 1 + \lambda h z_{n,j}$ and thus,

$$\lim z_{k'_n,j} = \frac{l'-1}{\lambda h},$$
$$\lim z_{k''_n,j} = \frac{l''-1}{\lambda h}.$$

Moreover, after some manipulation on the expression of $z_{n,j}$, we get

$$(r_2-r_1)l' \leq \frac{l'-1}{\lambda h} \leq (r_2-r_1)l'',$$
$$(r_2-r_1)l' \leq \frac{l''-1}{\lambda h} \leq (r_2-r_1)l'',$$

combining the last inequalities gives

$$(1-h|\lambda|(r_2-r_1))(l''-l') \leq 0, \tag{4.11}$$

since $h|\lambda|(r_2-r_1) < 1$, (4.11) is satisfied only for $l' = l'' = y^*$. Hence, letting $n \longrightarrow \infty$ in equation (4.4) gives $y^* = 1 + h\lambda(r2-r1)y^*$ and this yields the result stated in the theorem.$\square$

Now, we investigate the numerical stability of the proposed block by block method according to the following definition [32].

Definition 4.1. A numerical method is stable with respect to (4.1), when its application to (4.1) gives a numerical solution behaving like the continuous one.

Hence, we look for the conditions on the step size h and on the parameters of (4.1) that lead to a numerical solution y_{n,c_j} which replicates the global properties obtained in pervious subsection for the analytical solution $y(t)$. From the obtained results we immediately derive the following result.

Corollary 4.1. *Theorems and lemma in this section assure that, with the hypotheses (4.2), the analytical solution of (4.1) and its numerical solution furnished by the block by block method have the same behavior. In this sense we can claim that if (4.2) is satisfied, the block by block method is stable with respect to the test equation (4.1).*

5. Numerical examples

In this section we report some numerical experiments that show the performances of the block by block method (with h satisfying (3.1)). All results computed by programming in Maple 14.

Example 5.1. *([8]) Consider*

$$y(t) = 1/4(sin(2(t-\tau)) - sin(2t)) + cos(t) - 1/2\tau + \int_{t-\tau}^{t} y^2(s)ds, \tag{5.1}$$

with

$$y(t) = \phi(t) = cos(t) \qquad (-\tau \leq t \leq 0),$$
$$y(t) = cos(t), \qquad (t \geq 0).$$

Similar to [8], we choose a fixed delay $\tau = 1.0$ and various step sizes $h = 0.1, 0.05$ and 0.025 with $t \in [0, 5]$. In Table 1 we have presented a comparison between the absolute errors of the Trapezium (TM) and Simpson (SM) methods that reported in [8] and the block by block method (BBM). In the last row of table, we reported computing times (programming of the all methods have been done by using Maple package).

TABLE 1. *Absolute errors of (5.1)*

	$h = 0.1$			$h = 0.05$			$h = 0.025$		
t	TM	SM	BBM	TM	SM	BBM	TM	SM	BBM
1.0	$5.4e^{-3}$	$1.5e^{-5}$	$4.6e^{-11}$	$1.4e^{-3}$	$9.0e^{-7}$	$1.2e^{-13}$	$3.6e^{-4}$	$5.9e^{-8}$	$1.8e^{-15}$
2.0	$8.1e^{-5}$	$2.4e^{-7}$	$2.6e^{-12}$	$3.3e^{-5}$	$2.2e^{-8}$	$3.3e^{-14}$	$9.7e^{-6}$	$1.6e^{-9}$	$4.7e^{-16}$
3.0	$7.3e^{-4}$	$2.0e^{-6}$	$6.2e^{-12}$	$1.8e^{-5}$	$1.2e^{-7}$	$9.6e^{-14}$	$4.6e^{-5}$	$7.7e^{-9}$	$1.5e^{-16}$
4.0	$1.0e^{-4}$	$2.8e^{-7}$	$9.5e^{-12}$	$2.6e^{-4}$	$1.7e^{-8}$	$8.1e^{-14}$	$6.4e^{-6}$	$1.1e^{-9}$	$1.2e^{-15}$
5.0	$1.3e^{-3}$	$3.5e^{-6}$	$1.3e^{-11}$	$3.3e^{-3}$	$2.2e^{-7}$	$2.0e^{-13}$	$8.3e^{-5}$	$1.4e^{-8}$	$3.1e^{-15}$
time	$0.717s$	$0.640s$	$3.406s$	$1.375s$	$1.512s$	$5.023s$	$3.545s$	$3.451s$	$8.469s$

Example 5.2. *([20]) Consider*

$$y(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} (\sigma + \mu(t-s))(1 + y(s))^2 ds, \qquad t \in [0, T], \tag{5.2}$$

with $\tau_1 = 0.5$, $\tau_2 = 1$, $\sigma = 1$, $\mu = -1.2$ *and we choose* $g(t)$ *such that the exact solution of the (5.2) to be* $y(t) = sin(t)$ *and set* $\phi(t) = sin(t)$.

The absolute errors in some mesh points for different values of T and h are reported in Table 2.

TABLE 2. *Absolute error of (5.2).*

t_i	$N = 100$ $T = 1$
0.1	$2.881e^{-17}$
0.2	$2.929e^{-17}$
0.3	$2.852e^{-17}$
0.4	$2.653e^{-17}$
0.5	$2.340e^{-17}$
0.6	$2.880e^{-17}$
0.7	$3.542e^{-17}$
0.8	$4.304e^{-17}$
0.9	$5.136e^{-17}$
1	$5.998e^{-17}$
time	$10.140s$

t_i	$N = 400$ $T = 2$
0.1	$4.501e^{-19}$
0.2	$4.576e^{-19}$
0.5	$3.655e^{-19}$
0.8	$6.767e^{-19}$
1	$9.447e^{-19}$
1.2	$7.020e^{-19}$
1.4	$6.701e^{-19}$
1.5	$8.162e^{-19}$
1.8	$1.991e^{-18}$
2	$2.518e^{-18}$
time	$66.441s$

t_i	$N = 1000$ $T = 10$
1	$5.999e^{-17}$
2	$1.594e^{-16}$
3	$9.389e^{-16}$
4	$2.250e^{-15}$
5	$4.618e^{-16}$
6	$4.268e^{-17}$
7	$8.336e^{-17}$
8	$3.687e^{-16}$
9	$2.217e^{-16}$
10	$7.069e^{-15}$
time	$99.123s$

Remark 3. *We assume that error of the block by block method is $E(h) = ch^q$, where c is constant and q is order of the convergence. Then we obtain*

$$q = \frac{ln(\frac{E(h)}{c})}{ln(h)}. \tag{5.3}$$

By computing the order q from the relation (5.3) for the reported errors in Tables 1-2, we conclude that $q \geq 5$ and it shows that our method produces the desired order according to Theorems 3.1 and 3.2.

In the following, we consider some test problems for illustrate theoretical results of the Corollary 4.1; In these tests we set $\tau_1 = 0.5$ and $\tau_2 = 1$. In order to compare numerical results of the introduced method with the results of [33], we choose three examples from this reference:

Example 5.3. *Consider*

$$y(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} (\sigma + \mu(t-s))y(s)ds,$$

and choose $g(t)$ *in such way that the exact solution to be* $y(t) = tsin(t)$.

Figure 1 shows the behavior of the numerical solution ($*$) and exact solution (solid line) for $\mu = -1$, $N = 80$, $T = 20$ and different values of the parameter σ. If similar to [33] set (by first approximation) $\lambda \approx \sigma$, then from these figures it is clear that when condition (4.2) is satisfied, the numerical solution is perfectly matches the exact solution (I, II), while it may skip away from the exact solution when (4.2) is not satisfied (III, IV).

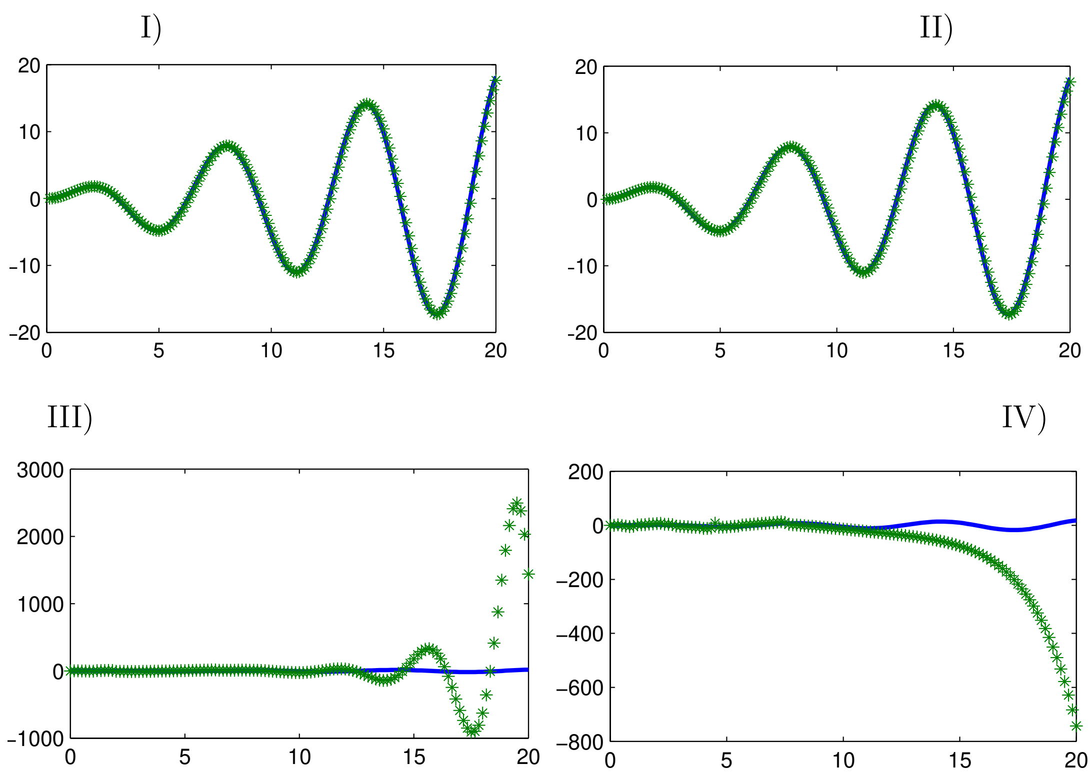

FIGURE 1. Solutions of Ex.5.3 $I)\sigma = -1, \lambda(\tau_2 - \tau_1) = -0.5$. $II)\sigma = 1, \lambda(\tau_2 - \tau_1) = 0.5$. $III)\sigma = -6, \lambda(\tau_2 - \tau_1) = -3$. IV)$\sigma = 6, \lambda(\tau_2 - \tau_1) = 3$.

Example 5.4. *Consider nonlinear delay equation introduced in Example 5.2, with* $\mu = -1$.

Figure 2 shows the behavior of the numerical solution $(*)$ and exact solution (solid line) for different values of the parameter σ and confirms the Corollary 4.1. In this case we set (from [33]) $\lambda \approx 2\sigma$.

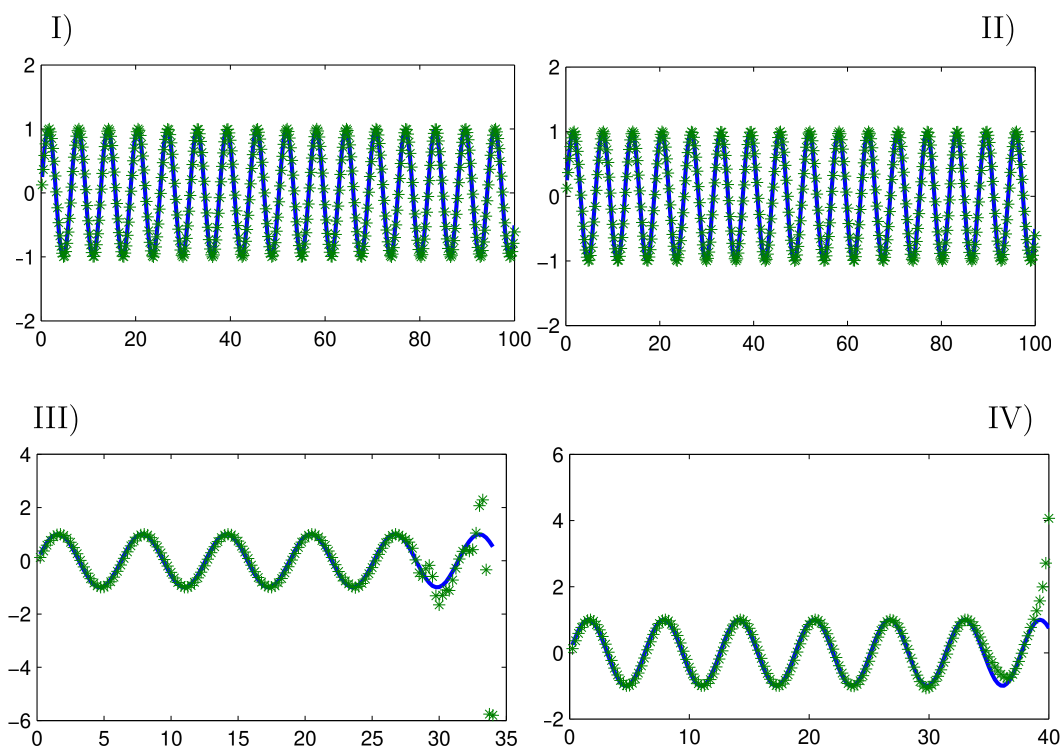

FIGURE 2. Solutions of Ex.5.4 $I)T = 100, N = 200, \sigma = -1/2, \lambda(\tau_2 - \tau_1) = -0.5$. $II)T = 100, N = 200, \sigma = 1/2, \lambda(\tau_2 - \tau_1) = 0.5$. $III)T = 34, N = 68, \sigma = -2, \lambda(\tau_2 - \tau_1) = -1$. $IV)T = 40, N = 80, \sigma = 3, \lambda(\tau_2 - \tau_1) = 1.5$.

Example 5.5. *In this example we show that condition (4.2) is a sufficient condition for the stability of the block by block method with respect to test problem (4.1) and it is not a necessary condition. To do this, we consider* $k(t,s,y) = (\sigma + \mu(t - s))exp(-y(s))y(s)$ *with exact solution* $y(t) = 1$.

Figure 3 shows the behavior of the numerical solution $(*)$ and exact solution (solid line) for $\mu = -1$, $T = 100$, $N = 200$ and different values of σ. It is clear that numerical and analytical solution are coincide even when (4.2) is not satisfied.

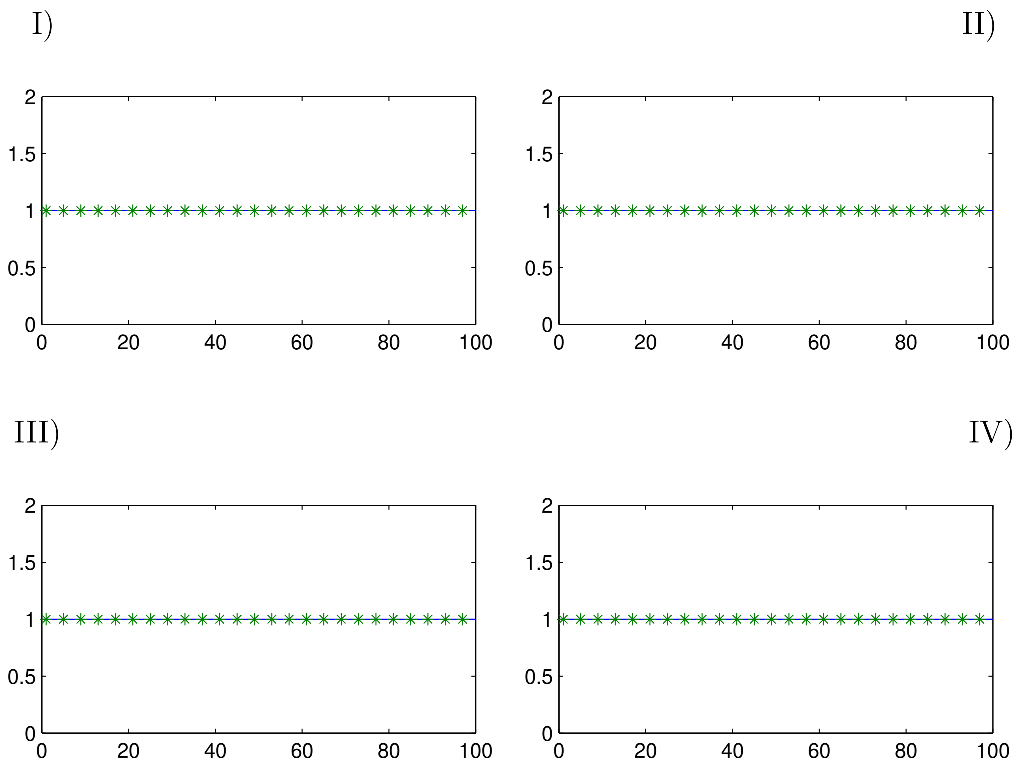

FIGURE 3. Solutions of Ex.5.5 $I)\sigma = -1, \lambda(\tau_2 - \tau_1) = -0.5$. $II)\sigma = 1, \lambda(\tau_2 - \tau_1) = 0.5$. $III)\sigma = -6, \lambda(\tau_2 - \tau_1) = -3$. $IV)\sigma = 6, \lambda(\tau_2 - \tau_1) = 3$.

6. Conclusion

In this paper, we proposed a block by block method for approximate solution of a class of the delay Volterra integral equations. We also discussed the convergence analysis with the order of convergence at least 5. Then, in order to complete the study of the proposed method, we analyzed the stability with respect to a class of test problems introduced in [33]. We found a sufficient condition for numerical stability which assure that the numerical solution inherits the asymptotic properties of the continuous one. The numerical results confirmed that our method gives fairly good results in addition to its simplicity and efficiency for large intervals.

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(1) Faculty of sciences, Yasouj University, Yasouj - Iran.

Email address: katani@yu.ac.ir

(2)Faculty of mathematical sciences, University of Tabriz, Tabriz - Iran.

Email address: shahmorad@tabrizu.ac.ir