Jordan Journal of Mathematics and Statistics (JJMS), 14(4), 2021, pp 707 - 719
DOI: https://doi.org/10.47013/14.4.7

CONVERTING THE PROPERTIES IN $\mathcal{B}(\mathcal{H})$ BY OPERATORS ON $\mathcal{B}(\mathcal{H})$

E. ANSARI-PIRI [(1)] , R. G. SANATI [(2)] AND S. PARSANIA [(3)]

Abstract. The pair of operators on $\mathcal{B}(\mathcal{H})$ which are related to each other with respect to a specific property on $\mathcal{B}(\mathcal{H})$, have been studied before. In this paper, we study a pair of operators φ_1, φ_2 on $\mathcal{B}(\mathcal{H})$ which can convert some suitable properties to each other. For instance, we show that $\varphi_1(T)$ is a compact operator if and only if $\varphi_2(T)$ is compact, whenever $\varphi_1(T)$ is a Fredholm operator if and only if $\varphi_2(T)$ is a semi-Fredholm operator.

1. Introduction

One of the most interesting topics in functional analysis which has attracted a lot of attention in the last years, is the preserving maps in the sense that how to characterize a map φ between two spaces $\mathcal{A}$ and $\mathcal{B}$ in such away that, φ preserves a certain property (such as invertibility of an operator, the spectrum of an operator, and etc.) from $\mathcal{A}$ into $\mathcal{B}$. The most commonly used types of spaces are Banach algebras of operators $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$ where $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $\mathcal{B}(\mathcal{X})$ denotes the algebra of all bounded linear operators on $\mathcal{X}$. Now there are two attitudes, where one of them is studying the maps preserving a certain property of an operator, and the other is studying about the maps that preserve a class of operators from $\mathcal{B}(\mathcal{X})$ into $\mathcal{B}(\mathcal{Y})$. For

2010 *Mathematics Subject Classification.* Primary: 47B05; Secondary: 47L05, 47A53.
Key words and phrases. Preserving map, converting map, Fredholm operator, semi-Fredholm operator.

Received: Jun 13, 2020 Accepted: Sept. 20, 2021 .

instance, Jafarian and Sourour [10] and Sourour [16] investigate how to characterize linear maps preserving the spectrum of operators and the invertibility, respectively. Actually, they prove that a surjective linear spectrum preserving map of $\mathcal{B}(\mathcal{X})$ onto $\mathcal{B}(\mathcal{Y})$, is either an algebra isomorphism or an anti-isomorphism and obtain the same result for a unital bijective linear map which preserves invertibility of an operator. Later, Aupetit and Mouton [3] extended the result of Jafarian and Sourour to a primitive Banach algebra with minimal ideals.

As more aspects of linear maps preserving a certain property of an operator, we refer to see [4, 6, 11, 17].

Moreover, many kinds of linear preserver problems can be raised when we consider $\mathcal{X}, \mathcal{Y}$ as a Hilbert space $\mathcal{H}$ and then attempt to study the linear maps preserving the certain class of operators from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. Among them, Mbekhta [12] characterize a surjective linear map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ preserving the set of Fredholm operators in both directions, where by in both directions, we mean that for every $T \in \mathcal{B}(\mathcal{H})$ the operator T has a property p if and only if $\varphi(T)$ has this property. Also Mbekhta and Šemrl [13], show that, for a surjective linear map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, φ preserves the set of all compact operators, when it preserves the set of all semi-Fredholm (generalized invertible) operators in both directions.

In a more general view, Aghasizadeh and Hejazian [1] studied linear preserver problem for two linear maps φ_1 and φ_2 on $\mathcal{B}(\mathcal{H})$ and for a certain property p on $\mathcal{B}(\mathcal{H})$, introduce an equivalence relation " $\sim_p$ " on $\mathcal{B}(\mathcal{H})$ for which, $\varphi_1 \sim_p \varphi_2$ if and only if $\varphi_1(T)$ has property p, whenever $\varphi_2(T)$ has this property.

In addition, there is a good motivation to study the maps preserving orthogonality property on a Hilbert space when we consider linear preserver problems on Hilbert spaces. So, many kinds of mappings preserving the orthogonality property have been studied, among other, by Chmieliński [5], Wójcik [19, 20], Moslehian and Zamani [21, 22], Ansari, Sanati and Kardel [2].

In this paper, we study a pair of mappings on $\mathcal{B}(\mathcal{H})$ which are related to each other by a pair of properties (p, q) on $\mathcal{B}(\mathcal{H})$. Then, we concentrate to study linear convertor problems instead of linear preserver problems.

In order to establish the theory of linear convertor mappings, first we give some definitions and notations.

Let p and q be two properties on $\mathcal{B}(\mathcal{H})$ and φ_1, φ_2 two maps on $\mathcal{B}(\mathcal{H})$. We say that φ_1 is $(p, q)-$ related to φ_2 if for each $T \in \mathcal{B}(\mathcal{H})$, $\varphi_2(T)$ has the property q, whenever $\varphi_1(T)$ has the property p, and we say φ_1 is $(p, q)-$ equivalent to φ_2, whenever φ_1 is $(p, q)-$ related to φ_2 and φ_2 is $(q, p)-$ related to φ_1. In this case, we use the notation $\varphi_1 \ {}_p\!\sim_q \ \varphi_2$.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. The ideals of compact operators and finite rank operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{K}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$, respectively. We recall that the Calkin algebra $\mathcal{C}(\mathcal{H})$ is the quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be Fredholm if its image is closed and both its kernel and co-kernel are finite-dimentional and is semi-Fredholm if its image is closed and its kernel or its co-kernel is finite-dimentional. The sets of all Fredholm and semi-Fredholm operators are denoted by $\mathcal{FR}(\mathcal{H})$ and $\mathcal{SF}(\mathcal{H})$, respectively. By Atkinson Theorem, [14, Theorem 1.4.16], if $\mathcal{H}$ is an infinite-dimentional Hilbert space, then $T \in \mathcal{B}(\mathcal{H})$ is Fredholm (resp. semi-Fredholm) if and only if its projection is invertible (resp. right or left invertible) in Calkin algebra $\mathcal{C}(\mathcal{H})$. For more details about Fredholm operators and Calkin algebra, see [14, 18].

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be generalized invertible if there is an operator $R \in \mathcal{B}(\mathcal{H})$ such that $TRT = T$. The set of all generalized invertible operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{G}(\mathcal{H})$. Note that $T \in \mathcal{G}(\mathcal{H})$ if and only if Im(T), the range of T, is closed [15].

Throughout this paper, we use the following notations for some specific properties:

(i) "f" is the property of " being finite rank. "

(ii) "fr" is the property of " being Fredholm. "

(iii) "sf" is the property of " being semi-Fredholm. "

(iv) "k" is the property of " being compact. "

(v) "g" is the property of " being generalized invertible. "

Example 1.1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space.*

1. *If φ is a map on $\mathcal{B}(\mathcal{H})$, then φ is $(fr, g)-$ related to itself.*

2. *If $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined by $\varphi(T) = P_T$, where P_T is the orthogonal projection on $Im(T)^{\perp}$, then $\varphi\ {}_{sf}\!\sim_{f} \mathcal{I}$. Because we have*

$$Ker(\varphi(T)) = \overline{Im(T)} = \mathcal{H}/Im(\varphi(T)).$$

Let p, q be two properties on $\mathcal{B}(\mathcal{H})$ and φ and $\mathcal{I}$ be an arbitrary and the identity operator on $\mathcal{B}(\mathcal{H})$, respectively. We say that φ is invariant on a pair (p, q), whenever $\varphi\ {}_{p}\!\sim_{q} \mathcal{I}$.

Now let φ, φ_1 and φ_2 be linear operators on $\mathcal{B}(\mathcal{H})$. If φ is invariant on a pair (p, p), then φ preserves the property p in both directions and if $\varphi_1\ {}_{p}\!\sim_{p} \varphi_2$, then $\varphi_1 \sim_p \varphi_2$. If ψ is a linear operator on $\mathcal{B}(\mathcal{H})$, which is invariant on a pair of properties (p, q), then $\psi\varphi\ {}_{p}\!\sim_{q} \varphi$, for each linear map φ on $\mathcal{B}(\mathcal{H})$. Also, if υ is a linear operator on $\mathcal{B}(\mathcal{H})$, which is not invariant on a pair of properties (p, q), then for each surjective linear map φ, $\upsilon\varphi\ {}_{p}\!\nsim_{q} \varphi$.

Recall that a linear map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to be surjective up to finite rank operators (resp. compact operators), if for each $T \in \mathcal{B}(\mathcal{H})$ there exists $A \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ $(resp.\ F \in \mathcal{K}(\mathcal{H}))$ such that $T = \varphi(A) + F$. Obviously, if φ is surjective up to finite-rank operators, then it is surjective up to compact operators and each surjective linear map satisfies both of these properties.

Take $C = \{T \in \mathcal{B}(\mathcal{H}) \mid$ for every operator $A \in \mathcal{B}(\mathcal{H})$ with $Im(A)$ not closed, there exists $\lambda \in \mathbb{C}$ such that $A + \lambda T \neq 0$ and $Im(A + \lambda T)$ is closed$\}$. It is proved in [8, Lemma 3.1] that $C = \mathcal{SF}(\mathcal{H})$.

Since $\mathcal{FR}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H}) = \varnothing$, $\mathcal{SF}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H}) = \varnothing$ and $\mathcal{SF}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}) = \varnothing$ and $\mathcal{FR}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}) = \varnothing$, there are many results. For instance,

$\varphi_1 \sim_{sf} \varphi_2 \Rightarrow \varphi_1 \,{}_{sf} \nsim {}_f \, \varphi_2, \varphi_1 \,{}_{sf} \nsim {}_k \, \varphi_2, \varphi_1 \,{}_{fr} \nsim {}_f \, \varphi_2, \varphi_1 \,{}_{fr} \nsim {}_k \, \varphi_2.$

$$\varphi \,{}_g \sim {}_f \, \varphi_2 \Rightarrow \begin{cases} \varphi_1 \,{}_g \nsim {}_{sf} \, \varphi_2, \; \varphi_1 \,{}_g \nsim {}_{fr} \, \varphi_2, \; \varphi_1 \,{}_f \nsim {}_{sf} \, \varphi_2, \; \varphi_1 \,{}_f \nsim {}_{fr} \, \varphi_2, \\ \varphi_1 \nsim_{sf} \varphi_2, \; \varphi_1 \,{}_{sf} \nsim {}_{fr} \, \varphi_2, \; \varphi_1 \,{}_{fr} \nsim {}_{sf} \, \varphi_2, \; \varphi_1 \nsim_{fr} \varphi_2. \end{cases}$$

It is clear that, if $\varphi_1 \sim_{sf} \varphi_2$, then φ_1 is $(fr, sf)-$ related to φ_2.
Now we give an example to show that the revers does not hold, in general.

Example 1.2. *Suppose that $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective linear map and $S \in \mathcal{B}(\mathcal{H})$ is a lower semi-Fredholm operator. Since φ is surjective, there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\varphi(T) = S$. Now if $A \in \mathcal{B}(\mathcal{H})$ is an upper semi-Fredholm operator, then φ is $(fr, sf)-$ related to $L_A\varphi$ but $\varphi \nsim_{sf} L_A\varphi$ because S is a semi-Fredholm operator but $L_A(S)$ is not semi-Fredholm. Here $L_A : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the left multiplier operator defined by $L_A(S) = AS$.*

In the next section, we study a pair of linear operators on $\mathcal{B}(\mathcal{H})$ which are related to each other with respect to a pair of properties (p, q) on $\mathcal{B}(\mathcal{H})$. For instance, we show that for linear mappings φ_1 and φ_2 on $\mathcal{B}(\mathcal{H})$ which are surjective up to compact operators, if $\varphi_1 \,{}_{fr} \sim {}_{sf} \, \varphi_2$, then $\varphi_1 \sim_k \varphi_2$, also if φ_1 is $(sf, f)-$ related to φ_2, then φ_1 is $(k, f)-$ related to φ_2. Moreover, for linear mappings φ_1 and φ_2 on $\mathcal{B}(\mathcal{H})$ which are surjective up to finite-rank operators, we proved that, if $\varphi_1 \sim_g \varphi_2$, then $\varphi_1 \,{}_{sf} \nsim {}_f \, \varphi_2$. We also show that, if $\varphi_1 \,{}_{sf} \sim {}_f \, \varphi_2$ on $\mathcal{B}(\mathcal{H}) \backslash \{0\}$ under a certain conditions, there exists a linear bijective continuous map from $\mathcal{B}(\mathcal{H})$ onto Calkin algebra $\mathcal{C}(\mathcal{H})$ which is homomorphism or anti-homomorphism. Finally, we give other versions of proposition 2.5 of [1].

2. Linear maps converting the properties in $\mathcal{B}(\mathcal{H})$

The following two lemmas play an important role in this paper.

Lemma 2.1. ([12, Lemma 2.2], [13, Lemma 2.2]) *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent.*
(i) T is compact.
(ii) For every $R \in \mathcal{FR}(\mathcal{H})$, we have $R + T \in \mathcal{FR}(\mathcal{H})$.
(iii) For every $R \in \mathcal{SF}(\mathcal{H})$, we have $R + T \in \mathcal{SF}(\mathcal{H})$.

To state the next lemma, we recall that a C^*-algebra is of real rank zero, if the set of all real combinations of orthogonal Hermitian idempotents is dense in the set of all its Hermitian elements. In particular, $\mathcal{B}(\mathcal{H})$ is a real rank zero C^*- algebra.
Note that an algebra $\mathcal{A}$ is prime if and only if $a\mathcal{A}b = 0$ implies that $a = 0$ or $b = 0$, where $a, b \in \mathcal{A}$. (see page 449 of [9]).

Lemma 2.2. ([7, Theorem 3.1]) *Let $\mathcal{A}$ be a unital C^*- algebra of real rank zero and $\mathcal{B}$ a unital semi-simple complex Banach algebra. Let $Sp(.)$ denote the spectral function. Suppose that $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map such that $Sp(\varphi(a)) \subseteq Sp(a)$ for every $a \in \mathcal{A}$. Then φ is a Jordan homomorphism and if $\mathcal{B}$ is prime, then φ is either a homomorphism or an anti-homomorphism.*

We recall that the essential spectrum of an operator $T \in \mathcal{B}(\mathcal{H})$ is the spectrum of $T + \mathcal{K}(\mathcal{H})$ in the Calkin algebra $\mathcal{C}(\mathcal{H})$.
In the theory of linear preserver problems, one of the most important results was obtained by Mbekhta [12], where he characterized linear maps on $\mathcal{B}(\mathcal{H})$ preserving the set of compact operators.

Theorem 2.1 (Mbekhta)**.** ([12, Theorem 3.2]) *Let $\mathcal{H}$ be an infinite-dimentional Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map. Assume that ϕ is surjective*

up to compact operators. Then the following are equivalent:
(i) ϕ preserves the set of Fredholm operators in both directions and $\phi(I) = I - K$, where $K \in \mathcal{K}(\mathcal{H})$;
(ii) ϕ preserves the essential spectrum;
(iii) $\phi(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$ and the induced map $\varphi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$, $\varphi o \pi = \pi o \phi$ is either an automorphism or an anti-automorphism.

Let φ_1, φ_2 be two linear maps on $\mathcal{B}(\mathcal{H})$. We can write $\varphi_1 \sim_k \mathcal{I}$, whenever φ_1 preserves the set of compact operators in both directions. In this section, we substitute $\mathcal{I}$, the identity map on $\mathcal{B}(\mathcal{H})$, with φ_2 and we obtain a certain condition under which $\varphi_1 \sim_k \varphi_2$.
It is well known that if $T \in \mathcal{G}(\mathcal{H})$, then for each finite rank operator $F \in \mathcal{F}(\mathcal{H})$, we have $T + F \in \mathcal{G}(\mathcal{H})$.

Theorem 2.2. *Let φ_1 and φ_2 be two linear maps on $\mathcal{B}(\mathcal{H})$.*
Assume that φ_1 and φ_2 are surjective up to compact operators. Then
(i) If $\varphi_1 {}_{fr} \sim_{sf} \varphi_2$, then $\varphi_1 \sim_k \varphi_2$.
(ii) If φ_1 is $(sf, f)-$ related to φ_2, then φ_1 is $(k, f)-$ related to φ_2.
Assume that φ_1 and φ_2 are surjective up to finite rank operators, then
(iii) If $\varphi_1 \sim_g \varphi_2$, then $\varphi_1 {}_{sf} \nsim_f \varphi_2$.

Proof. (i) Suppose that $\varphi_1(T) \in \mathcal{K}(\mathcal{H})$, but $\varphi_2(T)$ is not compact. Therefore, there exists $S \in \mathcal{SF}(\mathcal{H})$ such that $S+\varphi_2(T)$ is not semi-Fredholm. Since φ_2 is surjective up to compact operators, there exists $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ such that $S = \varphi_2(A)+K$. So $\varphi_2(A + T) + K$ is not semi-Fredholm, which implies that $\varphi_2(A + T) \notin \mathcal{SF}(\mathcal{H})$. Therefore by assumption we have $\varphi_1(A + T)$ is not Fredholm operator, which is a contradiction because $\varphi_2(A) \in \mathcal{SF}(\mathcal{H})$ and by the fact $\varphi_1 {}_{fr} \sim_{sf} \varphi_2$, we have $\varphi_1(A) \in FR(H)$. Thus, $\varphi_1(A + T)$ is a Fredholm operator. By similar proof, we obtain that φ_2 is $(k,k)-$ related to φ_1.

(ii) Suppose that $\varphi_1(T)$ is compact. Let S be an arbitrary semi-Fredholm operator. Then $S + \varphi_1(T) \in \mathcal{SF}(\mathcal{H})$. Since φ_1 is surjective up to compact operators, there exists $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ such that $S = \varphi_1(A) + K$. So $\varphi_1(A)$ is semi-Fredholm and by the fact φ_1 is $(sf, f)-$ related to φ_2, we have $\varphi_2(A)$ is finite rank. On the other hand, $\varphi_1(A+T) + K \in \mathcal{SF}(\mathcal{H})$, therefore $\varphi_1(A+T) \in \mathcal{SF}(\mathcal{H})$. Thus, $\varphi_2(A+T) \in \mathcal{F}(\mathcal{H})$ and $\varphi_2(T)$ is finite rank.

(iii) suppose that $\varphi_1(T)$ is a semi-Fredholm and $\varphi_2(T)$ is a finite rank operator. Since $\varphi_1(T) \in C$, for every operator $A \in \mathcal{B}(\mathcal{H})$ with $Im(A)$ not closed, there exists $\lambda \in \mathbb{C}$ such that $A + \lambda\varphi_1(T) \neq 0$ and $Im(A + \lambda\varphi_1(T))$ is closed. By the assumption φ_1 is surjective up to $\mathcal{F}(\mathcal{H})$, for $A+\lambda\varphi_1(T) \in \mathcal{B}(\mathcal{H})$, there exists $B \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ such that $A+\lambda\varphi_1(T) = \varphi_1(B)+F$. So $Im(\varphi_1(B-\lambda T))$ is not closed and by the fact $\varphi_1 \sim_g \varphi_2$, we have $\varphi_2(B-\lambda T) \notin \mathcal{G}(\mathcal{H})$. Therefore, $\varphi_2(B) \notin \mathcal{G}(\mathcal{H})$ which implies that $\varphi_1(B) \notin \mathcal{G}(\mathcal{H})$. That is a contradiction because $\varphi_1(B) + F \in \mathcal{G}(\mathcal{H})$, so $Im(\varphi_1(B))$ is closed. □

Corollary 2.1. *Suppose that ϕ is a linear map on $\mathcal{B}(\mathcal{H})$ which is surjective up to compact operators and $\mathcal{I}$ is the identity map on $\mathcal{B}(\mathcal{H})$ such that $\phi\ _{fr} \sim\ _{sf}\ \mathcal{I}$. Then by Theorem 2.2 (i), ϕ preserves the set of compact operators in both directions. Now by Mbekhta's theorem which is expressed in the Theorem 2.1, the induced map $\varphi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$, $\varphi o \pi = \pi o \phi$ is either an automorphism or an anti-automorphism if and only if ϕ preserves the essential spectrum if and only if ϕ preserves the set of Fredholm operators in both directions and $\phi(I) = I - K$, where $K \in \mathcal{K}(\mathcal{H})$.*

Corollary 2.2. *Suppose that φ_1 and φ_2 are bijective linear maps on $\mathcal{B}(\mathcal{H})$ such that $\varphi_1\ _{fr} \sim\ _{sf}\ \varphi_2$. Take $\psi = \varphi_1\varphi_2^{-1}$ and suppose that the induced map $\hat{\psi} : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ be an automorphism or an anti-automorphism. Then by Theorem 2.2 (i), ψ preserves the set of compact operators in both directions. Therefore, $\psi(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$ and* *hence ψ preserves the essential spectrum, and preserves the set of Fredholm operators in both directions.*

Remark 1. *By similar proof of Theorem 2.2 (ii), we obtain the following results: Let φ_1 and φ_2 be two linear maps on $\mathcal{B}(\mathcal{H})$ which are surjective up to compact operators. Then*
(i) If φ_1 is $(fr, f)-$ related to φ_2, then φ_1 is $(k, f)-$ related to φ_2.
(ii) If φ_1 is $(sf, k)-$ related to φ_2, then φ_1 is $(k, k)-$ related to φ_2.
(iii) If φ_1 is $(fr, k)-$ related to φ_2, then φ_1 is $(k, k)-$ related to φ_2.

Theorem 2.3. *Let φ_1, φ_2 be two linear maps on $\mathcal{B}(\mathcal{H})$ which are surjective up to compact operators. If $Sp(\varphi_1(T)) \subseteq Sp(T)$ and $\varphi_1 \,_{sf}\sim_f\, \varphi_2$ on $\mathcal{B}(\mathcal{H})\backslash\{0\}$, then there exists a bijective continuous map from $\mathcal{B}(\mathcal{H})$ onto Calkin algebra $\mathcal{C}(\mathcal{H})$ which is homomorphism or anti-homomorphism.*

Proof. Define $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ by $\tau(T) = \varphi_1(T)+\mathcal{K}(\mathcal{H})$. First we show that τ is one-to-one. For this purpose, suppose that there exists $T \in \mathcal{B}(\mathcal{H})\backslash\{0\}$ such that $\varphi_1(T)$ is a compact operator, then Theorem 2.2 (*ii*) implies that $\varphi_2(T)$ is finite rank, so by the fact $\varphi_1 \,_{sf}\sim_f\, \varphi_2$, $\varphi_1(T)$ is semi-Fredholm which is a contradiction. Hence, τ is one-to-one. Also if $T+\mathcal{K}(\mathcal{H}) \in \mathcal{C}(\mathcal{H})$, then there exist $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ such that $T = \varphi_1(A) + K$. Now $\tau(A) = \varphi_1(A) + \mathcal{K}(\mathcal{H}) = \varphi_1(A) + K + \mathcal{K}(\mathcal{H}) = T + \mathcal{K}(\mathcal{H})$. This shows that τ is onto. On the other hand, $\mathcal{B}(\mathcal{H})$ is a unital C^*-algebra of real rank zero and $\mathcal{C}(\mathcal{H})$ is a unital semi-simple Banach algebra and so Lemma 2.2 implies that τ is a continuous Jordan homomorphism. But $\mathcal{C}(\mathcal{H})$ is a prime algebra (see [7]), hence τ is a homomorphism or anti-homomorphism. □

Corollary 2.3. *Let I be the identity map on $\mathcal{H}$. With the stated assumptions in the above theorem we have, $\varphi_1(I) \in \mathcal{FR}(\mathcal{H})$.*

Proof. Since τ is Jordan homomorphism, we have:

$$\tau(I) = I + \mathcal{K}(\mathcal{H}) = \varphi_1(I) + \mathcal{K}(\mathcal{H}).$$

This shows that there is an operator $W \in \mathcal{K}(\mathcal{H})$ such that $\varphi_1(I) = I + W$ which implies that $\varphi_1(I)$ is a Fredholm operator. □

The following proposision was proved in [1]

Proposition 2.1. *Let $\varphi_1,\ \varphi_2$ be two linear maps on $\mathcal{B}(\mathcal{H})$ which are surjective up to compact operators and $\varphi_2 = L_A R_B \varphi_1 + \lambda$, where $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is a linear map and $A, B \in \mathcal{B}(\mathcal{H})$. $\big($Here L_A(resp. R_B) is the left (resp. right) multiplier operator on $\mathcal{B}(\mathcal{H})\big)$. If $\varphi_1 \sim_k \varphi_2$, then A and B are Fredholm operators and $\varphi_1 \sim_{fr} \varphi_2$ and $\varphi_1 \sim_{sf} \varphi_2$.*

In the following theorem, we show that if A is in the center of $\mathcal{B}(\mathcal{H})$, then automatically we have $\varphi_1 \sim_k \varphi_2$. So A and B are Fredholm operators.

Theorem 2.4. *Let $\varphi_1,\ \varphi_2$ be two linear maps on $\mathcal{B}(\mathcal{H})$ which are surjective up to compact operators and $\varphi_2 = L_A R_B \varphi_1 + \lambda$, where $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is a linear mapping. If A is in the center of $\mathcal{B}(\mathcal{H})$, then $\varphi_1 \sim_k \varphi_2$.*

Proof. For $i = 1, 2$, define $\tau_i : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ by $\tau_i(T) = \varphi_i(T) + \mathcal{K}(\mathcal{H})$. Since $\varphi_1,\ \varphi_2$ are surjective up to compact operators, τ_1 and τ_2 are surjective. Let $a = A + \mathcal{K}(\mathcal{H})$ and $b = B + \mathcal{K}(\mathcal{H})$. Then trivially, $\tau_2(T) = a\tau_1(T)b$. Now let I be the identity operator on $\mathcal{H}$, since τ_2 is surjective, there exists $T \in \mathcal{B}(\mathcal{H})$ such that $a\tau_1(T)b = I + \mathcal{K}(\mathcal{H})$ or equivalently, $A\varphi_1(T)B + \mathcal{K}(\mathcal{H}) = I + \mathcal{K}(\mathcal{H})$. So there is an operator $W \in \mathcal{K}(\mathcal{H})$ such that $A\varphi_1(T)B - I = W$, hence $A\varphi_1(T)B = I + W \in \mathcal{FR}(\mathcal{H})$. This shows that A and B are semi-Fredholm operators.

If $\varphi_1(T)$ is a compact operator, then trivially $\varphi_2(T)$ is compact. Now let π be the canonical map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{C}(\mathcal{H})$ and $\varphi_2(T)$ be compact. We have $\varphi_2(T)-\lambda(T) = A\varphi_1(T)B$. Therefore,

$$\pi(A)\pi(\varphi_1(T))\pi(B) = \pi(A\varphi_1(T)B) = \pi(\varphi_2(T) - \lambda(T)) = 0 \quad (1).$$

Let $\pi(\varphi_1(T)) \neq 0$ and M be the left ideal generated by $\pi(\varphi_1(T))$. Since $\mathcal{C}(\mathcal{H})$ is a simple algebra, $\overline{M} = \mathcal{C}(\mathcal{H})$. Note that A is in the center of $\mathcal{B}(\mathcal{H})$, so (1) implies that $\pi(A)\overline{M}\pi(B) = 0$ or equivalently, $\pi(A)\mathcal{C}(\mathcal{H})\pi(B) = 0$. But $\mathcal{C}(\mathcal{H})$ is a prime algebra and so $\pi(A) = 0$ or $\pi(B) = 0$ which is a contradiction, because A and B are semi-Fredholm operators. Thus, $\pi(\varphi_1(T))$ must be zero i.e. $\varphi_1(T)$ is compact. $\square$

In the next proposition, we prove the Proposition 2.1 with the assumption that φ_2 is $(k, f)-$ related to φ_1 instead of $\varphi_1 \sim_k \varphi_2$.

Proposition 2.2. *Let φ_1, φ_2 be two linear maps on $\mathcal{B}(\mathcal{H})$ which are surjective up to compact operators and $\varphi_2 = L_A R_B \varphi_1 + \lambda$, where $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is a linear mapping. If φ_2 is $(k, f)-$ related to φ_1, then A and B are Fredholm operators, and hence $\varphi_1 \sim_{fr} \varphi_2$ and $\varphi_1 \sim_{sf} \varphi_2$.*

Proof. By similar proof of what we saw in the first paragraph of Theorem 2.4, we obtain that A and B are semi-Fredholm operators. Now we show that A and B are Fredholm. The condition φ_2 is $(k, f)-$ related to φ_1 says that, if $\tau_2(T) = 0$, then $\tau_1(T) = 0$ (*i.e. if* $a\tau_1(T)b = 0$, *then* $\tau_1(T) = 0$, *for* $a = A + \mathcal{K}(\mathcal{H})$ *and* $b = B + \mathcal{K}(\mathcal{H})$). Since τ_1 is onto, this in turn says that with $x \in \mathcal{C}(\mathcal{H})$, if $axb = 0$, then $x = 0$. If A were not Fredholm, then (since a is right invertible), we must have $null(A) = \infty$. Let $P \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection from $\mathcal{H}$ onto $Ker(A)$ and π be the canonical map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{C}(\mathcal{H})$. Then $apb = \pi(APB) = \pi(0B) = 0$ but $p = \pi(P) \neq 0$, which is a contradiction. Thus A is Fredholm. Finally, since τ_2 is onto, $azb = \pi(I)$, for some $z \in \mathcal{C}(\mathcal{H})$. Thus, there exists $Z \in \mathcal{B}(\mathcal{H})$ such that AZB

is a Fredholm operator. Therefore, $B^*Z^*A^*$ is also a Fredholm operator. The same argument implies that B^*, and hence B are Fredholm operators. □

Acknowledgement

The authors sincerely thank the editor and the referees for the valuable comments.

References

[1] T. Aghasizadeh, S. Hejazian, *Some equivalence classes of operators on $B(H)$*, Bull. Iran Math. Soc. Vol. **37** No. **1**(2011), 225–233

[2] E. Ansari-piri, R. G. Sanati, M. Kardel, *A characterization of orthogonality preserving operators*, Bull. Iran Math. Soc. vol. **43** No. **7**(2017), 2495–2505

[3] B. Aupetit, H. du T. Mouton, *Spectrum-preserving linear mappings in Banach algebras*, Studia Math. **109**(1994), 91–100

[4] M.Brešar, P. Šemrl, *Invertibility preserving maps preserve idempotents*, Michigan J. Math. **45**(1998), 483–488

[5] J. Chmieliński, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **304**(2005), 158–169

[6] M-D. Choi, D. Hadwin, E. Nordgren, H. Radjavi, P. Rosenthal, *On positive linear maps preserving invertibility*, J. Funct. Anal. **59**(1984), 462–469

[7] J. Cui, J. Hou, *Linear maps between Banach algebras compressing certain spectral functions*, Rocky Mountain J. Math. Vol. **34** No. **2**(2004), 565–584

[8] J. Hou, J. Cui, *Linear maps preserving essential spectral functions and closeness of operator ranges*, Bull. Lond. Math. Soc. **39**(2007), 575–582

[9] T. W. Hungerford, *Algebra*, Springer–Verlag, New York, Inc., 1974

[10] A. A. Jafarian, A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. **66**(1986), 255–261

[11] I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, Amer. Math. Soc., Providence, 1970

[12] M. Mbekhta, *Linear maps preserving the set of Fredholm operators*, Proc. Amer. Math. Soc. **135**(2007), 3613–3619

[13] M. Mbekhta, P. Šemrl, *Linear maps preserving semi-Fredholm operators and generalized invertibility*, Linear Multilinear Algebra **57**(2009), 55–64

[14] G. J. Murphy, *C^*-Algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990

[15] M. Z. Nashed (ed.), *Generalized inverses and applications*, Univ. Wisconsin Math., Res. Center Publ. No. **32**, Academic press, New York, 19761

[16] A. R. Sourour, *Invertibility preserving linear maps on L(X)*, Trans. Amer. Math. Soc. **348**(1996), 13–30

[17] Q. Wang, J. Hou, *Point-spectrum preserving elementary operators on $B(H)$*, Proc. Amer. Math. Soc. **126: 7**(1998), 2083–2088

[18] N. E. Wegge-Olsen, *K-Theory and C*-Algebras*, a Friendly Approach, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993

[19] P. Wójcik, *Linear mappings preserving ρ-orthogonality*, J. Math. Anal. Appl. **386** No. **1**(2012), 171–176

[20] P. Wójcik, *Linear mappings approximately preserving orthogoality in real normed spaces*, Banach J. Math. Anal. **9**(2015), 134–141

[21] A. Zamani, M.S. Moslehian, *Approximate Robert orthogonality*, Aequationes Math. **89** No. **3**(2015), 529–541

[22] A. Zamani, M.S. Moslehian, M. Frank, *Angle preserving mappings*, Z. Anal. Anwend. **34**(2015), 485–500.

(1) Faculty of Mathematics, University of Guilan, P.O. Box 1914, Rasht, Iran.
Email address: eansaripiri@gmail.com

(2) Faculty of Mathematics, University of Guilan, P.O. Box 1914, Rasht, Iran.
Email address: reza_sanaaty@gmail.com

(3) Faculty of Mathematics, University of Guilan, P.O. Box 1914, Rasht, Iran.
Email address: sparsania512@gmail.com