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STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

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ABSTRACT. In this paper, we show that a double sequence in a topological space satisfies certain conditions, which are capable of generating a topology on a nonempty set. Also, we used the idea of statistical limit and statistical cluster point to establish some properties in the sense of double sequences. One of our main interest is to investigate the relationship between statistical limit, s-limit and statistical cluster points of double sequences.

1. Introduction

The notion of statistical convergence of sequences was given by Fast [8] and Schoenberg [21] as an extension of the notion of convergence of a real sequence. In 2012, M. Mursaleen et al. [16] introduced the notion of weighted A-statistical convergence of a sequence, where A represents the nonnegative regular matrix. Kostyrko et al. [11] introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of this convergence. The concept of $\mathcal{I}$-convergence is a generalization of statistical convergence and it is dependent on the notion of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers. In [17], a decomposition theorem for $\mathcal{I}$-convergent

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sequences is given and also they introduced the notions of $\mathcal{I}$-Cauchy sequence and studied their certain properties. Further this concept was studied by [2], [15], [19] and many others. In 2008, Das et al. [5] considered the notions of $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences in real line as well as in general metric spaces.

The notion of statistical convergence was introduced for double sequences by Mure-saleen and Edely [14] (also by Ḿoricz [13] who introduced it for multiple sequences). Further, this concept was studied by Banerjee [1], Fridy [9, 10], Cakalh [3], Maio [6] and many others. Recently, Mohiuddine et al. [12] studied statistically convergent, statistically bounded and statistically Cauchy double sequences in locally solid Riesz spaces. Throughout the paper, X denotes a topological space, $\mathbb{R}$ stands for the set of all real numbers, $\mathbb{N}$ stands for the set of all natural numbers, $X \setminus A$ denotes the complement of the set A. All spaces are assumed to be Hausdorff.

2. Preliminaries

A double sequence (x_{mn}) in a topological space (X, τ) is said to *converge to a point* $\xi \in X$ in Pringsheims sense [20] if for every open set U containing ξ, there exists a $k \in \mathbb{N}$ such that $x_{mn} \in U$ for all $m > k$ and $n > k$. The element ξ is called Pringsheim limit of the double sequence (x_{mn}) and is denoted by $P\text{-}\lim\limits_{m,n\to\infty} x_{mn} = \xi$.

Let (X, τ) be a topological space and let ϕ be lexicographically ordered set $\mathbb{N} \times \mathbb{N}$ into X considered as a subset of $\mathbb{N} \times \mathbb{N} \times X$. Then the set ϕ, ordered by $((m, n), \phi((m, n)))$ $<$ $((r, s), \phi((r, s)))$ if and only if $(m, n) < (r, s)$ is called a *double sequence* [18]. Denote $\phi((m, n)) = x_{mn}$ and $\phi = \langle x_{mn}\rangle$. For a fixed $m \in \mathbb{N}$, the simple sequence $\langle x_{mn}\rangle_{n=1}^{\infty}$ is called the *m-th straight-sequence* and its subsequences are called *straight-subsequences* [18] in $\phi = \langle x_{mn}\rangle$; we condense $\langle x_{mn}\rangle_{n=1}^{\infty}$ to $\langle x_{mn}\rangle$. For a map $g : \mathbb{N} \to \mathbb{N}$, the simple sequence $(x_{mg(m)})$ is called a *cross-sequence* [18] and its subsequences are called cross-subsequences in (x_{mn}). It should be also consider that the notion of the set $\mathbb{N} \times \mathbb{N}$ with respect to the relation by Das and Malik [4] given as follows.

$(m,n) < (m_1,n_1)$ if $m+n < m_1+n_1$, or $m < m_1$ when $m+n = m_1+n_1$ and $(m,n) = (m_1,n_1)$ if $m = m_1, n = n_1$.

The following lemma will be useful in the sequel.

Lemma 2.1. [18, Lemma 2] Let (x_{mn}) be a double sequence of points of a set X. Then there exists a one-to-one double subsequence of (x_{mn}) if and only if there is a double subsequence (y_{ij}) such that $\{y_{ij} \mid j \in \mathbb{N}\}$ are infinite sets for all $i \in \mathbb{N}$.

3. Properties of s-convergence

In this paper, the cardinality of the set B is denoted by $|B|$. The definition of statistical convergence of double sequences is based on the notion of asymptotic density of a set $A \subset \mathbb{N}\times\mathbb{N}$.

Definition 3.1. Let $A \subset \mathbb{N}\times\mathbb{N}$. Put $A(m,n) = \{(k,l) \in A \mid k \le m, l \le n\}$. Then we call $\underline{\delta}(A) = P\text{-}\liminf\limits_{m,n\to\infty} \dfrac{|A(m,n)|}{mn}$ and $\bar{\delta}(A) = P\text{-}\limsup\limits_{m,n\to\infty} \dfrac{|A(m,n)|}{mn}$, the *lower and upper asymptotic density* of A, respectively. If $\underline{\delta}(A) = \bar{\delta}(A)$, then $d(A) = \delta(A) = P\text{-}\lim\limits_{m,n\to\infty} \dfrac{|A(m,n)|}{mn}$ is called the *double asymptotic density* (or *double natural density)* of A.

Definition 3.2. A subset K of $\mathbb{N}\times\mathbb{N}$ is said to be *statistically dense* if $d(K) = 1$. Note that $d((\mathbb{N}\times\mathbb{N}) \setminus A) = 1 - d(A)$ for $A \subset \mathbb{N}\times\mathbb{N}$.

Example 3.1. Consider the sequence $(x_{mn})_{m,n\in\mathbb{N}}$ in $\mathbb{R}$ defined by

$$(x_{mn}) = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are prime numbers} \\ 0 & \text{otherwise} \end{cases}$$

Let U be a neighborhood of 0.

Then $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \notin U\}) = \lim\limits_{m,n\to\infty} \frac{|\{(m,n)\in\mathbb{N}\times\mathbb{N} | x_{mn}\notin U\}|}{mn} = 0$. Therefore, the set of prime natural numbers has asymptotic density 0 and so this sequence statistically converges to 0.

Definition 3.3. A double sequence (x_{mn}) in a topological space (X, τ) is said to *converge statistically* (or shortly, *s-converge*) to $x \in X$, if for every neighborhood U of x, $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \notin U\}) = 0$. In this case, we write $x = s\text{-}\lim_{m,n\to\infty} x_{mn}$ or $(x_{mn}) \xrightarrow{s} x$.

For $X = \mathbb{R}$, equivalently this definition says that there exists a subset $A = \{(m,n)\}$ of $\mathbb{N} \times \mathbb{N}$ with $d(A) = 1$ such that a double sequence (x_{mn}) converges to x, that is, for every neighborhood U of x, there is $m_0 \in \mathbb{N}$ such that $m, n \geq m_0$ and $(m,n) \in A$ implies that $x_{mn} \in U$.

Definition 3.4. We write $\{x_{mn} \mid (m,n) \in \mathbb{N} \times \mathbb{N}\}$ to denote the range of a sequence S. If $(x_{m_k n_l})$ is a subsequence of S and $K = \{(m_k, n_l) \mid (k,l) \in \mathbb{N} \times \mathbb{N}\}$, then we abbreviate $(x_{m_k n_l})$ by $\{S\}_K$. If $d(K) = 0$, then $\{S\}_K$ is called a *subsequence of density zero*, or a *thin subsequence.* On the other hand, $\{S\}_K$ is a *non-thin subsequence* of S if K does not have density zero.

Definition 3.5. A subsequence $(x_{m_k n_k})$ of a double sequence (x_{mn}) is *statistically dense* in (x_{mn}) if the set of indices (m_k, n_k) is a statistically dense subset of $\mathbb{N} \times \mathbb{N}$.

Definition 3.6. A double sequence (x_{mn}) in a topological space (X, τ) is said to *s^*-converge* to $x \in X$, if there is $A \subset \mathbb{N} \times \mathbb{N}$ with $d(A) = 1$ such that $\lim_{\substack{m,n\to\infty \\ (m,n)\in A}} (x_{mn}) = x$. We write $s^*\text{-}\lim x_{mn} = x$.

The following Lemma 3.1 shows that every convergent sequence is s-convergent in the sense of double sequences. But converse is not true as shown by Example 3.2.

Lemma 3.1. *If a double sequence (x_{mn}) in a topological space (X, τ) converges to $x \in X$, then (x_{mn}) s-converges to x.*

Proof. Let U be a neighborhood of x. Since (x_{mn}) converges to x, there exists $n_0 \in \mathbb{N}$ such that $x_{mn} \in U$ for all $m, n \geq n_0$. Let $A = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}$. Then

$\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \notin U\} \subset (\mathbb{N} \times \mathbb{N}) \setminus A$. Since $d(A) = 1$, $d((\mathbb{N} \times \mathbb{N}) \setminus A) = 1 - d(A) = 1 - 1 = 0$. Therefore, $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \notin U\}) = 0$. Hence (x_{mn}) s-converges to x. □

Example 3.2. [6] Consider $\mathbb{R}$ with usual topology. Let (x_{mn}) be a divergent sequence in $\mathbb{R}$ defined by

$$(x_{mn}) = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are prime} \\ 0 & \text{otherwise} \end{cases}$$

Since the set of prime natural numbers has natural density 0, this sequence statistically converges to 0. But (x_{mn}) is not a convergent sequence.

Theorem 3.1. *If a double sequence (x_{mn}) is s-convergent to x in (X, τ), then each non-thin subsequence $(x_{m_k n_l})$ of (x_{mn}) is also s-convergent to x in (X, τ).*

Proof. Let $\{x_{m_k n_l} \mid (k,l) \in \mathbb{N} \times \mathbb{N}\}$ be any non-thin subsequence of (x_{mn}) and U be any open neighborhood of x. Then $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \notin U\}) = 0$. Let $y_{kl} = x_{m_k n_l}$ and f and g be injective functions such that $g(y_{kl}) = x_{m_k n_l}$. We prove that (y_{kl}) statistically converges to x. Let $z_{ts} = y_{k_t l_s} \in U$, that is, $f(z_{ts}) = y_{k_t l_s}$.
Let $d(\{(m_k, n_l) \in \mathbb{N} \times \mathbb{N} \mid y_{kl} \in g^{-1}(x_{m_k n_l})\}) = r$.
Since (x_{mn}) statistically converges to x, $d(\{(m_k, n_l) \in \mathbb{N} \times \mathbb{N} \mid g(y_{kl}) = x_{m_k n_l} \in U\}) = r$, that is, $d(\{(m_{k_t}, n_{l_s}) \in \mathbb{N} \times \mathbb{N} \mid z_{ts} \in (g \circ f)^{-1}(x_{m_{k_t} n_{l_s}})\}) = r$
$d(\{(m_{k_t}, n_{l_s}) \in \mathbb{N} \times \mathbb{N} \mid z_{ts} \in (g \circ f)^{-1}(x_{m_{k_t} n_{l_s}})\}) = r$
Thus, $r = \lim\limits_{p,q \to \infty} \frac{|\{(m_{k_t}, n_{l_s}) \in \mathbb{N} \times \mathbb{N} \mid z_{ts} \in (g \circ f)^{-1}(x_{m_{k_t} n_{l_s}}), m_{k_t} \le p,\ n_{l_s} \le q\}|}{p \times q}$
If $|\{(m_k, n_l) \in \mathbb{N} \times \mathbb{N} \mid y_{kl} \in g^{-1}(x_{m_k n_l}), m_{k_t} \le p,\ n_{l_s} \le q\}| = kl$, then
$\{(m_{k_t}, n_{l_s}) \in \mathbb{N} \times \mathbb{N} \mid z_{ts} \in (g \circ f)^{-1}(x_{m_{k_t} n_{l_s}})\} = \{(k_t, l_s) \in \mathbb{N} \times \mathbb{N} \mid k_t \le k, l_s \le l,$
$z_{ts} \in f^{-1}(y_{k_t l_s})\}$.

$$r = \lim_{m,n \to \infty} \frac{|\{(m_{k_t}, n_{l_s}) \in \mathbb{N} \times \mathbb{N} \mid z_{ts} \in (g \circ f)^{-1}(x_{m_{k_t}, n_{l_s}}), m_{k_t} \le m, n_{l_s} \le n\}||\{(m_k, n_l) \in \mathbb{N} \times \mathbb{N} \mid y_{kl} \in g^{-1}(x_{m_k n_l}), m_k \le m, n_l \le n\}|}{m \times n \times k \times l}$$

$$r = \lim_{k,l \to \infty} \frac{|\{(k_t, l_s) \in \mathbb{N} \times \mathbb{N} \mid k_t \le k, l_s \le l, z_{ts} \in f^{-1}(y_{k_t l_s})\}|}{k \times l} \times \lim_{m,n \to \infty} \frac{|\{(m_k, n_l) \in \mathbb{N} \times \mathbb{N} \mid y_{kl} \in g^{-1}(x_{m_k n_l}), m_k \le m,\ n_l \le n\}|}{m \times n}$$

$r = \lim_{k,l\to\infty} \frac{|\{(k_t,l_s)\in\mathbb{N}\times\mathbb{N} \mid k_t\le k, l_s\le l, z_{ts}\in f^{-1}(y_{k_t l_s})\}|}{k\times l} \times r$

$\lim_{k,l\to\infty} \frac{|\{(k_t,l_s)\in\mathbb{N}\times\mathbb{N} \mid k_t\le k, l_s\le l, z_{ts}\in f^{-1}(y_{k_t l_s})\}|}{k\times l} = 1$

That is, $d(\{(k,l)\in\mathbb{N}\times\mathbb{N} \mid y_{kl}\in U\}) = 1$.

Therefore, (y_{kl}) statistically converges to x. □

The following Example 3.3 shows that the condition that non-thin on the subsequence cannot be dropped in Theorem 3.1.

Example 3.3. Consider the sequence (x_{mn}) in $\mathbb{R}$ defined by

$$(x_{mn}) = \begin{cases} mn & \text{if } m \text{ and } n \text{ are square} \\ \frac{1}{mn} & \text{otherwise} \end{cases}$$

Here the sequence $(x_{mn}) \xrightarrow{s} 0$. Consider the sequence $(x_{m_k n_l}) = kl$, that is, k and l are square numbers. Then this subsequence is a thin sequence, as $d(K) = 0$ where $K = \{(k^2, l^2) \mid k, l \in \mathbb{N}\}$. Thus, this subsequence of a statistically convergent sequence is not statistically convergent.

Lemma 3.2. *A double sequence (x_{mn}) is statistically convergent if and only if any of its statistically dense subsequence is statistically convergent.*

Proof. Sufficient part is trivial, since every sequence is statistically dense in itself. Necessity, follows from Theorem 3.1. □

Lemma 3.3. *If a double sequence (x_{mn}) s^*-converges to x in a topological space (X,τ), then (x_{mn}) s-converges to x.*

Proof. Let U be a neighborhood of x. Since (x_{mn}) s^*-converges to x, there exists a subset A of $\mathbb{N}\times\mathbb{N}$ with $d(A) = 1$, $n_0 = n_0(U)$ and $m_0 = m_0(U)$ such that $m \ge m_0$, $n \ge n_0$ implies that $x_{mn} \in U$. Then $\{(m,n)\in\mathbb{N}\times\mathbb{N} \mid x_{mn}\notin U\} \subset \{(m_0,n_0)\in\mathbb{N}\times\mathbb{N}\}\cup\{(\mathbb{N}\times\mathbb{N})\setminus A\}$. Since $d(\{(m_0,n_0)\in\mathbb{N}\times\mathbb{N}\}\cup\{(\mathbb{N}\times\mathbb{N})\setminus A\}) = 0$, $d(\{(m,n)\in\mathbb{N}\times\mathbb{N} \mid x_{mn}\notin U\}) = 0$. Therefore, (x_{mn}) s-converges to x. □

However, the converse of Theorem 3.3 is not true as shown in the following Example 3.4.

Example 3.4. Consider $X = M \cup \{0, 1\}$ where M is an uncountable set with cofinite topology. Let $A \subset X$ and let (x_{mn}) be a divergent sequence in A defined by

$$(x_{mn}) = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are infinite numbers} \\ 0 & \text{otherwise} \end{cases}$$

Since the set of natural numbers has natural density 1, $d(A) = 1$ and hence this sequence statistically converges to 0. But (x_{mn}) is not a convergent sequence. Therefore, (x_{mn}) is s-convergent sequence but not s^*-convergent sequence.

But the following Theorem 3.2 shows that the converse of the Lemma 3.3 holds if X is first countable.

Theorem 3.2. *Let (X, τ) be a first countable topological space. If a double sequence (x_{mn}) in X s-converges to x, then (x_{mn}) s^*-converges to x.*

Proof. Let $U_1 \supset U_2 \supset ...$ be a countable local base at x. For every $j \in \mathbb{N}$, set $A_j = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U_j\}$. Then $A_1 \supset A_2 \supset ...$ and $d(A_j) = 1$ for each $j \in \mathbb{N}$. Let $(s_1, t_1) \in A_1$. There is $(s_2, t_2) \in A_2, (s_2, t_2) \geq (s_1, t_1)$ such that for every $(m, n) \geq (s_2, t_2)$,

$$\frac{|A_2(m, n)|}{mn} = \frac{|\{(p, q) \in A_2 \mid p \leq m,\ q \leq n\}|}{mn} > \frac{1}{2},$$

since $d(A_2) = 1$ and so on. We have $(s_j, t_j) \in A_j, (s_1, t_1) \leq (s_2, t_2) \leq ... \leq (s_j, t_j)$ such that for every $(m, n) \geq (s_j, t_j)$,

$$\frac{|A_j(m, n)|}{mn} = \frac{|\{(p, q) \in A_j \mid p \leq m,\ q \leq n\}|}{mn} > 1 - \frac{1}{j}.$$

Define the set $A \subset \mathbb{N} \times \mathbb{N}$ for each $(s, t) \leq (s_1, t_1), (s, t) \in A$, if $j \geq 1$ and $(s_j, t_j) < (s, t) \leq (s_{j+1}, t_{j+1})$, then $(s, t) \in A$ if and only if $(s, t) \in A_j$. Let $A = \{(m_1, n_1) < (m_2, n_2) < ...\}$.

If $(m,n) \in \mathbb{N} \times \mathbb{N}$ and $(s_j, t_j) < (m,n) \leq (s_{j+1}, t_{j+1})$, then

$$\frac{|A(m,n)|}{mn} \geq \frac{|A_j(m,n)|}{mn} > 1 - \frac{1}{j}.$$

and hence $d(A) = 1$.

We prove that $\lim_{\substack{m,n\to\infty \\ (m,n)\in A}} x_{mn} = \lim_{j\to\infty} x_{m_j n_j} = x$.

Let V be a neighborhood of x and $U_j \subset V$. If $(m,n) \in A, (m,n) \geq (s_j, t_j)$, then there exists $i \geq j$ with $(s_i, t_i) \leq (m,n) \leq (s_{j+1}, t_{i+1})$.

Hence by the definition of A, $(m,n) \in A_i$. Therefore, for each $(m,n) \in A, (m,n) \geq (s_j, t_j)$, we have $x_{mn} \in U_i \subset U_j \subset V$. Hence $\lim_{j\to\infty} x_{m_j n_j} = x$ or $(x_{m_j n_j}) \xrightarrow{s^*} x$. □

Lemma 3.4. *Let (x_{mn}) be a s-convergent double sequence in (X, τ) and let s-$\lim\ x_{mn} = x$. Then there is a cross-sequence in (x_{mn}) which s-converges to the point x.*

Proof. If $x_{mn} = x$ for infinitely many indexes, then the proof is trivial. Suppose not, there is a double subsequence of points $s_{pq} \neq x$ and so by Lemma 2.1, there is a one-to-one simple sequence of points of the set $\{x_{mn} \mid (m,n) \in \mathbb{N}\times\mathbb{N}\}$ s-converging to the point x, since $\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U \text{ and } x_{mn} \neq x\} \cup \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} = x\} = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid s_{pq} \in U\}$ and hence $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \notin U\}) = 0$. Consider the functions $f, g : \mathbb{N} \to \mathbb{N}$ and distinct pairs $(f(i), g(i))$ such that $(x_{f(i)g(i)})$ is constant or one-to-one sequence s-converging to x, $d(\{(f(i), g(i)) \in \mathbb{N}\times\mathbb{N} \mid x_{f(i)g(i)} \notin U\}) = 0$. Let $x_{f(r)g(r)} = t_r \in U$. Since $(x_{f(i)g(i)})$ s-converges to x, $d(\{(f(i), g(i)) \in \mathbb{N} \times \mathbb{N} \mid t_r \notin U\}) = 0$. Two cases are possible.

Case (i): There is a non-thin subsequence (r_i) of r such that $f(r_1) < f(r_2) < ...$ and so on. $(x_{f(r_i)g(r_i)})$ is a cross-subsequence in (x_{mn}) and a non-thin subsequence of (t_r). Hence $s\text{-}lim\ x_{f(r_i)g(r_i)} = r$.

Case (ii): There are positive integers p, r_0 such that $f(r) = p,\ r > r_0$. Since the pairs $(f(r), g(r))$ are distinct, there is a non-thin subsequence (s_i) of r such that $g(s_1) <$

$g(s_2) < ...$. Hence we have a sequence $(x_{pg(s_i)}) = (t_{s_i})$. It is a cross-subsequence in (x_{mn}) and a non-thin subsequence of (t_r) as well. Therefore, *s-lim* $x_{pg(s_i)} = x$. □

Theorem 3.3. *In a topological space (X, τ), the following statements hold for a s-convergent double sequence.*

(a) *For every $x \in X$, the double sequence $(x_{mn}) = x$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, s-converges to x.*

(b) *Addition of finite number of terms to a s-convergent double sequence affects neither its s-convergence nor s-limit to which it s-converges.*

(c) *If (x_{mn}) is a s-convergent double sequence in $A \cup B$ which s-converges to x, where A and B are two non-empty disjoint subsets in X, then there exists a double sequence (y_{mn}) either in A or in B consisting of infinitely many terms of (x_{mn}) s-converging to x.*

(d) *If (x_{mn}) is a double sequence s-converging to a point x and (x_p) be formed such that, for each $p \in \mathbb{N}$, x_p equals to some x_{mn} where $m > p$ and $n > p$, then the double sequence (y_{mn}) where for each $m \in \mathbb{N}$, $y_{mn} = (x_m)$ for all $n \in \mathbb{N}$, s-converges to x*

Proof. (a) Any open set containing x, contains all the terms of the double sequence (x_{mn}). So it s-converges to x.

(b) Let (x_{mn}) be a double sequence s-convergent to x and U be any open neighborhood of x. Then $d(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) = 1$. Now let r terms $y_1, y_2, ..., y_r$ be added to the double sequence (x_{mn}) and denote the new double sequence formed by (z_{mn}). Let $y_k = z_{m_k n_k}$, that is, $g(y_k) = z_{m_k n_k}$ for each k, $d(\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid z_{ij} \in U\}) = d(\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid z_{m_j n_j} \in U\}) + d(\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid x_{ij} \in U\}) = 0 + 1 = 1$, since density of finite subset of $\mathbb{N}$ is zero and cardinality of $\mathbb{N} \times \mathbb{N}$ is same as the cardinality of $\mathbb{N}$. Therefore, $d(\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid z_{ij} \in U\}) = 1$. Hence (z_{mn}) s-converges to x.

(c) Let (x_{mn}) be a double sequence in $A \cup B$ s-converging to x where A and B are two non empty disjoint subsets of X. Consider the sequence (x_m) where $x_m = x_{mn}$ for all $m \in \mathbb{N}$. Let U be any open neighborhood of x. Clearly, $d(\{m \in \mathbb{N} \mid x_m \in U\}) = 1$ and $(x_m) \in A \cup B$. By Theorem 3.1, at least one of A or B must contain a non-thin subsequence of (x_m) which s-converges to x. Suppose A contains a non-thin subsequence (x_{m_k}) of (x_m) s-converging to x. Consider a double sequence (y_{kn}) where for each $k \in \mathbb{N}$, $y_{kn} = x_{m_k}$ for all $m, n \in \mathbb{N}$. By Theorem 3.1, every non-thin subsequence of the s-convergent sequence is s-convergent and so $(x_{m_k}) \xrightarrow{s} x$. Hence $d(\{n \in \mathbb{N} \mid y_{kn} \notin U\}) = 0$. Therefore, (y_{kn}) s-converges to x.

(d) Let U be an open set containing x. Since $(x_{mn}) \xrightarrow{s} x$, $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U\}) = 1$. Hence $d(\{i \in \mathbb{N} \mid x_i \in U\}) = 1$, since $x_i = x_{mn}$ for some $m, n \in \mathbb{N}$. Let $y_{mn} = x_m$. Then $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid y_{mn} \notin U\}) = d((\mathbb{N} \times \mathbb{N}) \setminus \{i \in \mathbb{N} \mid x_i \in U\}) = 1 - d(\{i \in \mathbb{N} \mid x_i \in U\}) = 1 - 1 = 0$. Hence the double sequence (y_{mn}) s-converges to x.

□

Theorem 3.4. *Let X be a given set and let ϕ be a class of double sequences over X. Let the members of ϕ be called s-convergent double sequences and let each s-convergent double sequence be associated with an element of X called the s-limit of the s-convergent double sequence subject to the conditions (a) to (d) as stated in Theorem 3.3. Now, let a subset A of X be called open if and only if no s-convergent double sequence lying in $X \setminus A$ has any s-limit in A. Then the collection of open sets thus obtained forms a topology τ on X.*

Proof. X and $\emptyset$ are open sets, since no convergent double sequence can lie outside X and $\emptyset$ contains no element.

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets, where Λ is an arbitrary indexing set. Let $A = \bigcup_{\alpha\in\Lambda} A_\alpha$ and consider a s-convergent double sequence (x_{mn}) in $X \setminus A$, where x is a s-limit point in $X \setminus A$. Then $(x_{mn}) \in X \setminus A_\alpha$ for all $\alpha \in \Lambda$. Hence (x_{mn}) cannot have a s-limit in A_α for all $\alpha \in \Lambda$. Therefore, x cannot have a s-limit in A. Thus, A is open.

Now let A and B be two open sets and suppose (x_{mn}) be a s-convergent double sequence in $X \setminus (A \cap B)$, which has a s-limit x in $A \cap B$. So by the condition (c), at least one of $X \setminus (A \cup B)$ and $(A \cap (X \setminus B)) \cup ((X \setminus A) \cap B)$ must contain a s-convergent double sequence whose range set is a subset of range set of (x_{mn}) and which has a s-limit x. If a double sequence $(y_{mn}) \in X \setminus (A \cup B)$ whose range set is a subset of the range set of (x_{mn}) and which has a s-limit x in $A \cap B \subset A \cup B$, then it will contradict the fact that $A \cup B$ is open. Therefore, $(A \cap (X \setminus B)) \cup ((X \setminus A) \cap B)$ contains a s-convergent double sequence (y_{mn}) whose range set is a subset of range set of (x_{mn}) and which has a s-limit x. Then by the condition (c) at least one of $(A \cap (X \setminus B))$ and $((X \setminus A) \cap B)$ must contain a s-convergent double sequence, say (z_{mn}), whose range set is a subset of the range set of (y_{mn}) and which has a s-limit x. Without any loss of generality, assume that $(A \cap (X \setminus B))$ contains the s-convergent double sequence (z_{mn}). Then (z_{mn}) is a s-convergent double sequence in $X \setminus B$ which has a s-limit x in B, which is a contradiction to B is open. Hence no s-convergent double sequence in $X \setminus (A \cap B)$ can have a s-limit in $A \cap B$. Therefore, $A \cap B$ is open. Thus, τ forms a topology on X. □

We call the topology defined above the s-convergence topology on X and (X, τ) is called the double s-limit space.

Theorem 3.5. *Let X be a double s-limit space with ν as the given collection of s-convergent double sequences and τ be the resulting s-convergence topology on X. Let*

σ be the family of all s-convergent double sequences determined by the topology τ on X. Then $\nu \subset \sigma$.

Proof. Let $x = (x_{mn}) \in \nu$ and a be a s-limit of x. Let G be an open set in the double s-limit space (X, τ) containing a. We prove that $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in G\}) = 1$. Suppose not. Then $d(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in G\}) \neq 1$. Therefore, we can construct a sequence (x_p) in $X \setminus G$ where for each $p \in \mathbb{N}$, x_p equals some x_{kl}, where $k > p$ and $l > p$. Therefore, by condition (d), the sequence (y_{kl}) where for each $k \in \mathbb{N}$, $y_{kl} = x_k$ for all $l \in \mathbb{N}$ s-converges to a. Thus, we obtain a double sequence (y_{kl}) in $X \setminus G$ which has a s-limit in G. This contradicts to the fact that G is open. Hence (x_{mn}) s-converges to a with respect to τ. Therefore, $x \in \sigma$. □

Theorem 3.6. *Let γ be the family of all s-convergent double sequences in a topological space (X, τ) and τ'_s be the s-convergence topology on X determined by the family γ. Then the following hold.*

(a) $\tau \subset \tau'_s$.

(b) *If $A \subset X$ is s-sequentially closed in (X, τ), then A is s-sequentially closed in (X, τ'_s).*

(c) *If (X, τ'_s) is s-sequential, then (X, τ) is also a s-sequential space.*

Proof. (a) Each member of γ satisfies (a) to (d) of Theorem 3.3. Let A be a τ-open set. Suppose A is not τ'_s-open. Then there exists a double sequence (x_{mn}) in $X \setminus A$ which has a s-limit x in A, by Theorem 3.4. Since A is τ-open and $x \in A$, $x_{mn} \in A$ which is a contradiction to $(x_{mn}) \in X \setminus A$. Hence A must be τ'_s-open.

(b) Suppose that A is s-sequentially closed in (X, τ). Let (x_{mn}) be a sequence in A s-converging to x in (X, τ'_s). Let U be a τ-open set containing x. Since $\tau \subset \tau'_s$, $U \in \tau'_s$. As $(x_{mn}) \xrightarrow{s} x$ in (X, τ'_s), we have $d(\{(m,n) \mid x_{mn} \in U\}) = 1$. This implies that $(x_{mn}) \xrightarrow{s} x$ in (X, τ). Since A is s-sequentially closed in (X, τ), $x \in A$. Hence A is s-sequentially closed in (X, τ'_s).

(c) Let A be a s-sequentially closed set in (X, τ). Then A is s-sequentially closed in (X, τ_s'), by (b). Since (X, τ_s') is a s-sequential space, A is closed in (X, τ_s'). Therefore, $A = cl_s(A)$ where $cl_s(A)$ is closure of A in τ_s'. Also, from (a), we have $\tau \subset \tau_s'$ and so $cl_s(A) \subset cl(A)$. Hence $A \subset cl(A)$ so that A is closed in (X, τ). Thus, (X, τ) is a s-sequential space. □

Theorem 3.7. *Let Γ be the family of all convergent double sequence in a topological space (X, τ). Let τ' be the convergence topology on X determined by the family Γ and γ be the family of all s-convergent double sequences in (X, τ). If τ_s' is the s-convergence topology on X determined by γ, then $\tau' \subset \tau_s'$*

Proof. Let A be a τ'-open set. Suppose A is not τ_s'-open. Then there exists a double sequence (x_{mn}) in $X \setminus A$ which has a s-limit x in A, by Theorem 3.4. Since A is τ'-open and $x \in A$, $x_{mn} \in A$ which is a contradiction to $(x_{mn}) \in X \setminus A$. Hence A must be τ_s'-open. □

The following Example 3.5 shows that the inequality in Theorem 3.6(a) and Theorem 3.7 is strict.

Example 3.5. (a) Let $X = \mathbb{R}$ and $\tau = \{\emptyset, X, \{0\}\}$. Let γ be a family of s-convergent double sequences. Consider τ_s' is the s-convergence topology on X determined by the family γ. Let $(x_{mn}) = \{0\} \cup \{\frac{1}{n} + \frac{1}{m} \mid (m, n) \in \mathbb{N} \times \mathbb{N}\}$. Since each neighborhood of the point 0 contains $\{0\} \cup \{\frac{1}{n} + \frac{1}{m} \mid (m, n) \in \mathbb{N} \times \mathbb{N}\}$, we have $\tau_s' \nsubseteq \tau$.

(b) Let Γ be the family of all convergent double sequences in the topological space (X, τ) and τ' be a convergence topology on X determined by the family Γ. Consider a divergent sequence (x_{mn}) in $\mathbb{R}$ defined by

$$(x_{mn}) = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are prime} \\ 0 & \text{otherwise} \end{cases}$$

Since the set of prime natural numbers has natural density 0, this sequence statistically converges to 0. But (x_{mn}) is not a convergent sequence. Therefore, $\tau_s' \nsubseteq \tau'$.

4. Relation between statistical limit points and statistical cluster points of a double sequence

We now present some additional properties related to statistical limit points and statistical cluster points of a double sequence.

Definition 4.1. A point x in a space X is called a *statistical limit point* of a double sequence (x_{mn}) in a topological space (X, τ) if there is a subset $\{(m_i, n_i) \mid m_i < m_j$ and $n_i < n_j, i < j\}$ of $\mathbb{N} \times \mathbb{N}$ whose asymptotic density is not zero (or greater than zero or does not exist, or equivalently, its upper asymptotic density is positive) such that $\lim_{k,l \to \infty} x_{m_k n_l} = x$. $\Lambda(x_{mn})$ denotes the set of all statistical limit point of (x_{mn}).

Definition 4.2. A point x in a space X is called a *statistical cluster point* of a double sequence (x_{mn}) if for each neighborhood U of x, the upper asymptotic density of the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}$ is positive. For a double sequence (x_{mn}) in a topological space (X, τ), let $L(x_{mn})$ denote the set $\{a \in X \mid$ for each neighborhood of U of "a" the set $\{(p, q) \in \mathbb{N} \times \mathbb{N} \mid x_{pq} \in U\}\}$ is infinite. $\Gamma(x_{mn})$ denotes the set of all statistical cluster points of (x_{mn}).

Lemma 4.1. *Let (X, τ) be a topological space and (x_{mn}) be a double sequence in X. Then the following hold.*

(a) $\Lambda(x_{mn}) \subset \Gamma(x_{mn})$.
(b) $\Gamma(x_{mn}) \subset L(x_{mn})$.

Proof. (a) Let $x \in \Lambda(x_{mn})$ and let $(x_{m_k n_l})$ be a subsequence of (x_{mn}) such that $\lim_{k,l \to \infty} x_{m_k n_l} = x$ and $\bar{d}(\{(m_k, n_l) \mid (k, l) \in \mathbb{N} \times \mathbb{N}\}) = \alpha > 0$. Let U be a neighborhood of x. Then for all but finitely many k's and l's, say $(k_1, k_2, ..k_{i_0})$ and $(l_1, l_2, ...l_{i_0})$, $x_{m_k n_l} \in U$.

We obtain $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\} \supset \{(m_k, n_l) \mid (k, l) \in \mathbb{N} \times \mathbb{N}\} \setminus$

$\{(m_{k_i}, n_{l_i}) \mid 1 \le i \le i_0\}$ and so $\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}$ is infinite. Thus,

$$\begin{aligned}
\bar{d}(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U\}) &= \lim_{p,q\to\infty} \sup \frac{|\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U, m \le p, n \le q\}|}{pq} \\
&\ge \lim_{p,q\to\infty} \sup \frac{|\{(m_k, n_l) \mid (k,l) \in \mathbb{N}\times\mathbb{N}\}|}{pq} - \frac{1}{pq}O(1) \\
&= \alpha - \frac{1}{pq}O(1) \\
&> \frac{\alpha}{2} \\
&> 0.
\end{aligned}$$

Here 'O' is standard Landau notation. Therefore, $x \in L(x_{mn})$.

(b) Let x be a point in $\Gamma(x_{mn})$ and let U be a neighborhood of x. Then $\bar{d}(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) > 0$. Also, this set consists of infinite number of elements. Therefore, $x \in Ł(x_{mn})$.

□

The following Example 4.1 and Example 4.2 below show that the converse inclusion in Lemma 4.1 does not hold in general.

Example 4.1. Consider $\mathbb{R}$ with usual topology. Let (x_{mn}) be a sequence in $\mathbb{R}$ defined by

$$(x_{mn}) = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are square} \\ 0 & \text{if otherwise} \end{cases}$$

Since the set of square numbers has density 0, $\Lambda(x_{mn}) = \{0\}$ and $L(x_{mn}) = \{0,1\}$. Therefore, $L(x_{mn}) \nsubseteq \Lambda(x_{mn})$.

Example 4.2. Consider $\mathbb{R}$ with usual topology. Let (x_{mn}) be a sequence in $\mathbb{R}$ defined by $x_{mn} = \frac{1}{p}$ where $m = 2^{p-1}(2q+1)$ and $n = 2^{p-1}(2q-1)$, here $p-1$ is the number of factors of 2 in the prime factorization of m and also n. Then for each p, $\bar{d}(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} = \frac{1}{p}\}) = 2^{-p} > 0$ and so $\frac{1}{p} \in \Gamma(x_{mn})$. Also, $\bar{d}(\{(m,n) \in$

$\mathbb{N} \times \mathbb{N} \mid 0 < x_{mn} < \frac{1}{p}\}) = 2^{-p}$. Thus, $0 \in Ł(x_{mn})$ and $L(x_{mn}) = \{0\} \cup \{\frac{1}{p}\}$. But $0 \notin \Gamma(x_{mn})$. Hence $Ł(x_{mn}) \nsubseteq \Gamma(x_{mn})$.

Theorem 4.1. *Let (X, τ) be a topological space and (x_{mn}) be a double sequence in X. If a point $x \in X$ is a statistical limit point, then x is a s-limit point.*

Proof. Let A be a subset of X and let $A = \{x_{mn} \mid (m, n) \in \mathbb{N} \times \mathbb{N}\}$. Suppose that A is an infinite set. Let U be a neighborhood of x. Then $d(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) = 1 > 0$, since x is a statistical limit point. Therefore, x is a s-limit point of A. Suppose that A is finite, then there exists $x \in X$ such that $x_{mn} = x$ for infinitely many $m, n \in \mathbb{N} \times \mathbb{N}$. Then for every open set U containing x, $d(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) = 1$. Hence x is a s-limit point of A. □

From Example 4.1, we observe that the set of all statistical cluster points need not be a closed point set. But the following Theorem 4.2 shows that the set of all statistical cluster points is a closed set.

Theorem 4.2. *For any topological space (X, τ) and any double sequence (x_{mn}) in X, the set $\Gamma(x_{mn})$ is closed.*

Proof. Let $x \in \overline{\Gamma(x_{mn})}$ and U be any neighborhood of x. Then there is a point $y \in U \cap \Gamma(x_{mn})$. Let V be a neighborhood of y such that $V \subset U$. Since $y \in \Gamma(x_{mn})$, $\bar{d}(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in V\})$ is positive. Now $\bar{d}(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) \geq \bar{d}(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in V\})$ and hence $\bar{d}(\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) > 0$. Therefore, $x \in \Gamma(x_{mn})$. □

Theorem 4.3. *Let (X, τ) be a first countable topological space. For any double sequence (x_{mn}) in X, the set $\Lambda(x_{mn})$ is an F_σ-set.*

Proof. For any $t \in \mathbb{N}$, let

$F_t = \{x \in X \mid$ there exists $((m_1, n_1) < (m_2, n_2) < ... < (m_k, n_k) < ...)$ and

$\lim x_{m_k n_l} = x\}$ and $\bar{d}(\{(m_k, n_l) \mid (k,l) \in \mathbb{N}\times\mathbb{N}\})\} \geq \frac{1}{t}$.

To prove that each F_t is a closed subset of X. Let $y \in cl(F_t)$ and U be a neighborhood of y. First we prove that F_t converges to y. There is a countable neighborhood basis $\{V_n \mid n \in \mathbb{N}\}$ at y, since X is first countable. Let $U_1 = V_1$, $U_2 = V_1 \cap V_2, \ldots,$ $U_n = V_1 \cap V_2 \cap, \ldots, \cap V_n$. Then $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ is a neighborhood basis at y. If $F_t \in U$ for all t, then U_n's form a neighborhood basis. It is clear that F_t is eventually in any neighborhood U of y, which means F_t converges to y. For every y_{pq}, choose a non-thin double subsequence of (x_{mn}) converging to y_{pq} whose set of indices M_{pq} has upper asymptotic density greater than or equal to $\frac{1}{t}$. Suppose we take a double sequence (z_{pq}) of positive real numbers converging to 0, then there is a double sequence $(i_1, j_1) < (i_2, j_2) < ...$ in $\mathbb{N}\times\mathbb{N}$ such that

$$\frac{|M_{pq} \cap A|}{i_p j_q} \geq \frac{1}{t} - z_{pq}, \quad (p,q) \in \mathbb{N}\times\mathbb{N}$$

where $A = \{(c,d) \mid (i_{p-1}, j_{q-1}) < (c,d) \leq (i_p, j_q)\}$. Put $M = \bigcup\{M_{pq} \cap A \mid (p,q) \in \mathbb{N}\times\mathbb{N}\}$. Then $\bar{d}(M) \geq \frac{1}{t}$.

Let $M = \{(r_1, s_1) < (r_2, s_2) < ... < (r_k, s_l)\}$. Since only finitely many y_{pq}, say $y_{11}, y_{12}, y_{21}, ..., y_{p_0 q_0}$ are not in U, $\lim\limits_{k,l\to\infty} x_{r_k s_l} = x$ as $M \subset \cup\{M_{pq} \mid p \geq p_0, q \geq q_0\} \subset \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U\} \setminus F$ where F is a finite set. Thus, $y \in F_t$. Therefore, F_t is closed. It follows that $\Lambda(x_{mn}) = \cup\{F_t \mid t \in \mathbb{N}\}$. Hence $\Lambda(x_{mn})$ is an F_σ-set in X. □

Theorem 4.4. *If (x_{mn}) and (y_{mn}) are double sequences in a topological space (X,τ) such that $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \neq y_{mn}\}) = 0$, then $\Gamma(x_{mn}) = \Gamma(y_{mn})$ and $\Lambda(x_{mn}) = \Lambda(y_{mn})$.*

Proof. Let $x \in \Gamma(x_{mn})$ and U be a neighborhood of x. Then $\bar{d}(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U\} > 0$. We have $\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U\} \setminus \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \neq y_{mn}\} \subseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid y_{mn} \in U\}$. Since $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \neq y_{mn}\}) = 0$, $\bar{d}(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid y_{mn} \in U\}) > 0$. Therefore, $x \in \Gamma(y_{mn})$ and hence $\Gamma(x_{mn}) \subset \Gamma(y_{mn})$.

By symmetry, $\Gamma(y_{mn}) \subset \Gamma(x_{mn})$. Therefore, $\Gamma(x_{mn}) = \Gamma(y_{mn})$. Let $x \in \Lambda(x_{mn})$. Then there is a subset $\{(m_k, n_l) \mid k, l \in \mathbb{N}\}$ such that $\lim_{k,l\to\infty} x_{m_k n_l} = x$. Therefore, $A = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid x_{m_k n_l} \notin U\}$ is finite and so $d(A) = 0$. Hence $d(\{(k,l) \in \mathbb{N}\times\mathbb{N} \mid x_{m_k n_l} \in U\}) = 1$. But $\{(k,l) \in \mathbb{N}\times\mathbb{N} \mid x_{m_k n_l} \in U\} \setminus \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid x_{m_k n_l} \neq y_{m_k n_l}\} \subseteq \{(k,l) \mid y_{m_k n_l} \in U\}$ and hence $\bar{d}(\{(k,l) \mid y_{m_k n_l} \in U\}) = 1 > 0$. Hence $x \in \Lambda(y_{mn})$. Therefore, $\Lambda(x_{mn}) \subseteq \Lambda(y_{mn})$. By similar argument, we can prove that $\Lambda(y_{mn}) \subset \Lambda(x_{mn})$. Hence $\Lambda(x_{mn}) = \Lambda(y_{mn})$. □

Theorem 4.5. *For any double sequence (x_{mn}) in a hereditarily Lindelöf space X, there is a double sequence (y_{mn}) such that $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \neq y_{mn}\}) = 0$ and $L(y_{mn}) = \Gamma(x_{mn})$.*

Proof. It is clear that if $\Gamma(x_{mn}) = L(x_{mn})$, then $L(y_{mn}) = L(x_{mn}) = \Gamma(x_{mn})$. Suppose that $\Gamma(x_{mn}) \subsetneq L(x_{mn})$. For each $x \in L(x_{mn}) \setminus \Gamma(x_{mn})$, let U_x be a neighborhood of x such that $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U\}) = 0$. From the open cover $\{U_x \mid x \in L(x_{mn}) \setminus \Gamma(x_{mn})\}$ of the set $L(x_{mn}) \setminus \Gamma(x_{mn})$, choose a countable sub cover $\{U_{x_i} \mid i \in \mathbb{N}\}$. Since $x_i \in L(x_{mn}) \setminus \Gamma(x_{mn})$ for each i, each set U_{x_i} contains a thin subsequence $(x_{m_{ik} n_{il}})$ of (x_{mn}) with $d(\{(m_{ik}, n_{il}) \mid (k,l) \in \mathbb{N}\times\mathbb{N}\}) = 0$. Put $A_i = \{(m_{ik}, n_{il}) \mid k, l \in \mathbb{N}\}$. Then A_i can be arranged in a set $A \subset \mathbb{N}\times\mathbb{N}$ such that $d(A) = 0$ and for each $i \in \mathbb{N}, \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U_{x_i}\} \setminus A$ is a finite set and so $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in U_{x_i}\} \setminus A) = 0$. Let $(\mathbb{N}\times\mathbb{N}) \setminus A = \{(k_m, l_n) \mid m, n \in \mathbb{N}\}$. Define the double sequence (y_{mn}) in the following sense

$$y_{mn} = \begin{cases} x_{k_m l_n} \text{ if } (m,n) \in A \\ x_{mn} \text{ if } (m,n) \in (\mathbb{N}\times\mathbb{N}) \setminus A \end{cases}$$

Then $d(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \neq y_{mn}\}) = 0$ and so $\Gamma(x_{mn}) = \Gamma(y_{mn})$, by Theorem 4.4. Next, we prove that $L(y_{mn}) = \Gamma(y_{mn})$. The subsequence $(x_{k_m l_n})$ of (y_{mn}) has no accumulation point in $L(x_{mn}) \setminus \Gamma(x_{mn})$ and also, has no statistical limit point of (y_{mn}). Therefore, $L(y_{mn}) = \Gamma(x_{mn})$. □

Theorem 4.6. *Let K be a compact subset of a topological space (X, τ). Then for every double sequence (x_{mn}) in X such that $\bar{d}(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in K\}) > 0, K \cap \Gamma(x_{mn}) \neq \emptyset$*

Proof. Suppose that $K \cap \Gamma(x_{mn}) = \emptyset$. Then for every $x \in K$, there is a neighborhood U_x of x such that the set $\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U_x\}$ has upper asymptotic density zero. From the open cover $\{U_x \mid x \in K\}$ of K, take a finite sub cover $\{U_{x_1}, ..., U_{x_r}\}$ of K. Since $\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in K\} \subset \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid x_{mn} \in \bigcup_{i=1}^{r} U_{x_i}\}, \bar{d}(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in K\}) = 0$, which is a contradiction. □

In [7], Engelking mentioned that if the density of the space is less than or equal to ω, then the space is separable. Also, in [6], $hcld(X) = \omega$ denotes the set of all closed subsets of the space X are separable. The following Theorem 4.7 gives the properties closed set and F_σ-set in terms of $L(x_{mn})$, $\Gamma(x_{mn})$ and $\Lambda(x_{mn})$ in a separable space.

Theorem 4.7. *Let (X, τ) be a topological space. Assume that every closed subsets of X are separable. Then the following hold.*

(a) *For each closed subset F of X, there is a double sequence (x_{mn}) in X such that $F = L(x_{mn})$.*

(b) *For each F_σ-set A in X, there exists a double sequence (x_{mn}) in X such that $A = \Lambda(x_{mn})$.*

(c) *For each closed set A in X, there exists a double sequence (x_{mn}) in X such that $A = \Gamma(x_{mn})$.*

Proof. (a) Let $A = \{a_i \mid i \in \mathbb{N}\}$ be a countable dense subset of F. Decompose $\mathbb{N} \times \mathbb{N}$ into pairwise disjoint infinite sets $\mathbb{N} \times \mathbb{N} = \bigcup_{i\in\mathbb{N}} (M_i \times N_i)$ and define a double sequence (x_{mn}) by $x_{mn} = a_i$ for each $(m,n) \in M_i \times N_i$. Clearly, $L(x_{mn}) \subset F$, since F is closed. Now, we prove that $F \subset L(x_{mn})$. Let $x \in F$ and U be a neighborhood of x. Choose a point $a_k \in U$. Since $x_{mn} = a_k$ for each $(m,n) \in M_k \times N_k$, there are infinitely many indexes (m,n) in $\mathbb{N} \times \mathbb{N}$ satisfying $x_{mn} \in U$. That is, $x \in L(x_{mn})$.

(b) Let $A = \bigcup\limits_{i=1}^{\infty} A_i$ where each A_i is a closed set in X. By (a), for every i, choose a double sequence $(x_{i,mn}) \subset A_i$ such that $L(x_{i,mn}) = A_i$. Decompose $\mathbb{N}\times\mathbb{N}$ into pairwise disjoint sets as $\mathbb{N}\times\mathbb{N} = \bigcup\limits_{i\in\mathbb{N}}(M_i \times N_i)$ where $M_i \times N_i = \{(2^{i-1}(2m-1), 2^{i-1}(2n+1)) \mid m,n \in \mathbb{N}\times\mathbb{N}\}$. Here $i-1$ is the number of factors of 2 in the prime factorization of $M_i \times N_i$. Therefore, for each i, $d(M_i \times N_i) = 2^{-i}$ and $d((\mathbb{N}\times\mathbb{N}) \setminus \bigcup\limits_{m=1,n=1}^{i}(M_m \times N_n))$ tends to 0 as i tends to ∞. Moreover, for each $i \in \mathbb{N}$, decompose $M_i \times N_i = \bigcup\limits_{(j,k)\in\mathbb{N}\times\mathbb{N}} S_{i,(j,k)}$ where $\bar{d}(S_{i,(j,k)}) = 2^{-i}$ for each $(j,k) \in \mathbb{N}\times\mathbb{N}$. Define a double sequence (x_{mn}) such that $x_{mn} = x_{i,mn}$ for each $(m,n) \in S_{i,(j,k)}$. To prove that $\Lambda(x_{mn}) = A$. First we prove that $\Lambda(x_{mn}) \subset A$. Suppose that there is a double subsequence $(x_{m_k n_l})$ of (x_{mn}) converging to $x \notin A$. Then for every i, the set $\bigcup\limits_{m=1,n=1}^{i}(M_m \times N_n)$ contains only finitely many (m_p, n_q). Hence $d(\{(m_1,n_2),(m_2,n_2),...,(m_p,n_q)\}) = 0$. Therefore, $x \notin \Lambda(x_{mn})$.

Next we prove that $A \subset \Lambda(x_{mn})$. If $x \in A$, then $x \in A_i$ for some i. If a double subsequence $(x_{i,m_k n_l})$ of $(x_{i,mn})$ converges to x, then for any $\epsilon > 0$ with $\epsilon < 2^{-i}$, there is a sequence (r_k, s_l) in $\mathbb{N}\times\mathbb{N}$ such that $\frac{|(S_{i,(m_k,n_l)})\cap B|}{r_k s_l} \geq 2^{-i} - \epsilon$ where $B = \{(c,d) \mid (r_{k-1}, s_{l-1}) < (c,d) \leq (r_k, s_l)\}$. Put $S = \bigcup\limits_{i}\{(S_{i,(m_k,n_l)}) \cap B \mid k,l \in \mathbb{N}\}$. Then $d(S) \geq 2^{-i} - \epsilon$ and $\lim\limits_{\substack{m,n\to\infty \\ (m,n)\in S}} x_{mn} = x$. Therefore, $x \in \Lambda(x_{mn})$. Hence $A = \Lambda(x_{mn})$.

(c) Let $B = \{b_i \mid i \in \mathbb{N}\}$ be a countable dense subset of A. As in (b), decompose $\mathbb{N}\times\mathbb{N}$ into pairwise disjoint sets $M_i \times N_i = \{(2^{i-1}(2m-1), 2^{i-1}(2n+1)) \mid m,n \in \mathbb{N}\times\mathbb{N}\}$, $i \in \mathbb{N}$, such that $d(M_i \times N_i = 2^{-i})$ for each $i \in \mathbb{N}$. Define a double sequence (x_{mn}) by $x_{mn} = b_i$ for each $(m,n) \in M_i \times N_i$.

Clearly, $\Gamma(x_{mn}) \subset A$, since A is closed. Let $x \in A$ and U be a neighborhood of x. Take a point $b_k \in U$. Since $x_{mn} = b_k$ for each $(m,n) \in M_k \times N_k$, we obtain $\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\} \supset M_k \times N_k$. Hence $\bar{d}(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x_{mn} \in U\}) > 0$. This implies that $x \in \Gamma(x_{mn})$.

□

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References

[1] A. K. Banerjee, R. Mondal, *A note on convergence of double sequences in a topological space*, Mat. Vesnik, **69 (2)** (2017), 144-152.

[2] V. Balaz, V. Cervenansky, J. Kostyrko, P. Salat, *$\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions*, In: Acta Mathematica **5**, Faculty of Natural Sciences, Constantine the Philosopher University, Nitra, 2004, 43-50.

[3] H. Cakalh, *A study on statistical convergence*, Funct. Anal. Approx. Comput., **1 (2)**(2009), 19-24.

[4] P. Das, P. Malik, *On the statistical version of double sequences,* Real Anal. Exchange, **33 (2)** (2008), 351-364.

[5] P. Das, P. Kostyrko, W. Wilczynski, P. Malik, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Slovaca, **58** (2008), 605-620. https://doi.org/10.2478/s12175-008-0096-x

[6] G. Di. Maio, D. R. Kocinac, *Statistical convergence in topology*, Topology Appl., **156** (2008), 28-45.

[7] R. Engelking, *General topology (revised and completed edition)*, Heldermann Verlag, Berlin, 1989.

[8] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241-244.

[9] J. A. Fridy, *On statistical convergence*, Analysis (Munich), **5** (1985), 301-313.

[10] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc., **118** (1993), 1187-1192.

[11] P. Kostyrko, T. Salat and W. Wilczynski, *I -Convergence*, Real Anal. Exchange, **26 (2)** (2000), 669-686.

[12] S.A. Mohiuddine, A. Alotaibi, M. Mursaleen, *Statistical convergence of double sequences in locally solid Riesz spaces*, Abstract Appl. Anal., **2012**, Article ID 719729, 9 pages.

[13] F. Ḿoricz, *Statistical convergence of multiple sequences*, Arch. Math. (Basel), **81** (2003), 82-89.

[14] M. Mursaleen, O.H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288** (2003), 223-231.

[15] M. Mursaleen, S.A. Mohiuddine, *On ideal convergence in probabilistic normed spaces*, Math. Slovaca, **62** (2012), 49-62.

[16] M. Mursaleen, V. Karakaya, M. Ertrk, F. Grsoy, *Weighted statistical convergence and its application to Korovkin type approximation theorem*, Appl. Math. Comput., **218** (2012), 9132-9137.

[17] A. Nabiev, S. Pehlivan, M. Gurdal, *$\mathcal{I}$-Cauchy Sequences*, Taiwanese J. Math., **11 (2)** (2007), 569-576.

[18] J. Novák, Praha, *Double convergence and products of Fréchet spaces*, Czechoslovak Math. J., **48 (123)** (1998), 207-227.

[19] S. K. Pal, *$\mathcal{I}$-Sequential topological spaces*, Appl. Math. E-Notes, **14** (2014), 236-241.

[20] A. Pringsheim, *Zur Theorie der zweifach unendlichen Zahlenfolgen,* Math. Ann., **53** (1900), 289-321.

[21] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361-375.

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