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UNIQUENESS AND TWO SHARED SET PROBLEMS OF L-FUNCTION AND CERTAIN CLASS OF MEROMORPHIC FUNCTION

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Abstract. Starting with a question of Yuan-Li-Yi [Value distribution of L-functions and uniqueness questions of F. Gross, Lithuanian Math. J., **58**(**2**)(2018), 249-262] we have studied the uniqueness of a meromorphic function f and an L-function $\mathcal{L}$ sharing two finite sets. At the time of execution of our work, we have pointed out a serious lacuna in the proof of a recent result of a of Sahoo-Halder [Some results on L-functions related to sharing two finite sets, Comput. Methods Funct. Theo., **19**(2019), 601-612] which makes most of the part of the Sahoo-Halder's paper under question. In context of our choice of sets, we have rectified Sahoo-Halder's result in a convenient manner.

1. Introduction

By a meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations of Nevanilinna theory of meromorphic functions as explained in [7]. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$, where $\mathbb{C}$ and $\mathbb{N}$, respectively, denote the set of all complex numbers and natural numbers and by $\mathbb{Z}$ we denote the set of all integers. For any non-constant meromorphic

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function $h(z)$ we define $S(r,h) = o(T(r,h)), (r \longrightarrow \infty, r \notin E)$ where E denotes any set of positive real numbers having finite linear measure.

Let for a non-constant meromorphic function f and $S \subset \overline{\mathbb{C}}$, $E_f(S) = \bigcup_{a\in S}\{(z,p) \in \mathbb{C}\times\mathbb{N} : f(z) = a \; with \; multiplicity \; p\}$ $\left(\overline{E}_f(S) = \bigcup_{a\in S}\{(z,1) \in \mathbb{C}\times\mathbb{N} : f(z) = a\}\right)$. Then we say f, g share the set S Counting Multiplicities or CM (Ignoring Multiplicities or IM) if $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$. When S contains only one element the definition coincides with the classical definition of value sharing.

This paper deals with the uniqueness problems of set sharing related to L-functions and an arbitrary meromorphic function in $\mathbb{C}$. In 1989, Selberg [18] found new class of Dirichlet series, called as Selberg class, which in course of time made a significant impact on the realm of research in analytic number theory. Throughout this paper an L-function means actually a Selberg class function with the Riemann zeta function as the prototype. The Selberg class $\mathcal{S}$ of L-functions is the set of all Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable s that satisfies the following axioms (see [18]):

(i) Ramanujan hypothesis: $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(ii) Analytic continuation: There is a nonnegative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional equation: $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s\prod_{j=1}^{K}\Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product hypothesis : $\mathcal{L}$ can be written over prime as

$$\mathcal{L}(s) = \prod_{p} \exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p.

The Ramanujan hypothesis implies that the Dirichlet series $\mathcal{L}$ converges absolutely in the half-plane $Re(s) > 1$ and then is extended meromorphically. The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2\sum_{j=1}^{K} \lambda_j,$$

where λ_j and K, respectively, be the positive real number and the positive integer as in axiom (iii) above.

For the last few years, the researchers have found an increasing interest on the value distributions of L-functions. Readers can make a glance over the references ([5], [12], [14], [19]). Like meromorphic function, for some $c \in \mathbb{C} \cup \{\infty\}$, the value distribution of an L-function $\mathcal{L}$ is actually the scattering of the roots of the equation $\mathcal{L}(s) = c$. By the sharing of sets of an L-function, we mean the same notion as mentioned in the first and second paragraph of this paper where all the definitions discussed also applicable to an L-function.

In 2007, in connection to Nevanlinna 5 point uniqueness theorem for meromorphic function, Steuding [p. 152, [19]] first studied the same problem of two $\mathcal{L}$ functions and obtained a remarkable result. In [19] it was shown that under certain hypothesis, only one shared value is enough to determine an $\mathcal{L}$ function. The result was as follows:

Theorem A. [19] *If two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Hu-Li [8] found a counterexample to show that *Theorem A* is not true when $c = 1$.

Since L-functions possess meromorphic continuations, researchers presumed that there might be an intimate relationship between L-function and arbitrary meromorphic function under sharing of values. In 2010, Li [12] exhibited the following example to show that for an L-function and a meromorphic function *Theorem 1.1* cease to hold.

Example 1.1. *For an entire function g, the functions ζ and ζe^g share 0 CM, but $\zeta \neq \zeta e^g$.*

However, corresponding to two distinct complex values, Li [12] was able to obtain the following uniqueness result.

Theorem B. [12] *Let f be a meromorphic function in $\mathbb{C}$ having finitely many poles and let a and b be any two distinct finite complex values. If f and a non constant L-function $\mathcal{L}$ share a CM and b IM, then $f = \mathcal{L}$.*

For three IM shared values, Li-Yi [14] obtained the following theorem.

Theorem C. [14] *Let f be a transcendental meromorphic function in C having finitely many poles in $\mathbb{C}$, and let $b_1,\ b_2,\ b_3$ be three distinct finite complex values. If f and a non-constant L-function $\mathcal{L}$ shares $b_1,\ b_2,\ b_3$ IM, then $\mathcal{L} \equiv f$.*

Inspired by the question of Gross [6], Yuan-Li-Yi [20] proposed the following question:

Question 1.1. *What can be said about the relationship between a meromorphic function f and an L-function $\mathcal{L}$ if f and $\mathcal{L}$ share one or two finite sets?*

In response to their own question Yuan-Li-Yi [20] proved the following uniqueness result.

Theorem D. [20] *Let $S = \{a_1, a_2, \ldots, a_l\}$, where $a_1, a_2, \ldots, a_l$ are all distinct roots of the algebraic equation $w^n + aw^m + b = 0$. Here l is a positive integer satisfying*

$1 \leq l \leq n$, n and m are relatively prime positive integers with $n \geq 5$ and $n > m$, and a, b, c are three nonzero finite constants, where $c \neq \alpha_j$ for $1 \leq j \leq l$. Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a non constant L-function. If f and $\mathcal{L}$ share S CM and c IM, then $f \equiv \mathcal{L}$.

In the mean time, considering the sharing of two finite sets Lin-Lin [13] proved the following theorem.

Theorem E. [13] *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, $S_1, S_2 \subset \mathbb{C}$ be two distinct sets such that $S_1 \cap S_2 = \phi$ and $\#(S_i) \leq 2$, $i = 1, 2$, where $\#(S)$ denotes the cardinality of the set S. Suppose for a finite set $S = \{a_i \mid i = 1, 2, \ldots, n\}$, $C(S)$ is defined by $C(S) = \frac{1}{n}\sum_{i=1}^{n} a_i$. If f and a non-constant L-function $\mathcal{L}$ share S_1 CM and S_2 IM, then (i) $\mathcal{L} = f$ when $C(S_1) \neq C(S_2)$ and (ii) $\mathcal{L} = f$ or $\mathcal{L} + f = 2C(S_1)$ when $C(S_1) = C(S_2)$.*

In the same paper Lin-Lin [13], asked the following question:

Question 1.2. *What can be said about the conclusions of Theorem E if max $\{\#(S_1), \#(S_2)\} \geq 3$?*

To provide an answer to the question of Lin-Lin [13], Sahoo-Halder [16] obtained the following result which is also pertinent to *Question 1.1*.

Theorem F. [16] *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, and $m(\geq 3)$ be a positive integer. Suppose that $S_1 = \{a_1, a_2, \ldots, a_m\}$, $S_2 = \{b_1, b_2\}$ be two subsets of $\mathbb{C}$ such that $S_1 \cap S_2 = \phi$ and $(b_1 - a_1)^2(b_1 - a_2)^2 \ldots (b_1 - a_m)^2 \neq (b_2 - a_1)^2(b_2 - a_2)^2 \ldots (b_2 - a_m)^2$. If f and a non-constant L-function $\mathcal{L}$ share S_1 IM and S_2 CM, then $\mathcal{L} = f$.*

The above theorem is one of the salient result in [16] and the proof of the same contains the major portion of the paper.

Remark 1. *In the proof of Theorem 1.2* [16] *[see p. 608, before (3.3)], the authors concluded that if* f *and* $\mathcal{L}$ *share* $S_1 = \{a_1, a_2, \ldots, a_m\}$ *IM and* $S_2 = \{b_1, b_2\}$ *CM, then* $P(f) = (f - a_1)(f - a_2)\ldots(f - a_m)$ *and* $P(\mathcal{L}) = (\mathcal{L} - a_1)(\mathcal{L} - a_2)\ldots(\mathcal{L} - a_m)$ *share the set* $S_3 = \{c_1, c_2\}$ *CM, where* $c_1 = (b_1 - a_1)(b_1 - a_2)\ldots(b_1 - a_m)$ *and* $c_2 = (b_2 - a_1)(b_2 - a_2)\ldots(b_2 - a_m)\}$*, with* $c_1^2 \neq c_2^2$*. With the help of this argument subsequently (see (3.4), under Case 2.1, in the proof of Theorem 1.2* [16]*) they set up an entire function* $V = e^u$ *and for some rational function* H*, they obtained* $T(r, e^u/H) = O(r)$*. Next using this, they proved the remaining part of the theorem.*
In general, from the basic definition of sharing of sets this argument is not true for any arbitrary f *and* $\mathcal{L}$*. Below we are explaining the facts:*
We first note that whenever f *and* $\mathcal{L}$ *share the set* $S_2 = \{b_1, b_2\}$ *CM, we have any* b_1 (b_2) *point of* f $(\mathcal{L})$ *of order say* p *becomes a* b_i $(i = 1, 2)$ *point of* $\mathcal{L}$ (f) *of order* p*. Then noting the definition of CM sharing of sets we know* $P(f)$ *and* $P(\mathcal{L})$ *will share the set* S_3 *CM only when the left hand side of the following equation*

$$P^2(h) - (c_1 + c_2)P(h) + c_1c_2 = 0$$

can be factorized in the form $(h - b_1)^m(h - b_2)^m$*, where* $h = f$ *or* $\mathcal{L}$ *with* $c_1^2 \neq c_2^2$*. But this is not always possible for any arbitrary choice of* a_i*'s,* $(i = 1, 2, \ldots, m)$ *and* b_i*'s,* $(i = 1, 2)$*. When* S_1 *contains one element say* $a_1 \neq \frac{b_1+b_2}{2}$ *then* $P^2(h) - (c_1 + c_2)P(h) + c_1c_2 = (h - a_1)^2 - (b_1 + b_2 - 2a_1)(h - a_1) + (b_1 - a_1)(b_2 - a_1) = (h - b_1)(h - b_2)$*. If* S_1 *contains two elements say* a_1*,* a_2*, then it is easy to verify*

$$P^2(h) - (c_1 + c_2)P(h) + c_1c_2 = (h - b_1)(h - a_1 - a_2 + b_1)(h - b_2)(h - a_1 - a_2 + b_2)$$

and so in this case also the arguments in [16] *does not hold. Next when* $m = 3$ *that is* S_1 *contains* 3 *elements, say* $a_1 = i$*,* $a_2 = -i$*,* $a_3 = -1$*, then considering* $b_1 = 1$*,* $b_2 = 0$ *it is easy to verify that* $c_1 = 4$ *and* $c_2 = 1$*. But*

$$P^2(h) - 5P(h) + 4 \neq h^3(h - 1)^3.$$

In fact, it is easy to verify that 1 *and* 0 *are the simple roots of the equation* $(t^2+1)^2(t+1)^2-5(t^2+1)(t+1)+4=0$. *So one can observe a big gap in the proof of Theorem 1.2* [16] *and the theorem cease to hold. Actually the condition* $c_1^2 \neq c_2^2$ *is not sufficient enough to factorize the expression* $P^2(t)-(c_1+c_2)P(t)+c_1c_2$ *into the form* $(t-b_1)^m(t-b_2)^m$ *except for the case* $m=1$ *with* $a_1 \neq \frac{b_1+b_2}{2}$.
As the entire analysis of Theorem 1.2 [16] *is depending upon the statement that if* f *and* $\mathcal{L}$ *share* S_1 *IM and* S_2 *CM, then* $P(f)$ *and* $P(\mathcal{L})$ *share* $S_3=\{c_1,c_2\}$ *CM, Theorem 1.2* [16] *is not valid.*

2. Motivation and Main results

In this paper though our prime intention is to provide an answer to the question of Yuan-Li-Yi [20], but at the same time we have somehow been able to present the corrected form of *Theorem F* concerning a special set introduced in [17] which in turn answer *Question 1.2*.
We require the following definitions for the main results of the paper.

Definition 2.1. [11] Let k be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g, respectively, share $(a,0)$ or (a,∞).

Definition 2.2. [10] For $S \subset \overline{\mathbb{C}}$ we define $E_f(S,k)=\cup_{a\in S}E_k(a;f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly, $E_f(S)=E_f(S,\infty)$ and $\overline{E}_f(S)=E_f(S,0)$. If $E_f(S,k)=E_g(S,k)$, we say that f and g share the set S with weight k.

Definition 2.3. [9] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f| \leq m)(N(r,a;f| \geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity. $\overline{N}(r,a;f \mid\leq m)$ and $\overline{N}(r,a;f \mid\geq m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.
Also $N(r,a;f \mid< m), N(r,a;f \mid> m), \overline{N}(r,a;f \mid< m)$ *and* $\overline{N}(r,a;f \mid> m)$ are defined analogously.

Definition 2.4. [1] Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the reduced counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r,a;f)$ the reduced counting function of those a-points of f and g where $p = q \geq 2$. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a \in \mathbb{C} \cup \{\infty\}$.

When f and g share (a,m), $m \geq 1$, then $N_E^{1)}(r,a;f) = N(r,a;f \mid= 1)$.

Definition 2.5. [10, 11] Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.
Clearly, $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$

Definition 2.6. [4] Let $P(z)$ be a polynomial such that $P^{'}(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities, $q_1, q_2, \ldots, q_k$, respectively. Then $P(z)$ is said to be a critically injective polynomial if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \cdots, k\}$.

From the definition it is obvious that $P(z)$ is injective on the set of distinct zeros of $P'(z)$ which are known as the critical points of $P(z)$. Thus a critically injective polynomial has at-most one multiple zero. We first invoke the following polynomial used in [17].

We denote by $P(z) = z^n + az^{n-m} + bz^{n-2m} + c$ and $\beta_i = -\left(c_i^n + ac_i^{n-m} + bc_i^{n-2m}\right)$, where $n, m \in \mathbb{N}$ and $a, b, c \in \mathbb{C}^*$ be such that $a^2 \neq 4b$, $\gcd(m,n) = 1$, $n > 2m$ and c_i be the roots of the equation

$$nz^{2m} + a(n-m)z^m + b(n-2m) = 0, \tag{2.1}$$

for $i = 1, 2, \ldots, 2m$. Note that when $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, then (2.1) reduces to the equation

$$n\left(z^m + \frac{a(n-m)}{2n}\right)^2 - \frac{a^2(n-m)^2}{4n} + b(n-2m) = 0;$$

i.e.,

$$n\left(z^m + \frac{a(n-m)}{2n}\right)^2 = 0. \tag{2.2}$$

Hence in this case (2.1) has m distinct roots c_i, $i = 1, 2, \ldots, m$ each being repeated twice.

In view of the above discussion, we have following theorems which are the main results of the paper.

Theorem 2.1. *Let $S = \{z : z^n + az^{n-m} + bz^{n-2m} + c = 0\}$, $S' = \{0, c_1, c_2, \ldots, c_m\}$, where $n \geq 2m + 3$, $\gcd(m,n) = 1$, $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$ and a, b, $c \in \mathbb{C}^*$ be such that $c \neq \beta_i, \frac{\beta_i\beta_j}{\beta_i+\beta_j}$. Let f be a non constant meromorphic function with finitely many poles and $\mathcal{L}$ be a non constant L-function such that $E_f(S,0) = E_{\mathcal{L}}(S,0)$, $E_f(S',\infty) = E_{\mathcal{L}}(S',\infty)$. Then for $n \geq \max\{2m+3, 7\}$ we get $f = \mathcal{L}$.*

Corollary 2.1. *Putting $a = -\frac{2n}{n-1}$, $b = \frac{n}{n-2}$, $c = \frac{2c'}{(n-1)(n-2)}$ and $m = 1$ in Theorem 2.1 we have $S = \{z : z^n - \frac{2n}{n-1}z^{n-1} + \frac{n}{n-2}z^{n-2} + \frac{2c'}{(n-1)(n-2)}\ (c' \neq 0, -1)\}$ and $S' = \{0, 1\}$. Clearly, if a nonconstant meromorphic function f with finitely many poles and a non* *constant L-function $\mathcal{L}$, such that $E_f(S,0) = E_{\mathcal{L}}(S,0)$, $E_f(S',\infty) = E_{\mathcal{L}}(S',\infty)$ then for $n \geq 7$ we will get $f = \mathcal{L}$. Hence for $m = 1$ we get a particular case of Theorem F.*

Theorem 2.2. *Let S and S' be defined as in* Theorem 2.1. *Let f be a non constant meromorphic function with finitely many poles and $\mathcal{L}$ be a non constant L-function such that $E_f(S,s) = E_{\mathcal{L}}(S,s)$, and $E_f(\{\alpha\},0) = E_{\mathcal{L}}(\{\alpha\},0)$ for some $\alpha \in S'$. For*
(I) $\alpha = 0$ and
(i) $s \geq 2$, $n \geq 2m+2$ or
(ii) $s = 1$, $n \geq 2m+3$ or
(iii) $s = 0$, $n \geq 2m+5$; we have $f = \mathcal{L}$.
Next suppose
(II) $\alpha \neq 0$. If
(i) $s \geq 1$ and $n \geq 2m+4$ or
(ii) $s = 0$ and $n \geq 2m+7$; then we have $f = \mathcal{L}$.

3. LEMMAS

Next, we present some lemmas that will be needed in the sequel. Henceforth, we denote by H, Φ the following functions:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{3.1}$$

and

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}. \tag{3.2}$$

Let f and g be two non-constant meromorphic functions and for an integer $n \geq 2m+1$

$$F = \frac{f^{n-2m}(f^{2m} + af^m + b)}{-c}, \quad G = \frac{g^{n-2m}(g^{2m} + ag^m + b)}{-c}. \tag{3.3}$$

Lemma 3.1. [21] *Let F and G share 1 IM and $H \not\equiv 0$. Then,*

$$N_E^{1)}(r,1;F) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 3.2. [15] *Let $P(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$, be an irreducible polynomial in f, with constants coefficient $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then*

$$T(r, P(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{m, n\}$.

Lemma 3.3. [2] *If F and G share $(1, s)$, $0 \leq s < \infty$, then*

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)+\left(s-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)-N_E^{1)}(r,1;F) \leq \frac{1}{2}\big(N(r,1;F)+N(r,1;G)\big).$$

Lemma 3.4. *Let F, G be given by (3.3) and $E_f(S,s) = E_g(S,s)$ where S is given as in* Theorem 2.1 *and $H \not\equiv 0$. Then we have*

$$\begin{aligned} N(r,\infty;H) \leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}\left(r,0;f^m + \frac{a(n-m)}{2n}\right) + \overline{N}\left(r,0;g^m + \frac{a(n-m)}{2n}\right) + \overline{N}_0(r,0;f') \\ & + \overline{N}_0(r,0;g') + S(r,f) + S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(nf^{2m} + (n-m)af^m + b(n-2m))(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. Since $E_f(S,s) = E_g(S,s)$, clearly, F and G share $(1,s)$.
Again from (3.3) and from the condition $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$ mentioned in Theorem 2.1 we get that

$$F' = \frac{f^{n-2m-1}(nf^{2m} + (n-m)af^m + b(n-2m))f'}{-c} = \frac{nf^{n-2m-1}(f^m + \frac{a(n-m)}{2n})^2 f'}{-c},$$

$$G^{'}=\frac{g^{n-2m-1}(ng^{2m}+a(n-m)g^{m}+b(n-2m))g'}{-c}=\frac{ng^{n-2m-1}(g^{m}+\frac{a(n-m)}{2n})^{2}g^{'}}{-c}.$$

then

$$\overline{N}(r,0;nf^{2m}+a(n-m)f^{m}+b(n-2m))=\overline{N}(r,0;f^{m}+\frac{a(n-m)}{2n}),$$

Similar result holds for g. Then, clearly, from the definition of H we have

$$\begin{aligned}N(r,H)\ &\leq\ \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_{*}(r,1;F,G)\\&\quad+\overline{N}\left(r,0;f^{m}+\frac{a(n-m)}{2n}\right)+\overline{N}\left(r,0;g^{m}+\frac{a(n-m)}{2n}\right)+\overline{N}_{0}(r,0;f^{'})\\&\quad+\overline{N}_{0}(r,0;g^{'})+S(r,f)+S(r,g).\end{aligned}$$

Hence the proof is complete. □

Lemma 3.5. *Let F, G be given by (3.3) and $E_f(S,0)=E_g(S,0)$, $E_f(S',\infty)=E_g(S',\infty)$, where S, S' be given as in* Theorem 2.1. *Suppose $H\not\equiv 0$. Then for $\frac{a^2}{4b}=\frac{n(n-2m)}{(n-m)^2}$, we have*

$$\begin{aligned}N(r,\infty;H)\ &\leq\ \chi_{n}\left(\overline{N}(r,0;f)+\overline{N}\left(r,0;f^{m}+\frac{a(n-m)}{2n}\right)\right)+\overline{N}(r,\infty;f)\\&\quad+\overline{N}(r,\infty;g)+\overline{N}_{*}(r,1;F,G)+\overline{N}_{0}(r,0;f^{'})+\overline{N}_{0}(r,0;g^{'}),\end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f(nf^{2m}+(n-m)af^{m}+b(n-2m))(F-1)$, $\overline{N}_0(r,0;g^{'})$ is similarly defined and $\chi_n=1$ when $n\neq 2m+3$ and $\chi_n=0$ when $n=2m+3$.

Proof. We omit this proof since it can be easily obtained from the proof of *Lemma 2.2* [17]. □

Lemma 3.6. *Let S, S' be defined as in* Theorem 2.1 *and F, G be given by (3.3). Suppose for two non-constant meromorphic functions f and g, $E_f(S,0)=E_g(S,0)$,*

$E_f(S',\infty)=E_g(S',\infty)$, and $\Phi\not\equiv 0$. Then for $\frac{a^2}{4b}=\frac{n(n-2m)}{(n-m)^2}$ with $n\geq 2m+3$, we have

$$\begin{aligned}&\overline{N}\left(r,0;f\right)+\overline{N}\left(r,0;f^m+\frac{a(n-m)}{2n}\right)\\ \leq\ &\frac{1}{2}\left(\overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+S(r,f)+S(r,g).\end{aligned}$$

Proof. We omit this proof since it can be easily obtained from the proof of *Lemma 2.5* [17]. □

Lemma 3.7. [3] *Let $\phi(z)=a^2(z^{n-m}-A)^2-4b(z^{n-2m}-A)(z^n-A)$, where $A,a,b\in\mathbb{C}^*$, $\frac{a^2}{4b}=\frac{n(n-2m)}{(n-m)^2}$,* gcd$(m,n)=1$, *$n>2m$. If ω^l is the m-th root of unity for $l=0,1,\ldots,m-1$, then*
i) $\phi(z)$ has no multiple zero, when $A\neq\omega^l$.
ii) $\phi(z)$ has exactly one multiple zero, when $A=\omega^l$ and that is of multiplicity 4.
In particular, when $A=1$, then the multiple zero is 1.

Lemma 3.8. [3] *Let $P(z)=z^n+az^{n-m}+bz^{n-2m}+c$, where $a,b\in\mathbb{C}^*$. Then the followings hold.*
i) β_i's are non-zero if $a^2\neq 4b$.
ii) $P(z)$ is critically injective polynomial if $\frac{a^2}{4b}=\frac{n(n-2m)}{(n-m)^2}$.

Lemma 3.9. *Let F, G be given by (3.3), $E_f(S,s)=E_g(S,s)$, where S is defined as in* Theorem 2.1. *Then*

$$\overline{N}_L(r,1;F)\leq\frac{1}{s+1}\left(\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right)+S(r,f).$$

Similar inequality holds for G.

Proof. Since $E_f(S,s)=E_g(S,s)$, clearly, F and G share $(1,s)$. From the choice of c, it is clear that the polynomial $P(z)=:z^n+az^{n-m}+bz^{n-2m}+c$ has no multiple zero, so we have

$$\begin{aligned}
\overline{N}_L(r,1;F) &\leq \overline{N}(r,1;F\mid\geq s+2)\\
&\leq \overline{N}(r,0;F'\mid\geq s+1;F=1)\\
&\leq \frac{1}{s+1}N(r,0;F'\mid\geq s+1;F=1)\\
&\leq \frac{1}{s+1}\big(N(r,0;f'\mid f\neq 0)-N_o(r,0;f')\big)\\
&\leq \frac{1}{s+1}\big(N(r,0;\frac{f'}{f})-N_o(r,0;f')\big)\\
&\leq \frac{1}{s+1}\big(\overline{N}(r,\infty;f)+\overline{N}(r,0;f)-N_o(r,0;f')\big)+S(r,f)\\
&\leq \frac{1}{s+1}\big(\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_o(r,0;f')\big)+S(r,f).
\end{aligned}$$

Here $N_o(r,0;f')=N(r,0;f'\mid f\neq 0,\alpha_1,\alpha_2,\ldots,\alpha_n)$, where $\alpha_1,\alpha_2,\ldots,\alpha_n$ are zeros of the polynomial $P(z)$. □

Lemma 3.10. *Let F, G be given by (3.3) and $\Phi\not\equiv 0$. Also let $E_f(S,s)=E_g(S,s)$, where S is defined as in* Theorem 2.1*, and f and g share* $(0,0)$ *then,*

$$\begin{aligned}
\overline{N}(r,0;f)=\overline{N}(r,0;g) &\leq \frac{1}{n-2m-1}(\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,\infty;F)\\
&\quad+\overline{N}(r,\infty;G))+S(r,F)+S(r,G).
\end{aligned}$$

Proof. Since f, g share $(0,0)$, it follows that

$$\begin{aligned}
\overline{N}(r,0;f)=\overline{N}(r,0;g) &\leq \frac{1}{n-2m-1}N(r,0;\Phi)\\
&\leq \frac{1}{n-2m-1}T(r,\Phi)+O(1)\\
&\leq \frac{1}{n-2m-1}N(r,\infty;\Phi)+S(r,F)+S(r,G)\\
&\leq \frac{1}{n-2m-1}\big(\overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\big)\\
&\quad+S(r,F)+S(r,G).
\end{aligned}$$

□

Lemma 3.11. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$ and S be defined as in Theorem 2.1 . If f and a non constant L-function $\mathcal{L}$ share the set S IM, then $\rho(f) = \rho(\mathcal{L}) = 1$.*

Proof. Adopting the same procedure as done in Theorem 5, {p. 6, [20]} we can easily obtain $\rho(f) = \rho(\mathcal{L}) = 1$. □

Lemma 3.12. [13] *If $\mathcal{L}$ is a non-constant L-function, then there is no generalized Picard exceptional value of $\mathcal{L}$ in the complex plane.*

4. Proofs of the theorems

Proof of Theorem 2.1. Let us consider

$$F = \frac{f^{n-2m}(f^{2m} + af^m + b)}{-c}, \quad G = \frac{\mathcal{L}^{n-2m}(\mathcal{L}^{2m} + a\mathcal{L}^m + b)}{-c}.$$

Clearly, F and G share $(1, 0)$. Since f has finitely many poles and $\mathcal{L}$ has at most one pole then $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; \mathcal{L}) = O(\log r)$. Also from *Lemma 3.11* we have $\rho(f) = \rho(\mathcal{L}) = 1$. Therefore it is obvious that, $S(r, f) = S(r, \mathcal{L}) = O(\log r)$.
Now from *Lemmas 3.1*, *3.2 3.5*, *3.6* and putting $s = 0$ in *Lemma 3.3* and by the second fundamental theorem we have

$$\begin{aligned}
& (n+m)(T(r,f) + T(r,\mathcal{L})) \\
\leq\ & \overline{N}(r,1;F) + \overline{N}(r,1;G) + \sum_{i=0}^{m} \overline{N}(r,c_i;f) + \sum_{i=0}^{m} \overline{N}(r,c_i;\mathcal{L}) + \overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L}) \\
& + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - N_0(r,0;f') - N_0(r,0;\mathcal{L}') + S(r,f) + S(r,\mathcal{L}).
\end{aligned}$$

i.e.,

$$\begin{aligned}(4.1)\qquad & \frac{n}{2}\left(T(r,f)+T(r,\mathcal{L})\right)\\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L})+\left(\frac{3}{2}+\frac{\chi_n}{2}\right)\left(\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\right)\\ & +O(\log r)\\ \leq\ & T(r)+\left(\frac{3}{2}+\frac{1}{2}\right)\left(\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\right)+O(\log r),\end{aligned}$$

where $T(r)=T(r,f)+T(r,\mathcal{L})$.

Clearly, when $n\geq 7$ in view of *Lemma 3.9*, from (4.1) we get a contradiction. Therefore $H\equiv 0$ and so integrating both sides we get,

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \tag{4.2}$$

where $A\neq 0$, B are two constants. From *Lemma 3.2* and (4.2) we have,

$$T(r,\mathcal{L})=T(r,f)+O(1). \tag{4.3}$$

We omit the rest of the proof of this theorem as it can be carried out in the line of proof of *Theorem 1.1* for $H\equiv 0$ [17]. Again if $\Phi\equiv 0$ then on integrating we have $F-1=C(G-1)$ and dealing exactly in the same way as in **Subcase-II-2.1.1 - 2.2** of *Theorem 2.2* we will get the result. □

Proof of Theorem 1.2. Let F and G be given as in the proof of *Theorem 2.1.* Since $E_f(S,s)=E_g(S,s)$ then, clearly, F and G share $(1,s)$. Also it is given that $E_f(\{\alpha\},0)=E_{\mathcal{L}}(\{\alpha\},0)$ where $\alpha\in S'$. Next we consider the following cases.

<u>**Case-I.**</u> Let us take $\alpha=0$. Considering $H\not\equiv 0$ and using the same argument as in *Lemma 3.4* we get

$$\begin{aligned}N(r,\infty;H)\ \leq\ & \overline{N}_*(r,0;f,\mathcal{L})+\overline{N}\left(r,0;f^m+\frac{a(n-m)}{2n}\right)+\overline{N}\left(r,0;\mathcal{L}^m+\frac{a(n-m)}{2n}\right)\\ & +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;\mathcal{L}')\\ & +S(r,f)+S(r,\mathcal{L}).\end{aligned}$$

Now proceeding same as in (4.1) we have

$$\frac{n}{2}T(r) \leq mT(r) + 3\overline{N}(r,0;f) + \left(\frac{3}{2} - s\right)\overline{N}_*(r,1;F,G) + O(\log r). \tag{4.4}$$

Next in view of *Definition 1.6*, using *Lemma 3.10* in (4.4) we get

$$\frac{n}{2}T(r) \leq mT(r) + \left(\frac{3}{2} - s + \frac{3}{n-2m-1}\right)\overline{N}_*(r,1;F,G) + O(\log r). \tag{4.5}$$

Clearly, when

$$\begin{array}{ll} (i) & s \geq 2,\ n \geq 2m+2 \ \ \textit{or} \ \ \textit{when} \\ (ii) & s = 1, \ \ n \geq 2m+3 \ \ \textit{or} \ \ \textit{when} \\ (iii) & s = 0, \ \ n \geq 2m+5, \end{array}$$

using *Lemma 3.9*, from (4.5) we get a contradiction.
Therefore $H \equiv 0$. Integrating both sides we get (4.2) and so from *Lemma 3.2* we again have (4.3).

<u>Case-I-1.</u> Suppose $B \neq 0$. Then from (4.2) we get

$$G - 1 \equiv \frac{F-1}{BF + A - B}. \tag{4.6}$$

<u>Subcase-I-1.1</u> If $A - B \neq 0$, then noting that $\frac{B-A}{B} \neq 0, 1, \infty$; from (4.1) we get

$$\overline{N}(r, \frac{B-A}{B}; F) = \overline{N}(r,\infty;G).$$

Therefore in view of *Lemma 3.3* and (4.3) the second fundamental theorem yields

$$\begin{aligned} nT(r,f) = T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r, \frac{B-A}{B}; F) + S(r,F) \\ &\leq (2m+1)T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{L}) + S(r,f) \\ &\leq (2m+1)T(r,f) + O(\log r), \end{aligned}$$

which is a contradiction for $n \geq 2m+2$.

Subcase-I-1.2. If $A - B = 0$, then from (4.6) we have

$$G - 1 = \frac{F-1}{BF}. \tag{4.7}$$

(4.7) implies that 0's of f and $(f^{2m} + af^m + b)$ contributes to the poles of G. Since $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$; i.e., $a^2 \neq 4b$, it follows that all the zeros of $z^{2m} + az^m + b$ are simple. Since $\overline{N}(r, \infty; G) = \overline{N}(r, \infty; \mathcal{L})$, $\mathcal{L}$ has at most one pole at $z = 1$ and $m \geq 2$, we arrive at a contradiction. When $m = 1$, let η_i $(i = 1, 2)$ be the zeros of $z^2 + az + b$ and so the $\{0, \eta_1, \eta_2\}$ points of f will be the poles of $\mathcal{L}$. First using the second fundamental theorem, it is easy to verify that among these $\{0, \eta_1, \eta_2\}$ points, f can not have two exceptional values, so f may have only one exceptional value which implies $\mathcal{L}$ has more than one pole. Hence we arrive at a contradiction again.

Case-I-2. Suppose $B = 0$. Then from (4.2) we get that

$$F - 1 = A(G - 1);$$

i.e.,

$$f^n + af^{n-m} + bf^{n-2m} \equiv A\left(\mathcal{L}^n + a\mathcal{L}^{n-m} + b\mathcal{L}^{n-2m} + c\frac{A-1}{A}\right) \tag{4.8}$$

and

$$f^n + af^{n-m} + bf^{n-2m} + c(1-A) \equiv A\left(\mathcal{L}^n + a\mathcal{L}^{n-m} + b\mathcal{L}^{n-2m}\right). \tag{4.9}$$

Since f and $\mathcal{L}$ share 0 IM and $\mathcal{L}$ has no exceptional value, from (4.8), (4.9) we get $A = 1$.

Subcase-I-2.1. When $A = 1$. Then we get $F \equiv G$; i.e.,

$$\mathcal{L}^{n-2m}(\mathcal{L}^{2m} + a\mathcal{L}^m + b) \equiv f^{n-2m}(f^{2m} + af^m + b). \tag{4.10}$$

From (4.10) we have f, $\mathcal{L}$ share 0 and ∞ CM. Then, clearly, $h=\frac{\mathcal{L}}{f}$ has no zero and no pole. Now putting $\mathcal{L}=fh$ in $F\equiv G$ we get

$$f^{2m}(h^{n}-1)+af^{m}(h^{n-m}-1)+b(h^{n-2m}-1)=0. \tag{4.11}$$

Subcase-I-2.1.1. If h is constant, then as f is non-constant so, $h^{n}=h^{n-m}=h^{n-2m}=1$. Since $\gcd(m,n)=1$, so $h=1$. Therefore $f\equiv\mathcal{L}$.

Subcase-I-2.1.2. If h is non-constant, then from (4.11), in view of *Lemma 3.7* we get

(4.12)

$$\left(f^{m}+\frac{a}{2}\frac{h^{n-m}-1}{h^{n}-1}\right)^{2}=\frac{\phi(h)}{4(h^{n}-1)^{2}}=\frac{a^{2}(h-1)^{4}(h-\nu_{1})(h-\nu_{2})\dots(h-\nu_{2n-2m-4})}{4(h^{n}-1)^{2}},$$

where ν_i's are the distinct simple zeros of $\phi(h)$ and each ν_i points of h are of multiplicities at least 2. Therefore by the second fundamental theorem we get

$$\begin{aligned}(2n-2m-4)T(r,h) &\leq \sum_{i=1}^{2n-2m-4}\overline{N}(r,\nu_i;h)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h)\\ &\leq (n-m-2)T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction for $n\geq 2m+2$.

Also if $\Phi\equiv 0$ then integrating we will have $F-1\equiv C(G-1)$ and since $f,\mathcal{L}$ share 0 IM then we will have $F\equiv G$ and hence $f\equiv\mathcal{L}$.

Case-II. Let us consider $\alpha(\neq 0)\in S'$.

Without loss of generality we may assume $\alpha = c_m$. Considering $H \not\equiv 0$ and by the same argument as in *Lemma 3.4* we get

$$\begin{aligned}
& N(r,\infty;H) \\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L})+\sum_{i=0}^{m-1}\overline{N}(r,c_i;f)+\sum_{i=0}^{m-1}\overline{N}(r,c_i;\mathcal{L})+\overline{N}_*(r,\alpha;f,\mathcal{L}) \\
& +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;\mathcal{L}^{'}) \\
& +O(\log r).
\end{aligned}$$

Now proceeding same as in (4.1) we have

$$\begin{aligned}
& \frac{n}{2}T(r) \\
\leq\ & (m-1)T(r)+2(\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}))+\left(\frac{3}{2}-s\right)(\overline{N}_L(r,1;F) \\
& +\overline{N}_L(r,1;G))+\overline{N}_*(r,\alpha;f,\mathcal{L})+O(\log r).
\end{aligned} \tag{4.13}$$

Now using *Lemma 3.9* in (4.13) we get

$$\begin{aligned}
& \frac{n}{2}T(r) \\
\leq\ & (m+1)T(r)+\frac{3-2s}{2(s+1)}\left(\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L})\right)+\overline{N}(r,\alpha;f)+O(\log r).
\end{aligned} \tag{4.14}$$

Clearly, when

$$(i) \quad s \geq 1,\ n \geq 2m+4 \ \ or \ \ when$$

$$(iii) \quad s = 0, \ \ n \geq 2m+7;$$

from (4.14) we get a contradiction.

Therefore $H \equiv 0$ and so integration again yields (4.2).

Case-II-1. Suppose $B \neq 0$. Then we again get (4.6). So we have

$$\overline{N}(r,\frac{B-A}{B};F)=\overline{N}(r,\infty;G),$$

where $A,\ A-B \neq 0$. Now we consider the following sub cases:

Subcase-II-1.1 Suppose that $\frac{B-A}{B} = \frac{\beta_m}{c}$ where $\alpha = c_m$. Since $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, then we have

$$F^{'} = n\frac{f^{n-2m-1}\left(\prod_{i=1}^{m}(f-c_i)\right)^2}{-c}f^{'}. \tag{4.15}$$

Again $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2} \neq 1$ implies $a^2 \neq 4b$. Therefore by *Lemma 3.8* we get $\beta_m \neq 0$ and $P(z)$ is critically injective. Since any critically injective polynomial can have at most one multiple zero, it follows that

$$f^n + af^{n-m} + bf^{n-2m} + \beta_m = (f-c_m)^3\prod_{j=1}^{n-3}(f-\xi_j), \tag{4.16}$$

where ξ_j's are $(n-3)$ distinct zeros of $z^n + az^{n-m} + bz^{n-2m} + \beta_m$ such that $\xi_j \neq c_m, 0$, $j = 1, 2, \ldots, n-3$. Then from (4.6) and (4.16) we have

$$B(G-1) \equiv \frac{-c(F-1)}{(f-c_m)^3\prod_{j=1}^{n-3}(f-\xi_j)}. \tag{4.17}$$

Since $E_f(\{c_m\}, 0) = E_g(\{c_m\}, 0)$, so c_m points of f are not poles of G and hence c_m is an e.v.P. of f and hence an e.v.P. of $\mathcal{L}$. Therefore from *Lemma 3.12* we arrive at a contradiction.

Subcase-II-1.2 Next suppose $\frac{B-A}{B} \neq \frac{\beta_m}{c}$. Since A and $A-B$ are non zero then adopting the same procedure as done in **Subcase-I-1.1** of this theorem again we can get a contradiction.

Subcase-II-1.3 If $A - B = 0$ then by **Subcase-I-1.2** we arrived at a contradiction.

Subcase-II-2 Assuming $B = 0$ we get

$$F - 1 = A(G-1)$$

and subsequently we can obtain (4.8), (4.9).

Subcase-II-2.1. Let $A \neq 1$. Then as $c \neq 0$, so $c\frac{(A-1)}{A} \neq 0$ and at the same time by

Lemma 3.8 we have $\beta_i \neq 0$. Therefore we have the following subcases.

Subcase-II-2.1.1. Suppose $c\frac{(A-1)}{A} = \beta_i$ for some $i \in \{1, 2, \ldots, m\}$. Then we claim that $c(1-A) \neq \beta_j$ for any $j \in \{1, 2, \ldots, m\}$. For if $c(1-A) = \beta_j$; then $A = \frac{c-\beta_j}{c}$ and since it is given that $c\frac{(A-1)}{A} = \beta_i$; i.e., $A = \frac{c}{c-\beta_i}$, it follows that $\frac{c-\beta_j}{c} = \frac{c}{c-\beta_i}$; i.e., $c = \frac{\beta_i\beta_j}{\beta_i+\beta_j}$, a contradiction. Thus $z^n + az^{n-m} + bz^{n-2m} + c(1-A) = 0$ has only simple roots say γ_i for $i = 1, 2, \ldots, n$. So from (4.9), (4.3) and by using the second fundamental theorem we get

$$\begin{aligned}(n-1)T(r,f) &\leq \sum_{i=1}^{n} \overline{N}(r,\gamma_i; f) + \overline{N}(r,\infty; f) + S(r,f)\\ &\leq (2m+1)T(r,\mathcal{L}) + O(\log r),\end{aligned}$$

gives a contradiction for $n \geq 2m+3$.

Subcase-II-2.1.2. Suppose $c\frac{(A-1)}{A} \neq \beta_i$ for all $i \in \{1, 2, \ldots, m\}$. So, $z^n + az^{n-m} + bz^{n-2m} + c\frac{(A-1)}{A} = 0$ has only simple roots say μ_i for $i = 1, 2, \ldots, n$. Therefore from (4.8), (4.3) and by the second fundamental theorem we have

$$\begin{aligned}(n-1)T(r,\mathcal{L}) &\leq \sum_{i=1}^{n} \overline{N}(r,\mu_i; \mathcal{L}) + \overline{N}(r,\infty; \mathcal{L}) + S(r,\mathcal{L})\\ &\leq (2m+1)T(r,f) + O(\log r),\end{aligned}$$

gives a contradiction for $n \geq 2m+3$.

Subcase-II-2.2. Suppose $A = 1$. Then we get $F = G$ and hence we obtain (4.10). Putting $\mathcal{L} = fh$ in (4.10) we get (4.11).

Now proceeding the same way as done in **Subcase-I-2.1.1-Case-I-2.1.2** of this theorem, we will get $f \equiv \mathcal{L}$, for $n \geq 2m+4$.

□

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