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SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR n-TIMES log-CONVEX FUNCTIONS

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Abstract. In this paper, some Hermite-Hadamard type inequalities for n-times differentiable log-convex functions are established.

1. Introduction

The following definitions are well know in the literature

Definition 1. [13] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2. [13] A positive function $f : I \to \mathbb{R}$ is said to be logarithmically convex, if

$$f(tx + (1-t)y) \ \leq [f(x)]^t\,[f(y)]^{1-t}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

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One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality, which can be formulated as follows: for each convex function f over the finite interval $[a,b]$, we have

$$f\left(\tfrac{a+b}{2}\right)\leq \tfrac{1}{b-a}\int\limits_{a}^{b} f(x)dx\leq \tfrac{f(a)+f(b)}{2}. \tag{1.1}$$

If the function f is concave, then (1.1) holds in the reverse direction (see [11]).

The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1.1) have appeared in the literature, see [3, 6, 10] and references therein.

In [2] Dragomir and Agarwal gave the following inequality connected with inequality (1.1)

$$\left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int_{a}^{b} f(x)\,dx\right|\leq \tfrac{b-a}{8}\left(|f'(a)|+|f'(b)|\right).$$

In [12] Pearce and Pečarić investigate the following inequality

$$\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{b-a}\int_{a}^{b} f(x)\,dx\right|\leq \tfrac{b-a}{4}\left(\tfrac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}.$$

In [4], Kavurmaci et al. gave the following generalized trapezoid type inequality

$$\begin{aligned}&\left|\tfrac{(b-x)f(b)+(x-a)f(a)}{b-a}-\tfrac{1}{b-a}\int\limits_{a}^{b} f(t)dt\right|\\ &\leq\tfrac{(x-a)^{2}}{b-a}\left(\tfrac{2|f'(a)|+|f'(x)|}{6}\right)+\tfrac{(b-x)^{2}}{b-a}\left(\tfrac{2|f'(b)|+|f'(x)|}{6}\right).\end{aligned}$$

Cerone and Dragomir [1], proved the following midpoint inequality for log-convex first derivative

$$\begin{aligned}&\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{b-a}\int_{a}^{b} f(t)\,dt\right|\\ &\leq\tfrac{b-a}{4}\left\{|f'(a)|\,\tfrac{\alpha\ln\alpha+1-\alpha}{(\ln\alpha)^{2}}+|f'(b)|\,\tfrac{\beta\ln\beta+1-\beta}{(\ln\beta)^{2}}\right\},\end{aligned}$$

where $\alpha=\frac{\left|f'(\frac{a+b}{2})\right|}{|f'(a)|}$ and $\beta=\frac{\left|f'(\frac{a+b}{2})\right|}{|f'(b)|}$.

In [15], Sarikaya et al. gave the following midpoint type inequlities for log-convex first derivative

$$\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|\leq (b-a)\left(\tfrac{|f'(b)|^{\frac{1}{2}}-|f'(a)|^{\frac{1}{2}}}{\ln|f'(b)|-\ln|f'(a)|}\right)^2.$$

In [9], Meftah et al. used the following identity for establishing some n-times Hermite-Hadamard type inequalities

Lemma 1. *[9] Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$ with $a<b$. Then the sum*

$$\begin{aligned}C(f,x,n,\lambda):&=\sum_{p=0}^{n-1}\tfrac{1}{(n-p)!(b-a)^p}\left(\left(\lambda-\tfrac{x-a}{b-a}\right)^{n-p}-(-1)^{n-p}(1-\lambda)^{n-p}\right)\\&\times f^{(n-1-p)}((1-\lambda)a+\lambda b)-\left(\left(\tfrac{b-x}{b-a}-\lambda\right)^{n-p}-(1-\lambda)^{n-p}\right)\\&\times f^{(n-1-p)}(\lambda a+(1-\lambda)b)+\tfrac{(-1)^n}{(b-a)^n}\int_a^b f(t)\,dt\end{aligned}$$

satisfies the equality

$$C(f,x,n,\lambda)=\int_a^b k_n(x,t)\,f^n(t)dt$$

for all $x\in[a,b]$ and $\lambda\in\left[\frac{1}{2},1\right]$, where $n\in\mathbb{N}$ and the kernel $k_n(x,t):\ [a,b]^2\longrightarrow\mathbb{R}$ is given by

$$k_n(x,t)=\begin{cases}\frac{1}{n!}\left(\frac{t-a}{b-a}\right)^n & \text{if } t\in[a,\lambda a+(1-\lambda)b)\\ \frac{1}{n!}\left(\frac{t-x}{b-a}\right)^n & \text{if } t\in[\lambda a+(1-\lambda)b,(1-\lambda)a+\lambda b)\\ \frac{1}{n!}\left(\frac{t-b}{b-a}\right)^n & \text{if } t\in[(1-\lambda)a+\lambda b,b].\end{cases}\tag{1.2}$$

Motivated by the above results in this paper, by using the identity given in Lemma 1, we establish some Hermite-Hadamard type inequalities for n-times differentiable log-convex functions.

2. Main results

In order to prove our results we need the following lemma

Lemma 2. *[6] Let $\mu > 0, \mu \neq 1$ and $n \in \mathbb{N}$. Then*

$$\int_0^1 t^n \mu^t dt = \tfrac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \tfrac{(-1)^k}{(n-k)!(\ln \mu)^{k+1}}.$$

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. If $\left|f^{(n)}\right|$ is* log*-convex function such that $f^{(n)}(\mu) \neq 0$ for all $\mu \in [a,b]$, then for all $x \in [\lambda a + (1-\lambda) b, (1-\lambda) a + \lambda b]$ and $\lambda \in [\frac{1}{2}, 1]$ the following inequalities hold*

for $\lambda = \frac{1}{2}$

$$\left|C\left(f,x,n,\tfrac{1}{2}\right)\right| \leq \tfrac{b-a}{n!2^{n+1}} \left(\left|f^{(n)}(a)\right| \varphi\left(\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|, \left|f^{(n)}(a)\right|\right)\right.$$
$$\left. + \left|f^{(n)}(b)\right| \varphi\left(\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|, \left|f^{(n)}(b)\right|\right)\right),$$

for $\lambda = 1$

$$\left|C(f,x,n,1)\right| \leq \tfrac{(x-a)^{n+1}\left|f^{(n)}(x)\right|}{n!(b-a)^n} \varphi\left(\left|f^{(n)}(a)\right|, \left|f^{(n)}(x)\right|\right)$$
$$+ \tfrac{(b-x)^{n+1}\left|f^{(n)}(x)\right|}{n!(b-a)^n} \varphi\left(\left|f^{(n)}(b)\right|, \left|f^{(n)}(x)\right|\right),$$

for $\lambda \in \left(\frac{1}{2}, 1\right)$

$$\left|C(f,x,n,\lambda)\right|$$
$$\leq \tfrac{(b-a)(1-\lambda)^{n+1}}{n!} \left|f^{(n)}(a)\right| \varphi\left(\left|f^{(n)}(\lambda a + (1-\lambda) b)\right|, \left|f^{(n)}(a)\right|\right)$$
$$+ \tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n} \left|f^{(n)}(x)\right| \varphi\left(\left|f^{(n)}(\lambda a + (1-\lambda) b)\right|, \left|f^{(n)}(x)\right|\right)$$
$$+ \tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n} \left|f^{(n)}(x)\right| \varphi\left(\left|f^{(n)}((1-\lambda) a + \lambda b)\right|, \left|f^{(n)}(x)\right|\right)$$
$$+ \tfrac{(b-a)(1-\lambda)^{n+1}}{n!} \left|f^{(n)}(b)\right| \varphi\left(\left|f^{(n)}((1-\lambda) a + \lambda b)\right|, \left|f^{(n)}(b)\right|\right),$$

where

$$\varphi(y,z)=\begin{cases}\dfrac{(-1)^{n+1}n!}{\left(\ln\left|f^{(n)}(y)\right|-\ln\left|f^{(n)}(z)\right|\right)^{n+1}} \\ +n!\dfrac{\left|f^{(n)}(y)\right|}{\left|f^{(n)}(z)\right|}\sum_{k=0}^{k=n}\dfrac{(-1)^k}{(n-k)!\left(\ln\left|f^{(n)}(y)\right|-\ln\left|f^{(n)}(z)\right|\right)^{k+1}} & if\ y\neq z \\ \dfrac{1}{n+1} & if\ y=z.\end{cases}$$

(2.1)

Proof. Using Lemma 1, and the properties of modulus, we have

$$\begin{aligned}&|C(f,x,n,\lambda)| \\ =&\left|\int_a^b k_n(x,t)\,f^{(n)}(t)dt\right| \\ \leq&\int_a^b |k_n(x,t)|\left|f^{(n)}(t)\right|dt \\ =&\tfrac{1}{(b-a)^n}\int_a^{\lambda a+(1-\lambda)b}\tfrac{(t-a)^n}{n!}\left|f^{(n)}(t)\right|dt+\tfrac{1}{(b-a)^n}\int_{\lambda a+(1-\lambda)b}^{x}\tfrac{(x-t)^n}{n!}\left|f^{(n)}(t)\right|dt \\ &+\tfrac{1}{(b-a)^n}\int_x^{(1-\lambda)a+\lambda b}\tfrac{(t-x)^n}{n!}\left|f^{(n)}(t)\right|dt+\tfrac{1}{(b-a)^n}\int_{(1-\lambda)a+\lambda b}^{b}\tfrac{(b-t)^n}{n!}\left|f^{(n)}(t)\right|dt \\ =&\tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|d\alpha \\ &+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|d\alpha \\ &+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|d\alpha \\ &+\tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|d\alpha.\end{aligned}\tag{2.2}$$

Using the log-convexity of $\left|f^{(n)}\right|$, (2.2) gives

$$\begin{aligned}
&\left|C\left(f,x,n,\lambda\right)\right| \\
&\leq \tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\left|f^{(n)}(a)\right|\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|}{\left|f^{(n)}(a)\right|}\right)^{\alpha}d\alpha \\
&\quad+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\left|f^{(n)}(x)\right|\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|}{\left|f^{(n)}(x)\right|}\right)^{\alpha}d\alpha \\
&\quad+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\left|f^{(n)}(x)\right|\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}((1-\lambda)a+\lambda b)\right|}{\left|f^{(n)}(x)\right|}\right)^{\alpha}d\alpha \\
&\quad+\tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\left|f^{(n)}(b)\right|\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}((1-\lambda)a+\lambda b)\right|}{\left|f^{(n)}(b)\right|}\right)^{\alpha}d\alpha. \qquad (2.3)
\end{aligned}$$

If $\lambda=\frac{1}{2}$, then (2.3) becomes

$$\begin{aligned}
\left|C\left(f,x,n,\tfrac{1}{2}\right)\right| &\leq \tfrac{b-a}{n!2^{n+1}}\left|f^{(n)}(a)\right|\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|}{\left|f^{(n)}(a)\right|}\right)^{\alpha}d\alpha \\
&+\tfrac{b-a}{n!2^{n+1}}\left|f^{(n)}(b)\right|\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|}{\left|f^{(n)}(b)\right|}\right)^{\alpha}d\alpha.
\end{aligned}$$

Using Lemma 2, the above inequality gives

$$\begin{aligned}
\left|C\left(f,x,n,\tfrac{1}{2}\right)\right| &\leq \tfrac{b-a}{n!2^{n+1}}\left(\left|f^{(n)}(a)\right|\varphi\left(\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|,\left|f^{(n)}\left(a\right)\right|\right)\right. \\
&\left.+\left|f^{(n)}\left(b\right)\right|\varphi\left(\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|,\left|f^{(n)}\left(b\right)\right|\right)\right), \qquad (2.4)
\end{aligned}$$

where $\varphi\left(.,.\right)$ is defined as in (2.1).

If $\lambda=1$, then from (2.3) we have

$$\left|C\left(f,x,n,1\right)\right|\leq\tfrac{(x-a)^{n+1}\left|f^{(n)}(x)\right|}{n!(b-a)^n}\int\limits_0^1\alpha^n\left(\tfrac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(x)\right|}\right)^{\alpha}d\alpha$$

$$+\frac{(b-x)^{n+1}\left|f^{(n)}(x)\right|}{n!(b-a)^{n}}\int_{0}^{1}\alpha^{n}\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(x)\right|}\right)^{\alpha}d\alpha. \tag{2.5}$$

Using Lemma 2, we get

$$\begin{aligned}|C(f,x,n,1)| &\leq \frac{(x-a)^{n+1}\left|f^{(n)}(x)\right|}{n!(b-a)^{n}}\varphi\left(\left|f^{(n)}(a)\right|,\left|f^{(n)}(x)\right|\right)\\ &+\frac{(b-x)^{n+1}\left|f^{(n)}(x)\right|}{n!(b-a)^{n}}\varphi\left(\left|f^{(n)}(b)\right|,\left|f^{(n)}(x)\right|\right),\end{aligned} \tag{2.6}$$

where $\varphi(.,.)$ is defined as in (2.1).

And if $\lambda\neq1$ and $\lambda\neq\frac{1}{2}$, from (2.3) and Lemma 2, we obtain

$$\begin{aligned}&|C(f,x,n,\lambda)|\\ \leq&\frac{(b-a)(1-\lambda)^{n+1}}{n!}\left|f^{(n)}(a)\right|\varphi\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|,\left|f^{(n)}(a)\right|\right)\\ &+\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^{n}}\left|f^{(n)}(x)\right|\varphi\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|,\left|f^{(n)}(x)\right|\right)\\ &+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^{n}}\left|f^{(n)}(x)\right|\varphi\left(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|,\left|f^{(n)}(x)\right|\right)\\ &+\frac{(b-a)(1-\lambda)^{n+1}}{n!}\left|f^{(n)}(b)\right|\varphi\left(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|,\left|f^{(n)}(b)\right|\right),\end{aligned} \tag{2.7}$$

where $\varphi(.,.)$ is defined as in (2.1). The desired result follows from (2.4), (2.6) and (2.7). □

Corollary 1. *In Theorem 1, if we take $n=\lambda=1$, we obtain the following generalized trapezoidal type inequalities*

$$\begin{aligned}&\left|\frac{b-x}{b-a}f(b)+\frac{x-a}{b-a}f(a)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\ \leq&\frac{b-a}{4}\left\{|f'(a)|\varphi\left(\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(a)|\right)+|f'(b)|\varphi\left(\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(b)|\right)\right\}.\end{aligned}$$

Hence, the explicit writing is

$$\left|\frac{b-x}{b-a}f(b)+\frac{x-a}{b-a}f(a)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$

$$\leq \begin{cases} \frac{(x-a)^2+(b-x)^2}{2(b-a)} \left|f'(x)\right| \text{ if } \left|f'(a)\right| = \left|f'(x)\right| = \left|f'(b)\right|, \\ \frac{(x-a)^2}{2(b-a)} \left|f'(x)\right| + \frac{(b-x)^2}{b-a} \left(\frac{|f'(b)|}{\ln|f'(b)|-\ln|f'(x)|} + \frac{|f'(x)|-|f'(b)|}{(\ln|f'(b)|-\ln|f'(x)|)^2} \right) \\ \text{if } \left|f'(a)\right| = \left|f'(x)\right| \neq \left|f'(b)\right|, \\ \frac{(x-a)^2}{b-a} \left(\frac{|f'(a)|}{\ln|f'(a)|-\ln|f'(x)|} + \frac{|f'(x)|-|f'(a)|}{(\ln|f'(a)|-\ln|f'(x)|)^2} \right) + \frac{(b-x)^2}{2(b-a)} \left|f'(x)\right| \\ \text{if } \left|f'(a)\right| \neq \left|f'(x)\right| = \left|f'(b)\right|, \\ \frac{(x-a)^2}{b-a} \left(\frac{|f'(a)|}{\ln|f'(a)|-\ln|f'(x)|} + \frac{|f'(x)|-|f'(a)|}{(\ln|f'(a)|-\ln|f'(x)|)^2} \right) \\ +\frac{(b-x)^2}{b-a} \left(\frac{|f'(b)|}{\ln|f'(b)|-\ln|f'(x)|} + \frac{|f'(x)|-|f'(b)|}{(\ln|f'(b)|-\ln|f'(x)|)^2} \right) \\ \text{if } \left|f'(a)\right| \neq \left|f'(x)\right| \neq \left|f'(b)\right|. \end{cases}$$

Corollary 2. *In Theorem 1, if we take $n = \lambda = 1$, and $x = \frac{a+b}{2}$, we obtain the following trapezoidal type inequalities*

$$\left| \tfrac{f(b)+f(a)}{2} - \tfrac{1}{b-a} \int_a^b f(t)\, dt \right|$$
$$\leq \tfrac{b-a}{4} \left\{ \left|f'(\tfrac{a+b}{2})\right| \varphi\left(\left|f'(a)\right|, \left|f'(\tfrac{a+b}{2})\right|\right) + \left|f'(\tfrac{a+b}{2})\right| \varphi\left(\left|f'(b)\right|, \left|f'(\tfrac{a+b}{2})\right|\right) \right\}.$$

Corollary 3. *In Theorem 1, if we take $n = 2\lambda = 1$, we obtain the following midpoint type inequalities*

$$\left| f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a} \int_a^b f(t)\, dt \right|$$
$$\leq \tfrac{b-a}{4} \left\{ \left|f'(a)\right| \varphi\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|, \left|f'(a)\right|\right) + \left|f'(b)\right| \varphi\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|, \left|f'(b)\right|\right) \right\}.$$

Remark 1. *The result in Corollary 3 is the same of Corollary 3 from [1].*

Remark 2. *Corollary 3 will be reduced to Corollary 1 from [15], if we assume that $\left|f'(a)\right| \neq \left|f'(x)\right| \neq \left|f'(b)\right|$ and we used the* log*-convexity of $|f'|$ i.e.$\left|f'\left(\frac{a+b}{2}\right)\right| \leq \sqrt{\left|f'(a)\right| \left|f'(b)\right|}$ (see [3]).*

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. If $\left|f^{(n)}\right|^{q}$ such that $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ is* log*-convex function, such that $f^{(n)}(\mu)\neq 0$ for all $\mu\in[a,b]$, then for all $x\in[\lambda a+(1-\lambda)b,(1-\lambda)a+\lambda b]$ and $\lambda\in[\frac{1}{2},1]$ the following inequalities hold*
for $\lambda=\frac{1}{2}$

$$\left|C\left(f,x,n,\tfrac{1}{2}\right)\right|\leq\frac{b-a}{n!2^{n+1}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(F\left(\left|f^{(n)}(\tfrac{a+b}{2})\right|^{q},\left|f^{(n)}(a)\right|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{n!2^{n+1}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(\tfrac{a+b}{2})\right|\left(F\left(\left|f^{(n)}(b)\right|^{q},\left|f^{(n)}(\tfrac{a+b}{2})\right|^{q}\right)\right)^{\frac{1}{q}},$$

for $\lambda=1$

$$\left|C\left(f,x,n,1\right)\right|\leq\frac{(x-a)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(F\left(\left|f^{(n)}(x)\right|^{q},\left|f^{(n)}(a)\right|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(x)\right|\left(F\left(\left|f^{(n)}(b)\right|^{q},\left|f^{(n)}(x)\right|^{q}\right)\right)^{\frac{1}{q}},$$

for $\lambda\in\left(\frac{1}{2},1\right)$

$$\left|C\left(f,x,n,\lambda\right)\right|$$
$$\leq\frac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(F\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^{q},\left|f^{(n)}(a)\right|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|\left(F\left(\left|f^{(n)}(x)\right|^{q},\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(x)\right|\left(F\left(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^{q},\left|f^{(n)}(x)\right|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left|f^{(n)}((1-\lambda)a+\lambda b)\right|\left(F\left(\left|f^{(n)}(b)\right|^{q},\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^{q}\right)\right)^{\frac{1}{q}},$$

where

$$F(y,z)=\begin{cases}\frac{y-z}{z(\ln y-\ln z)} & \text{if } y\neq z\\ 1 & \text{if } y=z.\end{cases}\qquad(2.8)$$

Proof. Using Lemma 1, the properties of modulus, and Hölder's inequality, we get

$$\left|C\left(f,x,n,\lambda\right)\right|$$

$$\leq \tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\left(\int_0^1 \alpha^{np}d\alpha\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\left(\int_0^1 (1-\alpha)^{np}d\alpha\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\left(\int_0^1 \alpha^{np}d\alpha\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!}\left(\int_0^1 (1-\alpha)^{np}d\alpha\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$=\tfrac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is log-convex, we deduce

$$\begin{aligned}(2.9)\qquad & \left|C\left(f,x,n,\lambda\right)\right| \\ \leq & \frac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q}{\left|f^{(n)}(a)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}} \\ & +\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}(x)\right|^q}{\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}} \\ & +\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(x)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q}{\left|f^{(n)}(x)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}} \\ & +\frac{(1-\lambda)^{n+1}(b-a)}{n!(np+1)^{\frac{1}{p}}}\left|f^{(n)}\left((1-\lambda)a+\lambda b\right)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}}.\end{aligned}$$

If $\lambda=\frac{1}{2}$, then from (2.9) we have

$$\begin{aligned}& \left|C\left(f,x,n,\lambda\right)\right| \\ \leq & \frac{b-a}{n!2^{n+1}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{\left|f^{(n)}(a)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}} \\ & +\frac{b-a}{n!2^{n+1}(np+1)^{\frac{1}{p}}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}} \\ = & \frac{b-a}{n!2^{n+1}(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(\digamma\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q,\left|f^{(n)}(a)\right|^q\right)\right)^{\frac{1}{q}} \\ (2.10)\qquad & +\frac{b-a}{n!2^{n+1}(np+1)^{\frac{1}{p}}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\left(\digamma\left(\left|f^{(n)}\left(b\right)\right|^q,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}.\end{aligned}$$

If $\lambda=1$, then from (2.9) we have

$$\left|C\left(f,x,n,\lambda\right)\right|$$

$$\leq \frac{(x-a)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}(x)\right|^q}{\left|f^{(n)}(a)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(x)\right|\left(\int_0^1\left(\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(x)\right|^q}\right)^{\alpha}d\alpha\right)^{\frac{1}{q}}$$

$$= \frac{(x-a)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(\digamma\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(a)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(x)\right|\left(\digamma\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(x)\right|^q\right)\right)^{\frac{1}{q}}. \tag{2.11}$$

And if $\lambda \neq 1$ and $\lambda \neq \frac{1}{2}$, from (2.9) we have

$$|C(f,x,n,\lambda)|$$

$$\leq \frac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left|f^{(n)}(a)\right|\left(\digamma\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q,\left|f^{(n)}(a)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|\left(\digamma\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left|f^{(n)}(x)\right|\left(\digamma\left(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q,\left|f^{(n)}(x)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left|f^{(n)}((1-\lambda)a+\lambda b)\right|\left(\digamma\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q\right)\right)^{\frac{1}{q}}. \tag{2.12}$$

The desired result follows from (2.10), (2.11) and (2.12). □

Corollary 4. *In Theorem 2, if we take $n=\lambda=1$, we obtain the following generalized trapezoidal type inequalities*

$$\left|\frac{b-x}{b-a}f(b)+\frac{x-a}{b-a}f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{(x-a)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left(\digamma\left(\left|f'(x)\right|^q,\left|f'(a)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left|f'(x)\right|\left(\digamma\left(\left|f'(b)\right|^q,\left|f'(x)\right|^q\right)\right)^{\frac{1}{q}}.$$

Corollary 5. *In Theorem 2, if we take $n = \lambda = 1$, and $x = \frac{a+b}{2}$, we obtain the following trapezoidal type inequalities*

$$\begin{aligned}
&\left|\tfrac{f(b)+f(a)}{2} - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq &\tfrac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left\{|f'(a)|\left(F\left(\left|f'(\tfrac{a+b}{2})\right|^q, |f'(a)|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left. + \left|f'(\tfrac{a+b}{2})\right|\left(F\left(|f'(b)|^q, \left|f'(\tfrac{a+b}{2})\right|^q\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 6. *In Theorem 2, if we take $n = 2\lambda = 1$, we obtain the following midpoint type inequalities*

$$\begin{aligned}
&\left|f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq &\tfrac{b-a}{4(p+1)^{\frac{1}{p}}}\left\{|f'(a)|\left(F\left(\left|f'(\tfrac{a+b}{2})\right|^q, |f'(a)|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left. + \left|f'(\tfrac{a+b}{2})\right|\left(F\left(|f'(b)|^q, \left|f'(\tfrac{a+b}{2})\right|^q\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 3. *In Corollary 6, if we assume that $|f'(a)| \neq |f'(x)| \neq |f'(b)|$ and we used the* log-*convexity of $|f'|$ i.e.$\left|f'\left(\frac{a+b}{2}\right)\right| \leq \sqrt{|f'(a)|\,|f'(b)|}$, we obtain*

$$\begin{aligned}
&\left|f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq &\tfrac{b-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}}\left(|f'(b)|^{\frac{1}{2}} + |f'(a)|^{\frac{1}{2}}\right)\left(\tfrac{|f'(b)|^{\frac{q}{2}} - |f'(a)|^{\frac{q}{2}}}{(\ln|f'(b)| - \ln|f'(a)|)}\right)^{\frac{1}{q}},
\end{aligned}$$

which is the correct result of Corollary 2 (also Theorem 11) from [15].

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. If $\left|f^{(n)}\right|^q$ with $q > 1$ is convex and positive functions, such that $f^{(n)}(\mu) \neq 0$ for all $\mu \in [a,b]$, then for all $x \in [\lambda a + (1-\lambda)b, (1-\lambda)a + \lambda b]$, and $\lambda \in [\frac{1}{2}, 1]$ the following inequalities hold*
for $\lambda = \frac{1}{2}$

$$|C(f, x, n, \lambda)|$$

$$\leq \frac{b-a}{n!2^{n+1}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(a)\right| \left(\varphi\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}, \left|f^{(n)}(a)\right|^{q}\right)\right)^{\frac{1}{q}}$$

$$+ \frac{b-a}{n!2^{n+1}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(b)\right| \left(\varphi\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}, \left|f^{(n)}(b)\right|^{q}\right)\right)^{\frac{1}{q}},$$

for $\lambda = 1$

$$|C(f,x,n,\lambda)| \leq \frac{(x-a)^{n+1}}{n!(b-a)^{n}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\varphi\left(\left|f^{(n)}(a)\right|^{q}, \left|f^{(n)}(x)\right|^{q}\right)\right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)^{n+1}}{n!(b-a)^{n}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\varphi\left(\left|f^{(n)}(b)\right|^{q}, \left|f^{(n)}(x)\right|^{q}\right)\right)^{\frac{1}{q}},$$

for $\lambda \in \left(\frac{1}{2}, 1\right)$

$$|C(f,x,n,\lambda)|$$

$$\leq \frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(a)\right| \left(\varphi\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^{q}, \left|f^{(n)}(a)\right|^{q}\right)\right)^{\frac{1}{q}}$$

$$+ \frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^{n}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\varphi\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^{q}, \left|f^{(n)}(x)\right|^{q}\right)\right)^{\frac{1}{q}}$$

$$+ \frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^{n}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\varphi\left(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^{q}, \left|f^{(n)}(x)\right|^{q}\right)\right)^{\frac{1}{q}}$$

$$+ \frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(b)\right| \left(\varphi\left(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^{q}, \left|f^{(n)}(b)\right|^{q}\right)\right)^{\frac{1}{q}},$$

where $\varphi(.,.)$ *is defined as in* (2.1).

Proof. Using Lemma 1, the properties of modulus, and power mean inequality, we get

$$|C(f,x,n,\lambda)|$$

$$\leq \frac{(b-a)(1-\lambda)^{n+1}}{n!} \left(\int_{0}^{1} \alpha^{n} d\alpha\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{0}^{1} \alpha^{n} \left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|^{q} d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\left(\int_0^1(1-\alpha)^n\,d\alpha\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\left(\int_0^1\alpha^n d\alpha\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)(1-\lambda)^{n+1}}{n!}\left(\int_0^1(1-\alpha)^n\,d\alpha\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}}.$$

Using the log-convexity of $\left|f^{(n)}\right|^q$, we get

(2.13) $\quad |C(f,x,n,\lambda)|$

$$\leq \frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(a)\right| \left(\int_0^1 \alpha^n \left(\frac{\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q}{\left|f^{(n)}(a)\right|^q} \right)^{\alpha} d\alpha \right)^{\frac{1}{q}}$$

$$+ \frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\int_0^1 \alpha^n \left(\frac{\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q}{\left|f^{(n)}(x)\right|^q} \right)^{\alpha} d\alpha \right)^{\frac{1}{q}}$$

$$+ \frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\int_0^1 \alpha^n \left(\frac{\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q}{\left|f^{(n)}(x)\right|^q} \right)^{\alpha} d\alpha \right)^{\frac{1}{q}}$$

$$+ \frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(b)\right| \left(\int_0^1 \alpha^n \left(\frac{\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q}{\left|f^{(n)}(b)\right|^q} \right)^{\alpha} d\alpha \right)^{\frac{1}{q}}.$$

If $\lambda = \frac{1}{2}$, then from (2.13) and Lemma 2, we have

$$\left|C(f,x,n,\lambda)\right|$$

$$\leq \frac{b-a}{n!2^{n+1}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(a)\right| \left(\int_0^1 \alpha^n \left(\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{\left|f^{(n)}(a)\right|^q} \right)^{\alpha} d\alpha \right)^{\frac{1}{q}}$$

$$+ \frac{b-a}{n!2^{n+1}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(b)\right| \left(\int_0^1 \alpha^n \left(\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{\left|f^{(n)}(b)\right|^q} \right)^{\alpha} d\alpha \right)^{\frac{1}{q}}$$

$$= \frac{b-a}{n!2^{n+1}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(a)\right| \left(\varphi\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q, \left|f^{(n)}(a)\right|^q \right) \right)^{\frac{1}{q}}$$

$$+ \frac{b-a}{n!2^{n+1}(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(b)\right| \left(\varphi\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q, \left|f^{(n)}(b)\right|^q \right) \right)^{\frac{1}{q}}, \tag{2.14}$$

where $\varphi(.,.)$ is defined as in (2.1).

If $\lambda = 1$, then from (2.13) and Lemma 2, we have

$$\left|C(f,x,n,\lambda)\right| \tag{2.15}$$

$$\leq \frac{(x-a)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\varphi\left(\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(x)\right|^q \right) \right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}} \left|f^{(n)}(x)\right| \left(\varphi\left(\left|f^{(n)}(b)\right|^q, \left|f^{(n)}(x)\right|^q \right) \right)^{\frac{1}{q}}.$$

If $\lambda \neq 1$ and $\lambda \neq \frac{1}{2}$ from (2.13), we have

$$
\begin{aligned}
(2.16)\qquad & \left|C\left(f,x,n,\lambda\right)\right| \\
\leq & \tfrac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left|f^{(n)}(a)\right|\left(\varphi\left(\left|f^{(n)}\left(\lambda a+(1-\lambda)b\right)\right|^{q},\left|f^{(n)}\left(a\right)\right|^{q}\right)\right)^{\frac{1}{q}} \\
& +\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^{n}(n+1)^{1-\frac{1}{q}}}\left|f^{(n)}(x)\right|\left(\varphi\left(\left|f^{(n)}\left(\lambda a+(1-\lambda)b\right)\right|^{q},\left|f^{(n)}\left(x\right)\right|^{q}\right)\right)^{\frac{1}{q}} \\
& +\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^{n}(n+1)^{1-\frac{1}{q}}}\left|f^{(n)}(x)\right|\left(\varphi\left(\left|f^{(n)}\left((1-\lambda)a+\lambda b\right)\right|^{q},\left|f^{(n)}\left(x\right)\right|^{q}\right)\right)^{\frac{1}{q}} \\
& +\tfrac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left|f^{(n)}\left(b\right)\right|\left(\varphi\left(\left|f^{(n)}\left((1-\lambda)a+\lambda b\right)\right|^{q},\left|f^{(n)}\left(b\right)\right|^{q}\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

The desired result follows from (2.14), (2.15) and (2.16). □

Corollary 7. *In Theorem 3, if we take $n=\lambda=1$, we obtain the following generalized trapezoidal type inequalities*

$$
\begin{aligned}
& \left|\tfrac{b-x}{b-a}f\left(b\right)+\tfrac{x-a}{b-a}f\left(a\right)-\tfrac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right| \\
\leq & \tfrac{(x-a)^{2}}{2^{1-\frac{1}{q}}(b-a)}\left|f'(x)\right|\left(\varphi\left(\left|f'\left(a\right)\right|^{q},\left|f'\left(x\right)\right|^{q}\right)\right)^{\frac{1}{q}} \\
& +\tfrac{(b-x)^{2}}{2^{1-\frac{1}{q}}(b-a)}\left|f'(x)\right|\left(\varphi\left(\left|f'\left(b\right)\right|^{q},\left|f'\left(x\right)\right|^{q}\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 8. *In Theorem 3, if we take $n=\lambda=1$, and $x=\frac{a+b}{2}$, we obtain the following trapezoidal type inequalities*

$$
\begin{aligned}
& \left|\tfrac{f(b)+f(a)}{2}-\tfrac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right| \\
\leq & \tfrac{(b-a)}{8}\left\{\left|f'(\tfrac{a+b}{2})\right|\left(2\varphi\left(\left|f'\left(a\right)\right|^{q},\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)\right)^{\frac{1}{q}}\right. \\
& \left.+\left|f'(\tfrac{a+b}{2})\right|\left(2\varphi\left(\left|f'\left(b\right)\right|^{q},\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Corollary 9. *In Theorem 3, if we take $n=2\lambda=1$, we obtain the following midpoint type inequalities*

$$
\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|
$$

$$\leq \frac{b-a}{8}\left\{|f'(a)|\left(2\varphi\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right)\right)^{\frac{1}{q}} + |f'(b)|\left(2\varphi\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(b)|^q\right)\right)^{\frac{1}{q}}\right\}.$$

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