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A SYMBOLIC METHOD FOR FINDING APPROXIMATE SOLUTION OF NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAYS

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Abstract. This paper presents a new symbolic method for finding an approximate solution of neutral functional-differential equations with proportional delays having variable coefficients in an algebraic setting. In several cases exact solution is obtained. This method is easy to apply for solving the multi-pantograph equations with variable coefficients. We introduce *iterative operator*. In the proposed method, the given problem is transformed into an operator based notation and again the solution of operator problem is translated into the solution of the given problem. The Maple implementation of the proposed algorithm is presented with sample computations. Various numerical examples are discussed to illustrate the efficiency of the proposed method, and comparisons are made to confirm the reliability of the method.

1. Introduction

Theory of differential equations is very important for scientific computing, engineering and modelling natural phenomena. We normally come across with differential equations in many theoretical and practical problems which cannot be solved by any

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of the known standard methods. In particular, functional-differential equations play an important role in the mathematical modelling of real world phenomena and the multi-pantograph equations arise in many applications such as probability theory on algebraic structures, electrodynamics, astrophysics, non-linear dynamical systems, quantum mechanics and cell growth, number theory etc. Properties of the analytic solution of these equations as well as numerical methods have been studied by several authors, for example, [7] deals with an approximate solution of multi-pantograph equation with variable coefficients in terms of Taylor polynomials, [24] deals with the variational iteration method to neutral functional-differential equations with proportional delays. However all have considered different methods in classical formulation. This paper deals with symbolic formulation and Maple implementation. Various Maple implementations of different algorithms are discussed in [9–21].

In this paper we consider the neutral functional-differential equation with proportional delays of the following form [5]

$$\begin{aligned} &(y(x)+\beta(x)y(q_nx))^{(n)} = \lambda y(x) + \sum_{j=0}^{n-1} \mu_j(x) y^{(j)}(q_jx) + f(x),\ x \geq 0, \\ &y^{(i)}(0) = c_i,\ i = 0, 1, \dots, n-1. \end{aligned} \tag{1.1}$$

where $\lambda, c_0, c_1, \dots, c_{n-1} \in \mathbb{C}$; $\mu_j(x)$ and $f(x)$ are analytic functions; $0 < q_1 < \cdots < q_{n-1} < 1$, and the superscripts indicate differentiation. In order to apply symbolic method, we write equation (1.1) as

$$\begin{aligned} &y^{(n)}(x) = \lambda y(x) - (\beta(x)y(q_nx))^{(n)} + \sum_{j=0}^{n-1} \mu_j(x) y^{(j)}(q_jx) + f(x),\ x \geq 0, \\ &y^{(i)}(0) = c_i,\ i = 0, 1, \dots, n-1. \end{aligned} \tag{1.2}$$

The multi-pantograph equation is a kind of delay differential equation of the form

$$
\begin{aligned}
& y'(x) = \lambda y(x) + \sum_{i=1}^{n} \mu_i(x) y(q_i x) + f(x), \;\; x \geq 0, \\
& y(0) = c_0,
\end{aligned} \tag{1.3}
$$

The main aim of this paper is to find an approximate solution of the given neutral functional-differential equations with proportional delays as well as the multi-pantograph equations with variable coefficients and a solution operator, so-called *iterative operator*. The key to find such an operator is the transformation of the given neutral functional-differential equation into an operator based notations and then we solve the operator problem using algebraic techniques. The solution of the operator problem is transformed again into the solution of the given neutral functional-differential equation. In Section 2, the proposed method is described. The rest of the paper is organized as follows: In Section 1.1, we recall the preliminary concepts related to integral and differential equations. Section 2 describes the algebraic and symbolic formulation of the proposed method, and its Maple implementation is presented in Section 2.1. Several examples are given in Section 3 to illustrate the efficiency and implementation of the method, sample computations are also presented using Maple implementation.

1.1. **Preliminaries.** For convenience of the reader, we briefly recall the basic concepts and results related to the differential and integral equations required for this paper (see [2, 4, 8]).

Definition 1.1. [4] A vector-valued function $f(x, y)$ is said to satisfy a Lipschitz condition in a region $\mathcal{R}$ if, for some constant L (called Lipschitz constant), we have

$$
\|f(x, y_1) - f(x, y_2)\| \leq L\|y_1 - y_2\|, \tag{1.4}
$$

whenever $(x, y_1), (x, y_2) \in \mathcal{R}$.

The following lemma illustrates the relation between integral equations and differential equation, see [8] for further details.

Lemma 1.1. *(Replacement Lemma) Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Then*

$$\int_a^x \int_a^{x_1} f(t)\ dt\ dx_1 = \int_a^x (x-t)f(t)\ dt,\ \ (x \in [a, b]).$$

Since the transformation of given neutral functional-differential equations arises in the form of initial value problem (IVP). The following theorem gives a solution of a IVP.

Theorem 1.1. *Let $f(x, y(x))$ be a continuous function on the interval $[a, b]$ with fundamental system $1, x, x^2, \ldots, x^{n-1}$ for L, a function $y(x)$ is a solution of the IVP*

$$\begin{aligned} &Ly(x) = f(x, y(x)), \\ &y(a) = c_0, y'(a) = c_1, y''(a) = c_2, \ldots, y^{n-1}(a) = c_{n-1}, \end{aligned} \tag{1.5}$$

if and only if it is a solution of the integral equation

$$y(x) = \sum_{i=0}^{n-1} c_i \frac{(x-a)^i}{i!} + \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(t, y(t))\ dt\ dx_1 \ldots\ dx_{n-1}. \tag{1.6}$$

Proof. Integrating the differential equation (1.5) n times over the interval (a, x) and substituting the initial conditions in the resultant integral equation, we get (1.6). Conversely by differentiating equation (1.6) n times, we get $Ly(x) = f(x, y(x))$ and setting $x = a$ in (1.6) yields $y^{(i)}(a) = c_i,$ for $i = 0, \ldots, n-1$. □

For a given function $y = f(x)$, if it is defined for all x in $|x-a| \leq m$ and is continuous, where $m > 0$ is a fixed constant, then we can define an operator P by

$$P(y) = \sum_{i=0}^{n-1} c_i \frac{(x-a)^i}{i!} + \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(t, y(t))\ dt\ dx_1 \ldots\ dx_{n-1}. \tag{1.7}$$

A solution of the integral equation (1.6) is a fixed point of the operator P, $y = P(y)$. By using the operator P, we can generate a sequence of functions $\{y_i\}$ by the successive iterations from the given initial conditions

$$y_0 \equiv c_0,\ y_i = P(y_{i-1}) = P_i(y_0),\ \text{for } i = 1, 2, \ldots. \tag{1.8}$$

This is called a *Picard's iteration.* We know that a Picard's iteration converges to a solution of the IVP (1.5) (see [2, 4, 8]). This can be stated as follows.

Theorem 1.2. *[4] Assume that $f(x, y)$ is continuous and satisfy the Lipschitz condition* (1.4) *on the interval $|x - a| \leq m$ for all x, y. Then the IVP* (1.5) *has the unique solution on the interval $|x - a| \leq m$.*

2. Algebraic Representation and Symbolic Formulation

In this section, we generalize the results from the Section 1.1 in symbolic formulation corresponding to special choice of $\mathcal{S} = C^\infty[0, 1]$ for simplicity. In Section 2.1 we demonstrate this method in Maple. Recall the equation (1.1) as follows.

$$\begin{aligned} &L(y(x) + \beta(x)y(q_n x)) = \lambda y(x) + \sum_{j=0}^{n-1} \mu_j(x) y^{(j)}(q_j x) + f(x), \\ &y(0) = c_0, \mathtt{D}y(0) = c_1, \mathtt{D}^2 y(0) = c_2, \ldots, \mathtt{D}^{(n-1)} y(0) = c_{n-1}, \end{aligned} \tag{2.1}$$

where $L : \mathcal{S} \to \mathcal{S}$ is a linear differential operator and $L = \mathtt{D}^n = \frac{d^n}{dx^n}$, $y \in \mathcal{S}$, and $c_0, \ldots, c_{n-1}$ are constants of $\mathbb{C}$. Analytically speaking, the differential operator L acts on the Banach space $(C[0, 1], \|\cdot\|_\infty)$ with dense domain of definition $C^n[0, 1]$ but algebraically speaking, the domain of L is the complex vector space $C^\infty[0, 1]$ without any prescribed topology. As mentioned in Section 1, we want to find an operator P which produces a sequence of functions by successive iterations from the given initial conditions and these functions satisfy the given differential equation and $\mathtt{D}^i P(0) = c_i$, for $i = 0, 1, \ldots, n-1$. This can be achieved by finding right inverse of the differential operator L as described below. Since the operator P involves integration, we introduce an operator $\mathtt{I}$, called *integral operator*, for computing the antiderivative

$$\mathtt{I}f(x) = \int_0^x f(t)\ dt,$$

such that the operator P can be rewritten in terms of $\mathtt{I}$ by replacing integral portion. Since $\mathtt{DI}f = f$, i.e., $\mathtt{DI} = 1$, we call $\mathtt{I}$ as *the right inverse* of $\mathtt{D}$. The powers of integral operator $\mathtt{I}$ define in the obvious way. Of course, every $\mathtt{I}^n f$ must be continuous. In particular,

$$\mathtt{I}^2 f(x) = \int_0^x \int_0^{x_1} f(t)\ dt\ dx_1.$$

From Lemma 1.1, we have

$$\mathtt{I}^2 f(x) = x\int_0^x f(t)\ dt - \int_0^x tf(t)\ dt. \tag{2.2}$$

Thus, equation (2.2) can be written in terms of $\mathtt{I}$ as follows

$$\mathtt{I}^2 f(x) = x\mathtt{I}f(x) - \mathtt{I}xf(x),$$

and as an operator we have $\mathtt{I}^2 = x\mathtt{I} - \mathtt{I}x$. moreover we can easily check that $\mathtt{D}^2\mathtt{I}^2 = 1$ and also $\mathtt{D}^2(x\mathtt{I} - \mathtt{I}x) = 1$. The operator $x\mathtt{I} - \mathtt{I}x$ is called *normal form* of $\mathtt{I}^2$. Lemma 1.1 can be restated in operator based notations as follows:

Lemma 2.1. *If $f(x)$ is integrable, then $\mathtt{I}^2 f(x) = x\mathtt{I}f(x) - \mathtt{I}xf(x)$ for all $x \in [0, 1]$.*

Proof. Result follows from integration by parts. □

Since $\mathtt{D}^n\mathtt{I}^n = 1$, $\mathtt{I}^n$ is the right inverse of $L = \mathtt{D}^n$ and by Lemma 2.1 one can easily compute the normal form of $\mathtt{I}^n$ by induction on n. Let the normal form of $\mathtt{I}^n$ be denoted by $L^\dagger$ and hence $LL^\dagger = 1$. The normal form of the right inverse of the differential operator L is also computed using the well-known *variation of parameters* formula (see [2]) as follows:

Lemma 2.2. *For a given differential operator $L = \mathtt{D}^n$ with the fundamental system $1, x, x^2, \ldots, x^{n-1}$, the normal form of the right inverse of L is given by*

$$L^\dagger = \sum_{i=1}^{n} \frac{x^{i-1}}{1!\ 2!\ \cdots\ (n-1)!}\ \mathtt{I}\ det(W_i), \tag{2.3}$$

where $\mathtt{I}$ is the integral operator $\int_0^x$, W_i is the Wronskian matrix associated with $1, x, x^2, \ldots, x^{n-1}$ whose i-th column is the n-th unit vector.

Putting $n = 2$ in the Lemma 2.2, we get $L^\dagger = \frac{1}{1}\mathtt{I}(-x) + \frac{x}{1}\mathtt{I}(1) = -\mathtt{I}x + x\mathtt{I}$ which is exactly as given in Lemma 2.1.

Now similar to the equation (1.7), the symbolic formulation for the approximate solution of (2.1) as iterative operator P is given by,

$$Py = -\beta(x)y(q_n x) + \sum_{i=0}^{n-1} c_i \frac{x^i}{i!} + L^\dagger g(x, y), \tag{2.4}$$

where $L^\dagger$ is the right inverse of the differential operator L as given in the equation (2.3) and $g(x, y) = \lambda y(x) + \sum_{j=0}^{n-1} \mu_j(x) y^{(j)}(q_j x) + f(x)$. The approximate solution of (2.1) after k number of iterations is given by

$$y_k(x) = -\beta(x)y(q_n x) + \sum_{i=0}^{n-1} c_i \frac{x^i}{i!} + L^\dagger \left(\lambda y(t) + \sum_{j=0}^{n-1} \mu_j(t) y^{(j)}(q_j t) + f(t) \right). \tag{2.5}$$

Since $y_k(x)$ is an approximate solution, the error involved due to the approximation is calculated as follows and it must be approximately zero, for $i = 0, 1, \ldots,$

$$E(x_i) = |y(x_i) - y_k(x_i)| \approxeq 0$$

or

$$E(x_i) = \left| \ (y_k(x_i) + \beta(x_i) y_k(q_n x_i))^{(n)} - \lambda y_k(x_i) - \textstyle\sum_{j=0}^{n-1} \mu_j(x_i) y_k^{(j)}(q_j x_i) + f(x_i) \ \right| \approxeq 0.$$

2.1. **Proposed Method in Maple.** In this section, we implement the proposed symbolic method for solving multi-pantograph delay equations with variable coefficients in Maple. For obtaining iterative operator of a given problem, we need to enter the order n of the differential operator L and initial condition values $c_0, \ldots, c_{n-1}$ as given in the following Maple procedure for getting iterative operator.

```
IterativeOperator:=proc (n, c::(seq(anything)))
local bval,nbv,RightInv,P,bpart;
bval:=[c];
nbv:=nops(bval);
if nbv <> n then
print(‘Invalid Equation‘)
else
RightInv:=1/((n-1)!)*I*(x-t)^ (n-1);
bpart:=sum(bval[i]*x^ (i-1)/(i-1)!,i=1..n);
P:=bpart+RightInv;
return P;
end if;
end proc:
```

For obtaining approximate solution of a given ODEs, we have the following procedure `ApproximateSolution(`k`,`n`,`$f(x,y)$`,`$c_0, \ldots, c_{n-1}$`)`, where $f(x, y)$ is right hand side of differential equation $Ly = f$ and k is the number of iterations required.

```
ApproximateSolution:=proc(k,n,fun,c::(seq(anything)))
local bval,nbv,bpart,y,i;
bval:=[c];
```

```
nbv:=nops(bval);
if nbv <> n then
print('Invalid IVP')
else
bpart:=sum(bval[i]*x^ (i-1)/(i-1)!,i=1..n);
f(x,y):=fun;
y[0](x):=bval[1];
for i from 1 to k do
y[i](x):=bpart+1/((n-1)!)*int((x-t)^ (n-1)*f(t,y[i-1]),t=0..x);
end do;
return y[k](x);
end if;
end proc:
```

In the above procedure we have used the following well-known identity to find the right inverse and one can obtain this identity from Lemma 2.2.

$$\int_0^x \int_0^{x_1} \dots \int_0^{x_{n-1}} f(t, y(t))\ dt\ dx_1 \dots\ dx_{n-1}$$
$$= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t, y)\ dx.$$

3. Numerical Examples

Example 3.1. *Consider a multi-pantograph equation*

$$y'(x) = \lambda y(x) + \mu y(\frac{x}{2}),$$
$$y(0) = 1,\ \ 0 \le x \le 1.$$

where $\lambda = \frac{a}{2}$ *and* $\mu = \frac{a}{2} e^{\frac{ax}{2}}$. *In symbolic notations, we have* $Ly(x) = f(x, y(x))$ *where* $L = \mathtt{D}$, $f(x, y(x)) = \frac{a}{2} e^{\frac{ax}{2}} y(\frac{x}{2}) + \frac{a}{2} y(x)$ *and* $c_0 = 1$. *The right inverse of* L, *as described in Section 2, is* $L^\dagger = \mathtt{I}$.

Now the iterative operator, for $k = 1, 2, \ldots,$ is

$$
\begin{aligned}
y_k = P(y_k) &= c_0 + L^{\dagger} f(x, y_{k-1}) \\
&= 1 + I\,\left(\frac{a}{2} e^{\frac{ax}{2}} y_{k-1}\left(\frac{x}{2}\right) + \frac{a}{2} y_{k-1}(x)\right) \\
\text{and } y_0 = y(0) = 1.
\end{aligned}
$$

First iteration i.e. $k = 1$

$$y_1(x) = e^{\frac{ax}{2}} + \tfrac{1}{2}ax.$$

Second iteration i.e. $k = 2$

$$y_2(x) = \tfrac{-1}{6} + \tfrac{2}{3}e^{\frac{3}{4}ax} + \tfrac{ax}{4}e^{\frac{ax}{2}} + \tfrac{1}{2}e^{\frac{ax}{2}} + \tfrac{1}{8}a^2x^2.$$

Similarly one can perform more number of iterations for better approximation. The analytic solution of given equation is $y(x) = e^{ax}$. Table 1 shows the numerical results of the analytical and the approximation solution when $a = 1$. The absolute errors due to the approximation are presented. Figure 1 shows the graphical comparison between analytic and approximate solution upto the fifth iteration using maple.

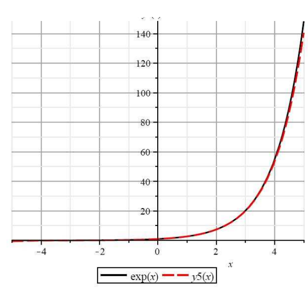


FIGURE 1. Comparison of $y_5(x)$ with analytic solution $y(x)$.

From Table 1, the errors are significantly very small; and from Figure 1, the approximate solution curve is very closer to the analytical solution curve. Therefore, the proposed symbolic method is efficient.

TABLE 1. Comparison between $y_5(x)$ and $y(x)$ with absolute errors.

x	*Analytic value* $y(x)$	*Approx. value* $y_5(x)$	*Absolute errors*
-1.00	0.9048374180	0.9048374179	1×10^{-10}
-0.70	0.4965853038	0.4965752455	1.00583×10^{-5}
-0.55	0.5769498104	0.5769473641	2.4463×10^{-6}
0.00	1.000000000	1.000000000	0
0.01	1.010050167	1.010050167	0
0.25	1.284025417	1.284025391	2.6×10^{-8}
0.37	1.447734615	1.447734333	2.82×10^{-7}
0.50	1.648721271	1.648719499	1.772×10^{-6}
0.78	2.181472265	2.181444819	2.7446×10^{-5}
1.00	2.718281828	2.718152636	1.29192×10^{-4}

Example 3.2. *[7] Consider the equation*

$$y'(x) = \lambda y(x) + \mu_1 y(\frac{x}{2}) + \mu_2(x) y(\frac{x}{4}),$$
$$y(0) = 1, \ 0 \le x \le 1,$$

where $\lambda = -1, \mu_1(x) = -e^{-\frac{x}{2}} \sin(\frac{x}{2})$ *and* $\mu_2(x) = -2e^{-\frac{3x}{4}} \cos(\frac{x}{2}) \sin(\frac{x}{4})$*. In symbolic notations,* $Ly(x) = f(x, y(x))$ *where* $L = \mathtt{D}$*,* $f(x, y(x)) = \lambda y(x) + \mu_1 y(\frac{x}{2}) + \mu_2(x) y(\frac{x}{4})$ *and* $c_0 = 1$*. The right inverse of* L *is* $L^\dagger = \mathtt{I}$*.*

Now the iterative operator, for $k = 1, 2, \ldots,$ *is*

$$\begin{aligned} y_k &= P(y_k) = c_0 + L^\dagger f(x, y_{k-1}) \\ &= 1 + \mathtt{I} \ \left(-y_{k-1}(x) - e^{-\frac{x}{2}} \sin(\frac{x}{2}) y_{k-1}(\frac{x}{2}) - 2e^{-\frac{3x}{4}} \cos(\frac{x}{2}) \sin(\frac{x}{4}) y_{k-1}(\frac{x}{4})\right) \end{aligned}$$

and $y_0 = y(0) = 1.$

First iteration i.e. $k = 1$

$$y_1(x) = -\tfrac{4}{15} - x + e^{-\frac{x}{2}}\cos(\tfrac{x}{2}) + e^{-\frac{x}{2}}\sin(\tfrac{x}{2}) + \tfrac{2}{3}e^{-\frac{3x}{4}}\cos(\tfrac{3x}{4}) + \tfrac{2}{3}e^{-\frac{3x}{4}}\sin(\tfrac{3x}{4}) - \tfrac{2}{5}e^{-\frac{3x}{4}}\cos(\tfrac{x}{4}) - \tfrac{6}{5}e^{-\frac{3x}{4}}\sin(\tfrac{x}{4}).$$

Better approximation can be obtained using more iterations. The analytic solution of given equation is $y(x) = e^{-x}\cos(x)$. Table 2 gives the absolute errors in different iterations. Figure 2 shows the error function for different iterations.

TABLE 2. Errors comparison

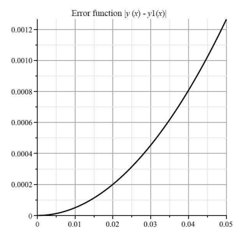
x_i	$\lvert y(x_i) - y_1(x_i)\rvert$	$\lvert y(x_i) - y_2(x_i)\rvert$	$\lvert y(x_i) - y_3(x_i)\rvert$
0.00	0	0	0
0.01	5.01236×10^{-5}	1.672×10^{-7}	3×10^{-10}
0.02	2.009780×10^{-4}	1.3414×10^{-6}	6.7×10^{-9}
0.03	4.532640×10^{-4}	4.5404×10^{-6}	3.39×10^{-8}
0.04	8.076505×10^{-4}	1.07930×10^{-5}	1.080×10^{-7}
0.05	1.2647737×10^{-3}	2.11401×10^{-5}	2.642×10^{-7}

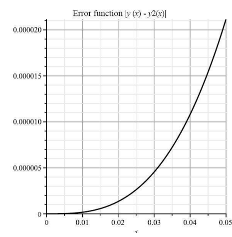
From Table 2 and Figure 2, it is clear that proposed method is more efficient and the approximate solution is almost equal to analytic solution.

Example 3.3. *[5] Consider the following neutral functional-differential equation*

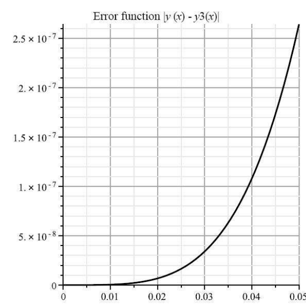
$$y''(x) = y'(\frac{x}{2}) - \frac{x}{2}y''(\frac{x}{2}) + 2,$$
$$y(0) = 1, y'(0) = 0,\ 0 \le x \le 1.$$

In symbolic notations, $Ly(x) = f(x, y(x))$ where $L = \mathtt{D}^2$, $f(x, y(x)) = y'(\frac{x}{2}) - \frac{x}{2}y''(\frac{x}{2}) + 2$ and $c_0 = 1, c_1 = 0$. The right inverse of L as described in Section 2 is $L^\dagger = x\mathtt{I} - \mathtt{I}x$.





(a) *Error function* $|y(x) - y_1(x)|$ (b) *Error function* $|y(x) - y_2(x)|$



(c) *Error function* $|y(x) - y_3(x)|$

FIGURE 2. Error functions

Now the iterative operator, for $k = 1, 2, \ldots,$ *is*

$$\begin{aligned} y_k &= P(y_k) = c_0 + L^\dagger f(x, y_{k-1}) \\ &= 1 + (x\mathtt{I} - \mathtt{I}x)\ \left(y'_{k-1}(\frac{x}{2}) - \frac{x}{2}y''_{k-1}(\frac{x}{2}) + 2\right) \\ \text{and } y_0 &= y(0) = 1, y'(0) = 0. \end{aligned}$$

First iteration i.e. $k = 1$

$$y_1(x) = 1 + x^2.$$

Second iteration i.e. $k = 2$

$$y_2(x) = 1 + x^2.$$

Since the last two iterations are same, the exact solution of the given equation is $y(x) = 1 + x^2$.

Example 3.4. *[6] Consider the following NFDE with proportional delay to compare with other existing methods.*

$$y'(x) = -y(x) + \frac{1}{2}y(\frac{x}{2}) + \frac{1}{2}y'(\frac{x}{2}),\ 0 \le t \le 1, \tag{3.1}$$

with initial conditions $y(0) = 1$. *It has the exact solution* $y(x) = e^{-x}$. *Using the proposed algorithm, we have the iterative operator, for* $k = 1, 2, \ldots,$ *as*

$$\begin{aligned} y_k = P(y_k) &= c_0 + L^{\dagger} f(x, y_{k-1}) \\ &= 1 + \mathcal{I}\left(-y_{k-1}(x) + \frac{1}{2}y_{k-1}(\frac{x}{2}) + \frac{1}{2}y'_{k-1}(\frac{x}{2})\right) \end{aligned}$$

and $y_0 = y(0) = 1.$

Following the proposed algorithm, compute the approximate solution for $k = 7$, *we have*

$$\begin{aligned} y_7(x) = 1 &- \frac{255}{256}x + \frac{32385}{65536}x^2 - \frac{680085}{4194304}x^3 + \frac{21082635}{536870912}x^4 - \frac{63247905}{8589934592}x^5 \\ &+ \frac{147578445}{137438953472}x^6 - \frac{63247905}{549755813888}x^7 + \frac{63247905}{8796093022208}x^8. \end{aligned}$$

In Table 3, we compare the absolute errors of the proposed method (P-M) for $k = 7$ *with those of the variational iteration method (VIM) [24] with* $n_i = 7$, *the Runge-Kutta method (R-KM) of [1,24] and the one-leg-*θ *method (OLM) [22,23] with* $\theta = 0.8$, *where* $h = 0.01$ *and homotopy perturbation method (HPM) [3] with* $n = 7$.

From Table 3, one can observe that the absolute errors obtained by proposed method are smaller than the other existing methods.

Example 3.5. *Consider an IVP of order four in Maple implementation as follows:*

$$y^{(4)}(x) = x + y,$$

$$y(0) = 1, y'(0) = 2, y''(0) = -1, y'''(0) = 1.$$

We have $Ly = f(x, y)$ *where* $L = \mathcal{D}^4$, $f(x, y) = x + y$, $c_0 = 1, c_1 = 2, c_2 = -1, c_3 = 1.$ *The right inverse of* L *is:*

TABLE 3. Absolute errors comparison for Example 3.4

x	*P-M*	*VIM*	*R-KM*	*OLM*	*HPM*
0.1	3.36×10^{-4}	7.43×10^{-4}	4.55×10^{-4}	2.57×10^{-3}	6.73×10^{-4}
0.2	5.80×10^{-4}	1.42×10^{-3}	8.24×10^{-4}	8.86×10^{-3}	1.16×10^{-3}
0.3	7.51×10^{-4}	2.02×10^{-3}	1.12×10^{-3}	1.72×10^{-2}	1.50×10^{-3}
0.4	8.64×10^{-4}	2.58×10^{-3}	1.33×10^{-3}	2.66×10^{-2}	1.73×10^{-3}
0.5	9.33×10^{-4}	3.07×10^{-3}	1.52×10^{-3}	3.63×10^{-2}	1.86×10^{-3}
0.6	9.69×10^{-4}	3.52×10^{-3}	1.66×10^{-3}	4.58×10^{-2}	1.94×10^{-3}
0.7	9.78×10^{-4}	3.93×10^{-3}	1.75×10^{-3}	5.47×10^{-2}	1.95×10^{-3}
0.8	9.62×10^{-4}	4.30×10^{-3}	1.81×10^{-3}	6.29×10^{-2}	1.93×10^{-3}
0.9	9.44×10^{-4}	4.64×10^{-3}	1.84×10^{-3}	7.02×10^{-2}	1.89×10^{-3}
1.0	9.10×10^{-4}	4.94×10^{-3}	1.85×10^{-3}	7.66×10^{-2}	1.82×10^{-3}

$$L^{\dagger} = \tfrac{1}{6}x^3\ I\ + \tfrac{1}{2}x\ I\ x^2 - \tfrac{1}{2}x^2\ I\ x - \tfrac{1}{6}\ I\ x^3.$$

Now the iterative operator, for $k = 1, 2, \ldots,$ is

```
IterativeOperator(4,1,2,-1,1);
```

$$1 + 2x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}\ I\ x^3 - \frac{1}{2}\ I\ x^2 t + \frac{1}{2}\ I\ x t^2 - \frac{1}{6}\ I\ t^3$$

```
f(x,y):= x + y;
```

$$(x, y) \to x + y$$

```
PicardsApproxSol(1,4,f(x,y),1,2,-1,1);
```

$$1 + 2x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

PicardsApproxSol(2,4,f(x,y),1,2,-1,1);

$$1 + 2x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{40}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9$$

Similarly one can find the required number of iterations by increasing the value of k for better approximate solution. The exact solution of the given IVP is $y(x) = -x + \cos(x) + e^x + \sin(x) - e^{-x}$. In Figure 3, two graphs show the comparison of the 5th and 20th iteration respectively with the exact solution. In the figures, red and black lines indicate the approximate and exact solutions respectively. One can

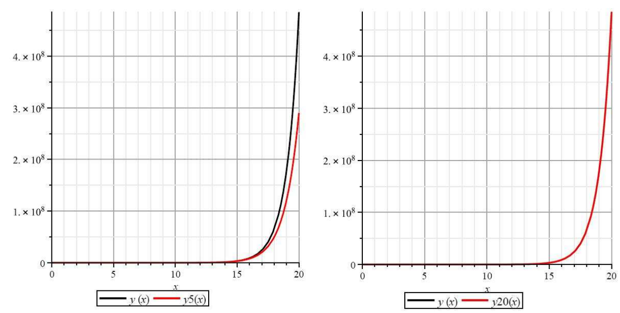


FIGURE 3. Comparison of approximate solution functions with analytic solution

easily observe from Figure 3 that the two lines are almost coincide each other in 20th iterations, i.e., the approximate solution using proposed method is very close to the analytic solution.

4. Conclusion

In this paper, we presented a symbolic method for finding an approximate solution of the neutral functional-differential equations with proportional delays having variable coefficients on the level of operators. In many cases we get exact solution in few steps as shown shown in example 3.3 and example 3.4. This method is also applicable to solve the multi-pantograph equations with variable coefficients. Efficiency of the proposed method is shown by considering several numerical examples.

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References

[1] A. Bellen, M. Zennaro: Numerical Methods for Delay Differential Equations, Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, NY, USA, (2003).

[2] E. A. Coddington and N. Levinson.: Theory of ordinary differential equations. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[3] J. Biazar, B. Ghanbari: The homotopy perturbation method for solving a neutral functional-differential equation with proportional delays, J. King Saud University-Science 24 33-37 (2012).

[4] Jishan Hu and Wei-Ping Li: Theory of ordinary differential equations, Existence, Uniqueness and Stability. Department of Mathematics, The Hong Kong University of Science and Technology, 2004.

[5] M. A. Jafari and A. Aminataei: Method of successive approximations for solving the multi-pantograph delay equations, Gen. Math. Notes, 8 (2012), 23–28.

[6] M. G. Sakar: Numerical solution of neutral functional-differential equations with proportional delays, International Journal of Optimization and Control: Theories & Applications, 7(2) (2017),186–194.

[7] M. Sezer, S. Yalcinbas and N. Sahin: Approximate solution of multi-pantograph equation with variable coefficients, Journal of Computational and Applied Mathematics, 241 (2008), 406–416.

[8] Peter J. Collins: Differential and integral equations. Oxford University Press, 2006.

[9] S. Thota: Initial value problems for system of differential-algebraic equations in Maple, BMC Research Notes 11:651 (2018).

[10] S. Thota: A Symbolic Algorithm for Polynomial Interpolation with Integral Conditions, Applied Mathematics & Information Sciences, 12 (5) (2018), 995–1000.

[11] S. Thota: On A New Symbolic Method for Initial Value Problems for Systems of Higher-order Linear Differential Equations, International Journal of Mathematical Models and Methods in Applied Sciences, 12 (2018), 194–202.

[12] S. Thota, S. D. Kumar: Maple Implementation of Symbolic Methods for Initial Value Problems, Research for Resurgence, 1 (1) (2018), 21–39. ISBN: 978-93-83861-12-5.

[13] S. Thota, S. D. Kumar: Symbolic algorithm for a system of differential-algebraic equations, Kyungpook Mathematical Journal, 56 (4) (2016), 1141–1160.

[14] S. Thota, S. D. Kumar: On a mixed interpolation with integral conditions at arbitrary nodes, Cogent Mathematics, 3(1) (2016), 1–10.

[15] S. Thota, S. D. Kumar: Solving system of higher-order linear differential equations on the level of operators, International journal of pure and applied mathematics, 106 (1) (2016), 11–21.

[16] S. Thota, P. Shanmugasundaram: On A Symbolic Method for Neutral Functional-Differential Equations with Proportional Delays, Cogent Mathematics & Statistics (2020) 7:1, 1813961.

[17] S. Thota: A New Symbolic Method for Solving Two-point Boundary Value Problems. The Hawassa Math&Stat Conference 2019, February 11-15, 2019, at Hawassa University, Hawassa, Ethiopia.

[18] S. Thota: An Introduction to Maple software, National Conference on Advances in Mathematical sciences at National Institute of Technology, Allahabad, 05-07, Oct. 2012.

[19] S. Thota: On A New Symbolic Method for Solving Two-point Boundary Value Problems with Variable Coefficients, International Journal of Mathematics and Computers in Simulation, 13 (2019), 160–164.

[20] S. Thota: A Symbolic Algorithm for Polynomial Interpolation with Stieltjes Conditions in Maple, Proceedings of the Institute of Applied mathematics, 8 (2) (2019), 112–120.

[21] S. Thota: On A Symbolic Method for Fully Inhomogeneous Boundary Value Problems, Kyungpook Mathematical Journal, 59 (1) (2019), 13–22.

[22] W. Wang, Y. Zhang, S. Li: Stability of continuous Runge-Kutta type methods for nonlinear neutral delay-differential equations, Appl. Math. Model. 33 (8) 3319–3329 (2009).

[23] W. Wang, S. Li: On the one-leg-θ methods for solving nonlinear neutral functional-differential equations, Appl. Math. Comput. 193 (1) 285-301 (2007).

[24] X. Chen and L. Wang: The variational iteration method for solving a neutral functional-differential equation with proportional delays, Computer and Mathematics with Applications, 59 (2010), 2696–2702.

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