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STRONG MODULAR SUMSET NUMBER OF GRAPHS WHEN VERTICES ARE ASSIGNED WITH SETS OF CARDINALITY TWO

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Abstract. For a positive integer n, let Z_n be the set of all non-negative integers modulo n and $\mathcal{P}(Z_n)$ be its power set. A graph that admits strong modular sumset labeling is called a strong modular sumset graph. The strong modular sumset number of a graph is the minimum cardinality required for the ground set Z_n so that the graph admits a strong modular sumset labeling and hence is a strong modular sumset graph. In this paper, we determine strong modular sumset labeling and the strong modular sumset number of graphs when vertices are assigned with sets of cardinality two.

1. Introduction

For all terms related to graph, not defined here, we refer to [3]. For graph labeling terminology, we refer to [4]. For concepts in number theory, we refer to [2]. The notion of set-valuation was introduced by Acharya in [1]. Further investigation by Sudev leads to extensive work in set-valuations of discrete structures [8] and introduction to the concept of modular sumset labeling of graphs in [5]. For the sumset valuation of graphs, we refer to [6]. For the notion of sumset, we refer to [7] and [9]. Let G be

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a simple, finite, undirected graph with no isolated vertices. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively.

Definition 1.1. [5] The modular sumset of two subsets A and B of Z_n is the set $A+B=\{x \in Z_n : a+b \equiv x \ (mod\ n)\ , \text{where } a \in A, b \in B\}$.

Definition 1.2. [5] Let G be a finite simple graph with no isolated vertices. For a function $f : V(G) \to \mathcal{P}(Z_n)$, define induced function $f^+ : E(G) \to \mathcal{P}(Z_n)$ as $f^+(uv) = f(u) + f(v)$. Then f is said to be a modular sumset labeling if f is injective. A graph G that admits a modular sumset labeling is called a modular sumset graph.

Definition 1.3. [5] A modular sumset indexer is a modular sumset labeling $f : V(G) \to \mathcal{P}(Z_n)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(Z_n)$ defined by $f^+(uv) = f(u) + f(v) = \{x \in Z_n : a+b \equiv x(mod\ n);\ a \in f(u), b \in f(v)\}$ is also injective.

Theorem 1.1. [5] *Every finite graph admits a modular sumset labeling (or a modular sumset indexer).*

Definition 1.4. [5] Let $f : V(G) \to \mathcal{P}(Z_n)$ be a modular sumset labeling defined on a given graph G. Then f is said to be a strong modular sumset labeling if for the associated function $f^+ : E(G) \to \mathcal{P}(Z_n), |f^+(uv)| = |f(u)|\,|f(v)|\,, \forall uv \in E(G)$. A graph that admits a strong modular sumset labeling is called a strong modular sumset graph.

Definition 1.5. [5] The difference set D_A of a given set A is defined as $D_A = \{|a-b| : a, b \in A; a \neq b\}$.

A necessary and sufficient condition for a graph to admit a strong modular sumset labeling is as follows :

Theorem 1.2. [5] *A modular sumset labeling* $f : V(G) \to \mathcal{P}(Z_n)$ *of a given graph* G *is a strong modular sumset labeling of* G *iff* $D_{f(u)} \cap D_{f(v)} = \emptyset, \forall uv \in E(G)$ *with* $|f(u)|\,|f(v)| \leq n$.

Definition 1.6. [5] A strong modular sumset number of graph G is the minimum cardinality required for the ground set Z_n so that G admits a strong modular sumset labeling.

Definition 1.7. The path P_m is the m-vertex graph with $m-1$ edges, all on a single open path.

Definition 1.8. The cycle C_m is the m-vertex graph with m edges, all on a single cycle.

Definition 1.9. The wheel graph $W_m = C_m + K_1$ is obtained by joining a vertex to each of the m vertices of the cycle graph C_m.

Definition 1.10. The helm graph H_m is obtained from the wheel W_m by attaching a pendant edge at each vertex of the m-cycle.

Definition 1.11. The closed helm graph CH_m is obtained from the helm H_m by joining each pendant vertex to form a cycle.

Definition 1.12. The gear graph G_m is obtained from the wheel W_m by adding a vertex between every pair of adjacent vertices of the m-cycle.

Definition 1.13. The friendship graph F_m is obtained by taking m copies of the cycle graph C_3 with a vertex in common.

2. MAIN RESULTS

Our aim in this section is to determine the strong modular sumset number of various graphs if the sets assigned to the vertices are of cardinality two. Since Z_n consists of

n elements, its power set $\mathcal{P}(Z_n)$ consists of 2^n elements. Out of these elements, there are precisely $\dfrac{n(n-1)}{2} = \dbinom{n}{2}$ of cardinality 2. The difference set of any such subset of $\mathcal{P}(Z_n)$ will contain exactly one element from the set $\{1, 2, \cdots, n-1\}$. There are exactly $n-i$ subsets with difference set containing only i, for $i = 1, 2, \cdots, n-1$.

Theorem 2.1. *The strong modular sumset number n of a path graph P_m, $m \geq 2$ is given by* $n = \left\lceil \sqrt{2m + \frac{1}{4}} + \frac{1}{2} \right\rceil$ *if sets assigned to the vertices are of cardinality two.*

Proof. We consider two cases :

Case (i) : $m = 2, 3$. It is easy to see that $n = 3$.

Case (ii) : $m \geq 4$. Since P_m has m vertices, there exist $n \in N$ such that $\dbinom{n-1}{2} < m \leq \dbinom{n}{2}$. Therefore,

$n(n-1) \geq 2m \Rightarrow \left(n - \frac{1}{2}\right)^2 \geq 2m + \frac{1}{4} \Rightarrow n \geq \sqrt{2m + \frac{1}{4}} + \frac{1}{2}$.

Hence, take

$$n = \left\lceil \sqrt{2m + \frac{1}{4}} + \frac{1}{2} \right\rceil \tag{2.1}$$

Let $f : V(P_m) \to \mathcal{P}(Z_n)$ be a modular sumset labeling of P_m such that $|f(u)| = |f(v)| = 2$, where n is as in (2.1).

Claim : f admits strong modular sumset labeling for a path graph P_m.

Since $m \geq 4$, $|f(u)|\,|f(v)| = 4 \leq n$. Now, we show that $D_{f(u)} \cap D_{f(v)} = \emptyset, \forall\ uv \in E(P_m)$. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ where $m = \dbinom{n-1}{2} + q$, $1 \leq q \leq n-1$.

Let A be a subset of $P(Z_n)$ with $|A| = 2$ and $D_A^{(i)}$ be its difference set containing only i where $i \in \{1, 2, \ldots, n-1\}$. We label the vertices by considering various stages. Suppose m_k represents the number of vertices which are not labeled at the k^{th} stage. In stage 1, assign the vertices v_{2j-1} , $j \in \{1, \ldots, n-1\}$ with $(n-1)$ subsets of $P(Z_n)$ having difference set $D_A^{(1)}$ and assign the vertex v_{2j} , $j \in \{1, \ldots, n-2\}$ with $(n-2)$

subsets of $P(Z_n)$ having difference set $D_A^{(2)}$.

So, $m_1 = m - (n-2+n-1) = \binom{n-1}{2} + q - (n-2+n-1) = (1+2+\ldots+n-2) + q - (n-2+n-1) = \binom{n-2}{2} + q - (n-1)$

In stage 2, assign the vertices v_{2j-1} , $j \in \{n, n+1, \ldots, 2n-4\}$ with $(n-3)$ subsets of $P(Z_n)$ having the difference set $D_A^{(3)}$ and assign the vertices v_{2j} , $j \in \{n-1, n, \ldots, 2n-6\}$ with $(n-4)$ subsets of $P(Z_n)$ having the difference set $D_A^{(4)}$

So, $m_2 = m_1 - (n-3+n-4) = \binom{n-2}{2} + q - (n-1) - (n-3+n-4) = (1+2+\ldots+n-4+n-3) + q - (n-1) - (n-3+n-4) = \binom{n-4}{2} + q - (n-1)$.

Continuing in this manner, we reach the k^{th} stage where $k = \left\lceil \frac{n-1}{4} \right\rceil$ if n is odd and $k = \left\lceil \frac{\left\lfloor \frac{n-1}{2} \right\rfloor + 1}{2} \right\rceil$ if n is even. Assign vertices v_{2j-1}, $j \in \{(k-1)n-(k-1)^2+1, \ldots, (k-1)n-(k-1)^2+n-(2k-1)\}$ with $(n-(2k-1))$ subsets of $P(Z_n)$ having the difference set $D_A^{(2k-1)}$ and assign the vertices v_{2j} , $j \in \{(k-1)n-(k-1)k+1, \ldots,$
$(k-1)n-(k-1)k+n-2k = kn-(k^2+k)\}$ with $(n-2k)$ subsets of $P(Z_n)$ having the difference set $D_A^{(2k)}$. So, $m_k = \binom{n-2k}{2} + q - (n-1)$ and observe that $m_k - q = \binom{n-2k}{2} - (n-1) < (n-2k-1) + (n-2k-2)$.

Now, we have two subcases :

Subcase (a) : $m_k \leq (n-2k-1) + (n-2k-2)$

Then at $(k+1)^{th}$ stage, the remaining m_k vertices can be assigned alternately with $(n-2k-1)$ subsets and $(n-2k-2)$ subsets of $P(Z_n)$ having the difference sets $D_A^{(2k+1)}$ and $D_A^{(2k+2)}$ respectively until all the vertices are assigned.

Subcase (b) : $m_k > (n-2k-1) + (n-2k-2)$

Then at $(k+1)^{th}$ stage, assign the vertices v_{2j-1} , $j \in \{kn-k^2+1, \ldots, kn-k^2+n-(2k+1)\}$ with $(n-(2k+1))$ subsets of $P(Z_n)$ having the difference set $D_A^{(2k+1)}$ and assign the vertices v_{2j} , $j \in \{kn-(k+1)k+1, \ldots, kn-(k+1)k+n-(2k+2) =$

$(k+1)n-(k+1)(k+2)\}$ with $(n-(2k+2))$ subsets of $P(Z_n)$ having the difference set $D_A^{(2k+2)}$.

So, $m_{k+1} = \binom{n-2k-2}{2} + q - (n-1)$ and $m_{k+1} < n-1$

Again we have two cases for m_{k+1} and we proceed as above. This may continue until $\frac{n-1}{2}$ stage if n is odd and $\left\lfloor \frac{n-1}{2} \right\rfloor + 1$ stage if n is even. When n is even, the last stage has only one vertex which is not labeled. Assign this vertex with the unique subset of $P(Z_n)$ having the difference set $D_A^{(n-1)}$. □.

Example 2.1. *Figure 1 illustrates a strong modular sumset labeling of P_{15}. By Theorem 2.1, we have $n=6$. Observe that labeling the vertices as in Figure 1 admits strong modular sumset labeling. Hence the strong modular sumset number is 6.*

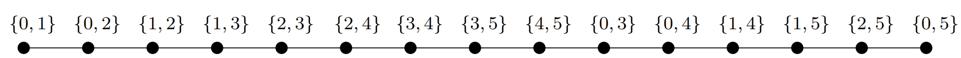

FIGURE 1. Strong modular sumset labeling of P_{15}

Theorem 2.2. *The strong modular sumset number n of a cycle graph C_m, $m \geq 3$ is given by $n = \left\lceil \sqrt{2m+\frac{1}{4}} + \frac{1}{2} \right\rceil$ if sets assigned to the vertices are of cardinality two.*

Proof. A similar argument as in Theorem 2.1 gives

$$n = \left\lceil \sqrt{2m+\frac{1}{4}} + \frac{1}{2} \right\rceil \tag{2.2}$$

For $m=3$, take $n=4$. Assume that $m \geq 4$. Let $f: V(C_m) \to \mathcal{P}(Z_n)$ be a modular sumset labeling of C_m such that $|f(u)| = |f(v)| = 2$, where n is as in (2.2).

Claim : f admits strong modular sumset labeling for cycle graph C_m .

Since $m \geq 4$, $|f(u)|\,|f(v)| = 4 \leq n$. We show that $D_{f(u)} \cap D_{f(v)} = \emptyset, \forall uv \in E(C_m)$.

Let $V(C_m) = \{v_1, v_2, \dots, v_m\}$ where vertices are labeled in clockwise direction. Consider the labeling as in path graph P_m with similar notations. We claim that it is the required labeling for cycle graph as well. Since the vertices v_1 and v_m are adjacent, it is enough to show that they are labeled with the subsets having disjoint difference sets.

Assign v_1 with the subset of $P(Z_n)$ having the difference set $D_A^{(1)}$ and note that v_{2n-3} is the last vertex assigned in this manner.

Observe that $m > \binom{n-1}{2}$ and $2n-3 < \binom{n-1}{2}$.

Hence, $2n-3 < \binom{n-1}{2} < m$. So, the vertices v_1 and v_m are labeled with the subsets of $P(Z_n)$ having disjoint difference sets. □.

Example 2.2. *Figure 2 illustrates a strong modular sumset labeling of C_{15}. By Theorem 2.2, we have $n = 6$. Hence, the strong modular sumset number is 6.*

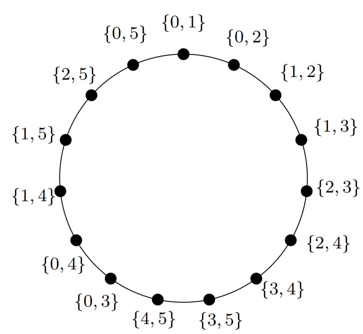

FIGURE 2. Strong modular sumset labeling of C_{15}

Theorem 2.3. *The strong modular sumset number n of a wheel $W_m = C_m + K_1, m \geq 3$ is given by $n = \left\lceil \sqrt{2m + \frac{9}{4}} + \frac{1}{2} \right\rceil$ if sets assigned to the vertices are of cardinality two.*

Proof. Since W_m has $m+1$ vertices, there exist $n \in N$ such that $\binom{n-1}{2} < m+1 \leq \binom{n}{2}$. Therefore, $n(n-1) \geq 2m+2 \Rightarrow \left(n - \frac{1}{2}\right)^2 \geq 2m + \frac{9}{4} \Rightarrow n \geq \sqrt{2m + \frac{9}{4}} + \frac{1}{2}$.

Hence, take

$$n = \left\lceil \sqrt{2m + \frac{9}{4}} + \frac{1}{2} \right\rceil \tag{2.3}$$

If m is $3, 4$ and 5, then take n as $5, 4$ and 5 respectively. Assume $m \geq 6$. Let $f : V(W_m) \to \mathcal{P}(Z_n)$ be a modular sumset labeling of W_m such that $|f(u)| = |f(v)| = 2$, where n is as in (2.3)

Claim : f admits strong modular sumset labeling for Wheel W_m .

Since $m \geq 6$, $|f(u)|\,|f(v)| = 4 \leq n$. Now, we show that

$D_{f(u)} \cap D_{f(v)} = \emptyset, \forall uv \in E(W_m)$. Let $V(W_m) = \{u_1, v_1, v_2, \ldots, v_m\}$ where u_1 be the apex vertex and $v_1, v_2, \ldots, v_m$ are the rim vertices labeled in clockwise direction . Consider the labeling as in the cycle graph C_{m+1}. It is enough to label the apex vertex with the subset of $P(Z_n)$ whose difference set is disjoint with all the other difference sets of the respective vertices. Hence, label the apex vertex u_1 with the subset of $P(Z_n)$ having the difference set $D_A^{(n-1)}$. Since such a set is unique, we have the required labeling for a wheel graph. $\square$.

Example 2.3. *Figure 3 illustrates a strong modular sumset labeling of* $W_{14} = C_{14} + K_1$*. By Theorem* 2.3*, we have* $n = 6$*. Hence, the strong modular sumset number is* 6*.*

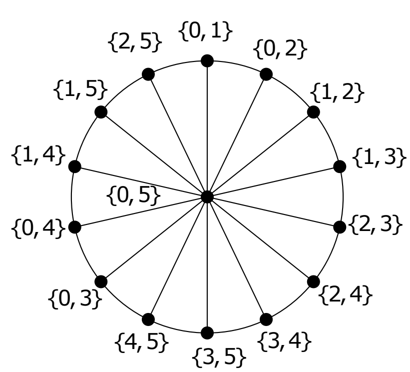

FIGURE 3. Strong modular sumset labeling of W_{14}

Theorem 2.4. *The strong modular sumset number n of a helm H_m, $m \geq 3$ is given by* $n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil$ *if sets assigned to the vertices are of cardinality two.*

Proof. Since H_m has $2m+1$ vertices, there exist $n \in N$ such that $\binom{n-1}{2} < 2m+1 \leq \binom{n}{2}$. Therefore, $n(n-1) \geq 4m+2 \Rightarrow \left(n - \frac{1}{2}\right)^2 \geq 4m + \frac{9}{4} \Rightarrow n \geq \sqrt{4m + \frac{9}{4}} + \frac{1}{2}$. Hence, take

$$n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil \tag{2.4}$$

Let $f : V(H_m) \to \mathcal{P}(Z_n)$ be a modular sumset labeling of H_m such that $|f(u)| = |f(v)| = 2$, where n is as in (2.4).

Claim : f admits strong modular sumset labeling for H_m.

For $m \geq 3, |f(u)|\,|f(v)| = 4 \leq n$. Now, we show that $D_{f(u)} \cap D_{f(v)} = \emptyset$, $\forall uv \in E(H_m)$. Let $V(H_m) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_m, u_0\}$ where $v_1, v_2, \ldots, v_m$ are the pendant vertices, $u_1, u_2, \ldots, u_m$ are the rim vertices and u_0 is the apex vertex. Assign vertex u_0 with the unique subset of $P(Z_n)$ having difference set $D_A^{(n-1)}$. Consider the sequence of vertices $v_1, u_1, u_2, v_2, v_3, u_3, u_4, v_4, \ldots,$ $v_{m-2}, u_{m-2}, u_{m-1}, v_{m-1}, v_m, u_m$ if m is odd and $v_1, u_1, u_2, v_2, v_3, u_3, u_4, v_4,$ $\ldots, v_{m-1},$ u_{m-1}, u_m, v_m if m is even. Depending on m, rename the vertices in the sequence as $w_1, w_2, \ldots, w_{2m-1}, w_{2m}$. We have three cases based on n.

Case (i) : n is even

Label the vertices as in cycle graph with $2m+1$ vertices.

Case (ii) : n is odd and $2m+1 = \binom{n-1}{2} + q,\ 1 \leq q < n-1$

Label the vertices as in cycle graph with $2m+1$ vertices.

Case (iii) : n is odd and $2m+1 = \binom{n-1}{2} + n - 1 = \binom{n}{2}$

Follow same procedure as in Case (ii) to label all the vertices except v_{m-1}, v_m, u_m

($i.e$ $w_{2m-2}, w_{2m-1}, w_{2m}$). We are left with two subsets of $P(Z_n)$ having difference set $D_A^{(n-2)}$ and one subset of $P(Z_n)$ having difference set $D_A^{(n-3)}$. Assign v_{m-1}, u_m with a subset of $P(Z_n)$having difference set $D_A^{(n-2)}$ and v_m with a subset of $P(Z_n)$ having difference set $D_A^{(n-3)}$.

For $m = 3$, we have $n = 5$. Assign u_0 with $\{0, 4\}$, v_1, v_2, v_3 with $\{0, 1\}, \{0, 2\}, \{0, 3\}$ respectively and u_1, u_2, u_3 with $\{1, 3\}, \{1, 4\}, \{1, 2\}$ respectively. For $m = 5$, we have $n = 6$. Assign u_0 with $\{0, 5\}$, v_i and u_i, $1 \leq i \leq 4$ as in the general case and v_5, u_5 with a subset of $P(Z_n)$ having difference set $D_A^{(3)}$ and $D_A^{(4)}$. □.

Example 2.4. *Figure 4 illustrates a strong modular sumset labeling of H_7. By Theorem 2.4, we have $n = 6$. Hence, the strong modular sumset number of H_7 is 6.*

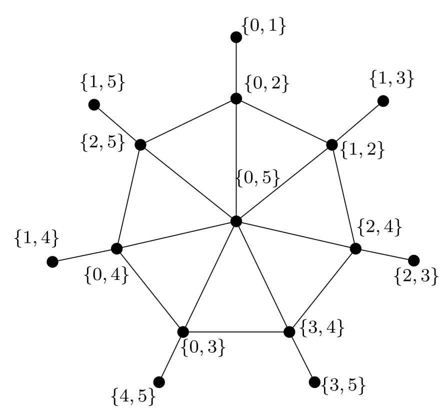

FIGURE 4. Strong modular sumset labeling of H_7

Theorem 2.5. *The strong modular sumset number n of a closed helm CH_m, $m \geq 3$ is given by* $n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil$ *if sets assigned to the vertices are of cardinality two.*

Proof. Argument is similiar to Theorem 2.4. □.

Example 2.5. *Figure 5 illustrates a strong modular sumset labeling of CH_7. By Theorem 2.5, we have $n = 6$. Hence, the strong modular sumset number of CH_7 is 6.*

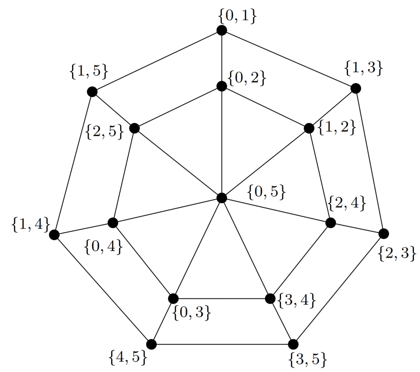

FIGURE 5. Strong modular sumset labeling of CH_7

Theorem 2.6. *The strong modular sumset number* n *of a gear graph* $G_m,\ m \geq 3$ *is given by* $n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil$ *if sets assigned to the vertices are of cardinality two.*

Proof. A similar argument as in Theorem 2.4 gives

$$n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil \tag{2.5}$$

Let $f : V(G_m) \to \mathcal{P}(Z_n)$ be a modular sumset labeling of G_m such that $|f(u)| = |f(v)| = 2$, where n is as in (2.5).

Claim : f admits strong modular sumset labeling for G_m.

For $m \geq 3, |f(u)|\,|f(v)| = 4 \leq n$. Now, we show that $D_{f(u)} \cap D_{f(v)} = \emptyset, \forall uv \in E(G_m)$. Let $V(G_m) = \{u_1, v_1, v_2, \ldots, v_{2m}\}$ where $\{v_1, v_2, \ldots, v_{2m}\}$ are the rim vertices and u_1 is the apex vertex. Consider the labeling as in wheel graph $W_{2m} = C_{2m} + K_1$. It is easy to see that this is the required labeling. $\square$.

Example 2.6. *Figure 6 illustrates a strong modular sumset labeling of* G_7*. By Theorem* 2.6,*we have* $n = 6$*. Hence, the strong modular sumset number of* G_7 *is* 6.

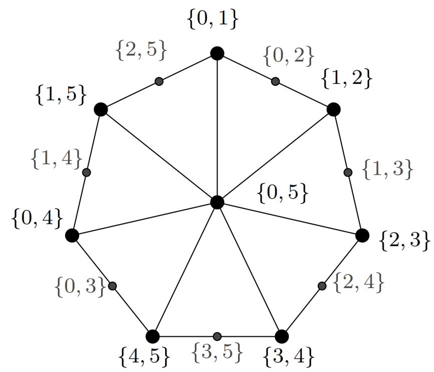

FIGURE 6. Strong modular sumset labeling of G_7

Theorem 2.7. *The strong modular sumset number* n *of a friendship graph* F_m, $m \geq 2$ *is given by* $n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil$ *if sets assigned to the vertices are of cardinality two.*

Proof. A similar argument as in Theorem 2.4, gives

$$n = \left\lceil \sqrt{4m + \frac{9}{4}} + \frac{1}{2} \right\rceil \tag{2.6}$$

Let $f : V(F_m) \to \mathcal{P}(Z_n)$ be a modular sumset labeling of F_m such that $|f(u)| = |f(v)| = 2$ where n is as in (2.6).

Claim : f admits strong modular sumset labeling for F_m.

For $m \geq 2, |f(u)|\,|f(v)| = 4 \leq n$. Now, we show that $D_{f(u)} \cap D_{f(v)} = \emptyset, \forall uv \in E(F_m)$. Let $V(F_m) = \{u_1, v_1, v_2, \ldots, v_{2m}\}$ where u_1 be the apex vertex and $\{v_1, v_2, \ldots, v_{2m}\}$ are the other vertices such that v_{2i-1} and v_{2i} are adjacent for $i = 1, 2, \ldots, m$. Consider the labeling as in wheel graph W_{2m}. It is easy to see that this is the required labeling. $\square$.

Example 2.7. *Figure 7 illustrates a strong modular sumset labeling of* F_7*. By Theorem* 2.7*, we have* $n = 6$*. Hence, the strong modular sumset number of* F_7 *is* 6.

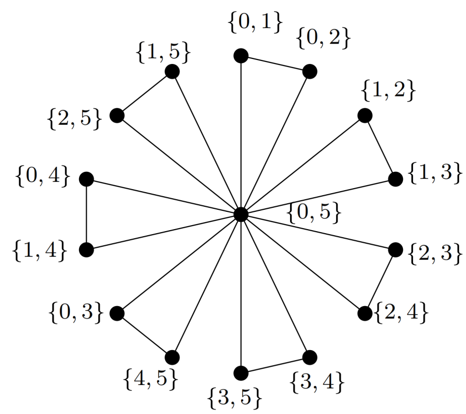

FIGURE 7. Strong modular sumset labeling of F_7

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