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COMPUTING CERTAIN TOPOLOGICAL INDICES OF INDU-BALA PRODUCT OF GRAPHS

SHREEKANT PATIL [(1)] AND B. BASAVANAGOUD [(2)]

Abstract. The Indu-Bala product $G_1 \blacktriangledown G_2$ of graphs G_1 and G_2 is obtained from two disjoint copies of the join $G_1 \vee G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2. In this paper we obtain the explicit formulae for certain degree and distance based topological indices viz. first Zagreb index, second Zagreb index, third Zagreb index, F-index, hyper-Zagreb index, harmonic index, first multiplicative Zagreb index, second multiplicative Zagreb index, modified first multiplicative Zagreb index, Wiener index, Harary index, sum-degree distance index, product-degree distance index, reciprocal sum-degree distance index and reciprocal product-degree distance index of Indu-Bala product of graphs. Also, we present the exact value of the distance based topological indices of graph G in terms of its order, size and Zagreb indices, when $diam(G) \leq 2$.

1. Introduction

Let G be a graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$, $|E(G)| = m$. As usual, n is order and m is size of G. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. The distance between two vertices u and v in G, denoted by $d_G(u, v)$ is the length of the shortest path

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between the vertices u and v in G. The shortest $u-v$ path is often called geodesic. The diameter $diam(G)$ of a connected graph G is the length of any longest geodesic. Any unexplained graph theoretical terminology and notation may be found in [2].

In structural chemistry and biology, molecular structure descriptors are utilized for modeling information of molecules, which are known as topological indices. Many topological indices are introduced to explain the physical and chemical properties of molecules (See [18]). We represent the selected degree-based topological indices in the following form [14].

$$TI(G) = \sum_{uv \in E(G)} F(d_G(u), d_G(v))$$

where the summation goes over all pairs of adjacent vertices u, v of the molecular graph G, and where $F = F(x, y)$ is an appropriately chosen function. In particular,

$$F(x,y) = x+y$$
$$F(x,y) = xy$$

for the first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$, respectively [17].

$$F(x,y) = x^2+y^2$$
$$F(x,y) = |x-y|$$

for the forgotten topological index $F(G)$ (or F-index) [13, 17] and the third Zagreb index $M_3(G)$ [11], respectively.

$$F(x,y) = \frac{2}{x+y}$$
$$F(x,y) = (x+y)^2$$

for the harmonic index $H(G)$ [12] and hyper-Zagreb index $HM(G)$ [29], respectively. Also, the logarithms of the three multiplicative Zagreb indices can be represented as follows:

$$F(x,y) = 2\left(\frac{\ln x}{x} + \frac{\ln y}{y}\right)$$
$$F(x,y) = \ln x + \ln y$$
$$F(x,y) = \ln(x+y)$$

for first multiplicative Zagreb index $\prod_1(G)$, second multiplicative Zagreb index $\prod_2(G)$ and modified first multiplicative Zagreb index $\prod_1^*(G)$, respectively [10, 15, 31, 32]. The oldest molecular index is the one put forward in 1947 by H. Wiener [34], nowadays referred to as the Wiener index and denoted by W. It is defined as the sum of distance between all pairs of vertices of a graph. Symbolically,

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

In 1993, Plavšić et al. [28] and Ivanciuc et al. [23] independently introduced the Harary index, named in honor of Frank Harary on the occasion of his 70^{th} birthday. Actually, the Harary index was first defined in 1992 by Mihalić and Trinajstić [26] as:

$$H^*(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$

The invariant sum-degree distance denoted by $DD_+(G)$ was first time introduced by Dobrynin and Kochetova [8], later the same quantity was examined under the name Schultz index [16] and defined as

$$DD_+(G) = \sum_{u,v \in V(G)} [d_G(u) + d_G(v)]d_G(u,v).$$

In [16], Gutman introduced another invariant named product-degree distance index and is defined as

$$DD_*(G) = \sum_{u,v \in V(G)} d_G(u)d_G(v)d_G(u,v).$$

Hua and Zhang [21] introduced a new graph invariant named reciprocal degree distance, which can be seen as a degree-weight version of Harary index, that is

$$RDD_{+}(G) = \sum_{u,v \in V(G)} \frac{d_G(u) + d_G(v)}{d_G(u,v)}.$$

In [30], Su et al. introduced the reciprocal product-degree distance of graphs, which can be seen as a product-degree-weight version of Harary index, that is

$$RDD_{*}(G) = \sum_{u,v \in V(G)} \frac{d_G(u) d_G(v)}{d_G(u,v)}.$$

The join $G_1 \vee G_2$ of graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Recently, Indulal and Balakrishnan [22] introduced a new graph operation named it as Indu-bala product of graphs and is defined as follows:

The Indu-Bala product $G_1 \blacktriangledown G_2$ of graphs G_1 and G_2 is obtained from two disjoint copies of the join $G_1 \vee G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2. Hence the number of vertices and edges in $G_1 \blacktriangledown G_2$ is given by, respectively, $|V(G_1 \blacktriangledown G_2)| = 2(n_1 + n_2)$ and $|E(G_1 \blacktriangledown G_2)| = 2(m_1 + m_2 + n_1 n_2) + n_2$. If u is a vertex of $G_1 \blacktriangledown G_2$ then

$$d_{G_1 \blacktriangledown G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & if\ u \in V(G_1) \\ d_{G_2}(u) + n_1 + 1 & if\ u \in V(G_2). \end{cases}$$

The Fig. 1 depicts an example of $G_1 \blacktriangledown G_2$. Motivated by the work on Indu-Bala prod-

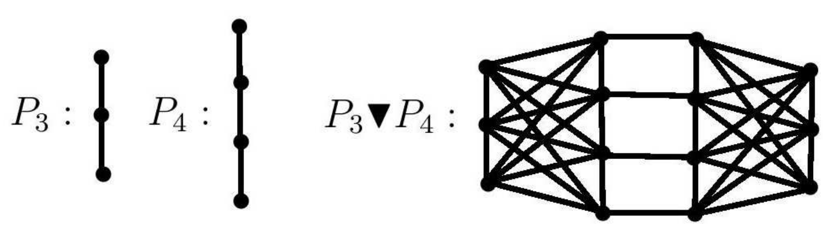

FIGURE 1. Graphs P_3, P_4 and their $P_3 \blacktriangledown P_4$

uct graphs [22, 27], we study the degree and distance based topological indices of this graph. The paper is organized as follows: In section 2, we compute the degree based topological indices viz. first Zagreb index, second Zagreb index, third Zagreb index, F-index, hyper-Zagreb index, harmonic index, first multiplicative Zagreb index, second multiplicative Zagreb index, modified first multiplicative Zagreb index and in section 3, we compute the distance based topological indices viz. Wiener index, Harary index, sum-degree distance index, product-degree distance index, reciprocal sum-degree distance index and reciprocal product-degree distance index of Indu-Bala product of graphs. Also, we present the exact value of the distance based topological indices of graph G in terms of its order, size and Zagreb indices, when $diam(G) \leq 2$. Readers interested in more information on computing topological indices of graph operations can be referred to [1, 3, 4, 5, 6, 7, 18, 20, 24, 25].

The following lemmas are useful for proving our results.

Lemma 1.1. *(AM-GM inequality) Let $x_1, x_2, ..., x_n$ be nonnegative numbers. Then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

holds with equality if and only if all the x_k's are equal.

Lemma 1.2. [19, 33] *Let G be a graph of order n and size m. Then $W(G) = n^2 - n - m$ if and only if $diam(G) \leq 2$.*

2. Degree Based Topological Indices

Operation considered is binary, hence we deal with two finite and simple graphs, G_1 and G_2. For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, and their cardinalities by n_i and m_i, respectively, where $i = 1, 2$.

Theorem 2.1. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned}\sum_{u\in V(G_1\blacktriangledown G_2)} d^{\alpha+1}_{G_1\blacktriangledown G_2}(u) &= 2\Bigg[\sum_{u\in V(G_1)} d^{\alpha+1}_{G_1}(u) \\ &+ \sum_{v\in V(G_2)} d^{\alpha+1}_{G_2}(v) + \binom{\alpha+1}{1}\Bigg[\sum_{u\in V(G_1)} d^{\alpha}_{G_1}(u)\cdot n_2 \\ &+ \sum_{v\in V(G_2)} d^{\alpha}_{G_2}(v)\cdot(n_1+1)\Bigg] + \binom{\alpha+1}{2}\Bigg[\sum_{u\in V(G_1)} d^{\alpha-1}_{G_1}(u)\cdot n_2^2 \\ &+ \sum_{v\in V(G_2)} d^{\alpha-1}_{G_2}(v)\cdot(n_1+1)^2\Bigg] + \cdots + n_1 n_2^{\alpha+1} + n_2(n_1+1)^{\alpha+1}\Bigg].\end{aligned}$$

Proof. Since $G_1\blacktriangledown G_2$ has $2(n_1+n_2)$ vertices, then we have

$$\sum_{u\in V(G_1\blacktriangledown G_2)} d^{\alpha+1}_{G_1\blacktriangledown G_2}(u) = 2\Bigg[\sum_{u\in V(G_1)} (d_{G_1}(u)+n_2)^{\alpha+1} + \sum_{v\in V(G_2)} (d_{G_2}(v)+(n_1+1))^{\alpha+1}\Bigg].$$

Using binomial theorem, expanding each term in right hand side of above equation, we get the following.

$$\begin{aligned}\sum_{u\in V(G_1\blacktriangledown G_2)} d^{\alpha+1}_{G_1\blacktriangledown G_2}(u) &= 2\Bigg[\sum_{u\in V(G_1)} d^{\alpha+1}_{G_1}(u) + \binom{\alpha+1}{1}\sum_{u\in V(G_1)} d^{\alpha}_{G_1}(u)\cdot n_2 \\ &+ \binom{\alpha+1}{2}\sum_{u\in V(G_1)} d^{\alpha-1}_{G_1}(u)\cdot n_2^2 + \cdots + n_1 n_2^{\alpha+1} \\ &+ \sum_{v\in V(G_2)} d^{\alpha+1}_{G_2}(v) + \binom{\alpha+1}{1}\sum_{v\in V(G_2)} d^{\alpha}_{G_2}(v)\cdot(n_1+1) \\ &+ \binom{\alpha+1}{2}\sum_{v\in V(G_2)} d^{\alpha-1}_{G_2}(v)\cdot(n_1+1)^2 + \cdots + n_2(n_1+1)^{\alpha+1}\Bigg].\end{aligned}$$

□

As a application of the above theorem, taking $\alpha = 1$ and $\alpha = 2$ leads to the expressions for the first Zagreb index and forgotten topological index of $G_1\blacktriangledown G_2$, which are given in following corollaries.

Corollary 2.1. *Let G_1 and G_2 be two graphs. Then* $M_1(G_1 \blacktriangledown G_2) = 2[M_1(G_1) + n_1 n_2^2 + 4n_2 m_1 + M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1)]$.

Corollary 2.2. *Let G_1 and G_2 be two graphs. Then* $F(G_1 \blacktriangledown G_2) = 2[F(G_1) + n_1 n_2^3 + 3n_2 M_1(G_1) + 6n_2^2 m_1 + F(G_2) + n_2(n_1+1)^3 + 3(n_1+1)M_1(G_2) + 6m_2(n_1+1)^2]$.

Theorem 2.2. *Let G_1 and G_2 be two graphs. Then* $M_2(G_1 \blacktriangledown G_2) = 2[M_2(G_1) + n_2 M_1(G_1) + m_1 n_2^2 + M_2(G_2) + (n_1+1)M_1(G_2) + m_2(n_1+1)^2 + 4m_1 m_2 + 2m_1 n_2(n_1 + 1) + 2m_2 n_1 n_2 + n_2^2 n_1(n_1+1)] + M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1)$.

Proof. By definition of second Zagreb index, we have

$$\begin{aligned}
& M_2(G_1 \blacktriangledown G_2) \\
= & \sum_{uv \in E(G_1 \blacktriangledown G_2)} d_{G_1 \blacktriangledown G_2}(u) d_{G_1 \blacktriangledown G_2}(v) \\
= & 2\Big[\sum_{uv \in E(G_1)} (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2) \\
& + \sum_{uv \in E(G_2)} (d_{G_2}(u) + n_1 + 1)(d_{G_2}(v) + n_1 + 1) \\
& + \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_1}(u) + n_2)(d_{G_2}(v) + n_1 + 1)\Big] \\
& + \sum_{v \in V(G_2)} (d_{G_2}(v) + n_1 + 1)^2 \\
= & 2\Big[\sum_{uv \in E(G_1)} (d_{G_1}(u) d_{G_1}(v) + n_2(d_{G_1}(u) + d_{G_1}(v)) + n_2^2) \\
& + \sum_{uv \in E(G_2)} (d_{G_2}(u) d_{G_2}(v) + (n_1+1)(d_{G_2}(u) + d_{G_2}(v)) + (n_1+1)^2) \\
& + \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_1}(u) d_{G_2}(v) + (n_1+1) d_{G_1}(u) + n_2 d_{G_2}(v) + n_2(n_1+1))\Big] \\
& + \sum_{v \in V(G_2)} (d_{G_2}^2(v) + (n_1+1)^2 + 2d_{G_2}(v)(n_1+1)).
\end{aligned}$$

□

Theorem 2.3. *Let G_1 and G_2 be two graphs. Then*

$$M_3(G_1 \blacktriangledown G_2) \leq 2[M_3(G_1) + M_3(G_2) + 2m_1n_2 + n_1n_2^2 + 2m_2n_1 + n_1n_2(n_1+1)].$$

Proof. Using the definition of third Zagreb index, we have

$$\begin{aligned} M_3(G_1 \blacktriangledown G_2) &= \sum_{uv \in E(G_1 \blacktriangledown G_2)} \mid d_{G_1 \blacktriangledown G_2}(u) - d_{G_1 \blacktriangledown G_2}(v) \mid \\ &= 2\Bigg[\sum_{uv \in E(G_1)} \mid d_{G_1}(u) + n_2 - d_{G_1}(v) - n_2 \mid \\ &\quad + \sum_{uv \in E(G_2)} \mid d_{G_2}(u) + n_1 + 1 - d_{G_2}(v) - n_1 - 1 \mid \\ &\quad + \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \mid d_{G_1}(u) + n_2 - d_{G_2}(v) - n_1 - 1 \mid \Bigg] \\ &\quad + \sum_{v \in V(G_2)} \mid d_{G_2}(v) + n_1 + 1 - d_{G_2}(v) - n_1 - 1 \mid \\ &\leq 2[M_3(G_1) + M_3(G_2) + 2m_1n_2 + n_1n_2^2 + 2m_2n_1 + n_1n_2(n_1+1)]. \end{aligned}$$

□

Theorem 2.4. *Let G_1 and G_2 be two graphs. Then* $HM(G_1 \blacktriangledown G_2) = 2[HM(G_1) + 4n_2^2m_1 + 4n_2M_1(G_1) + HM(G_2) + 4(n_1+1)^2m_2 + 4(n_1+1)M_1(G_2) + n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_1n_2(n_1+n_2+1) + 2(n_1+n_2+1)(2m_1n_2+2m_2n_1)] + 4[M_1(G_2) + (n_1+1)^2n_2 + 4m_2(n_1+1)]$.

Proof. By definition of hyper-Zagreb index, we have $HM(G_1 \blacktriangledown G_2) = F(G_1 \blacktriangledown G_2) + 2M_2(G_1 \blacktriangledown G_2)$. Hence the result follows from Corollary 3.3 and Theorem 3.4. □

Theorem 2.5. *Let G_1 and G_2 be two graphs. Then*

$$H(G_1 \blacktriangledown G_2) \geq 4\left[\frac{m_1}{m_1 + 2n_2 + 1} + \frac{m_2}{m_2 + 2n_1 + 3} + \frac{n_1n_2}{2n_1 + 2n_2 - 1}\right] + \frac{n_2}{n_1 + n_2}.$$

Proof. By definition of harmonic index, we have

$$\begin{aligned}
H(G_1 \blacktriangledown G_2) &= \sum_{uv \in E(G_1 \blacktriangledown G_2)} \frac{2}{d_{G_1 \blacktriangledown G_2}(u) + d_{G_1 \blacktriangledown G_2}(v)} \\
&= 2\Big[\sum_{uv \in E(G_1)} \frac{2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} \\
&\quad + \sum_{uv \in E(G_2)} \frac{2}{d_{G_2}(u) + n_1 + 1 + d_{G_2}(v) + n_1 + 1} \\
&\quad + \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1 + 1} \Big] \\
&\quad + \sum_{v \in V(G_2)} \frac{2}{2(d_{G_2}(v) + n_1 + 1)}.
\end{aligned}$$

For a graph G, we have $d_G(u) + d_G(v) \leq |E(G)| + 1$ for $uv \in E(G)$ and $d_G(u) \leq |V(G)| - 1$ for $u \in V(G)$.

$$\begin{aligned}
H(G_1 \blacktriangledown G_2) &\geq 2\Big[\sum_{uv \in E(G_1)} \frac{2}{m_1 + 2n_2 + 1} + \sum_{uv \in E(G_2)} \frac{2}{m_2 + 2(n_1 + 1) + 1} \\
&\quad + \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{n_1 - 1 + n_2 + n_2 - 1 + n_1 + 1} \Big] \\
&\quad + \sum_{v \in V(G_2)} \frac{1}{n_2 - 1 + n_1 + 1} \\
&= 2\left[\frac{2m_1}{m_1 + 2n_2 + 1} + \frac{2m_2}{m_2 + 2n_1 + 3} + \frac{2n_1 n_2}{2n_1 + 2n_2 - 1} \right] + \frac{n_2}{n_1 + n_2}.
\end{aligned}$$

□

Theorem 2.6. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned}
\prod_1 (G_1 \blacktriangledown G_2) &\leq \left[\frac{M_1(G_1) + n_1 n_2^2 + 4n_2 m_1}{n_1} \right]^{2n_1} \\
&\quad \times \left[\frac{M_1(G_2) + n_2(n_1 + 1)^2 + 4(n_1 + 1)m_2}{n_2} \right]^{2n_2}.
\end{aligned} \tag{2.1}$$

The equality holds in (2.1) if and only if G_1 and G_2 are regular graphs.

Proof. Using the definition of first multiplicative Zagreb index, we have

$$\begin{aligned}
\prod_1(G_1\blacktriangledown G_2) &= \prod_{u\in V(G_1\blacktriangledown G_2)} d^2_{G_1\blacktriangledown G_2}(u)\\
&= \left[\prod_{u\in V(G_1)}(d_{G_1}(u)+n_2)^2 \prod_{u\in V(G_2)}(d_{G_2}(u)+n_1+1)^2\right]^2\\
&= \left[\prod_{u\in V(G_1)}(d^2_{G_1}(u)+n_2^2+2n_2d_{G_1}(u))\right.\\
&\quad \left.\times\prod_{u\in V(G_2)}(d^2_{G_2}(u)+(n_1+1)^2+2(n_1+1)d_{G_2}(u))\right]^2.
\end{aligned}$$

By Lemma 1.1, we have

$$\begin{aligned}
&\leq \left[\frac{\sum_{u\in V(G_1)}(d^2_{G_1}(u)+n_2^2+2n_2d_{G_1}(u))}{n_1}\right]^{2n_1}\\
&\quad\times\left[\frac{\sum_{u\in V(G_2)}(d^2_{G_2}(u)+(n_1+1)^2+2(n_1+1)d_{G_2}(u))}{n_2}\right]^{2n_2}\\
&= \left[\frac{M_1(G_1)+n_1n_2^2+4n_2m_1}{n_1}\right]^{2n_1}\left[\frac{M_1(G_2)+n_2(n_1+1)^2+4(n_1+1)m_2}{n_2}\right]^{2n_2}.
\end{aligned}$$

Moreover, the above equality holds if and only if $d^2_{G_1}(u)+n_2^2+2n_2d_{G_1}(u) = d^2_{G_1}(v)+n_2^2+2n_2d_{G_1}(v)$ for $u,v\in V(G_1)$ and $d^2_{G_2}(r)+(n_1+1)^2+2(n_1+1)d_{G_2}(r) = d^2_{G_2}(s)+(n_1+1)^2+2(n_1+1)d_{G_2}(s)$ for $r,s\in V(G_2)$ by Lemma 1.1. Hence the equality holds in (2.1) if and only if both G_1 and G_2 are regular graphs. □

Theorem 2.7. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned}
(2.2)\quad \prod_2(G_1\blacktriangledown G_2) &\leq \left[\frac{M_2(G_1)+n_2M_1(G_1)+m_1n_2^2}{m_1}\right]^{2m_1}\\
&\quad\times\left[\frac{M_2(G_2)+(n_1+1)M_1(G_2)+m_2(n_1+1)^2}{m_2}\right]^{2m_2}\\
&\quad\times\left[\frac{4m_1m_2+2n_1n_2m_2+2m_1n_2(n_1+1)+n_1n_2^2(n_1+1)}{n_1n_2}\right]^{2n_1n_2}\\
&\quad\times\left[\frac{M_1(G_2)+4m_2(n_1+1)+n_2(n_1+1)^2}{n_2}\right]^{n_2}.
\end{aligned}$$

The equality holds in (2.2) if and only if G_1 and G_2 are regular graphs.

Proof. By definition of second multiplicative Zagreb index, we have

$$\begin{aligned}
&\prod_2(G_1\blacktriangledown G_2)\\
=&\prod_{uv\in E(G_1\blacktriangledown G_2)} d_{G_1\blacktriangledown G_2}(u)d_{G_1\blacktriangledown G_2}(v)\\
=&\Bigg[\prod_{uv\in E(G_1)}(d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2)\prod_{uv\in E(G_2)}(d_{G_2}(u)+n_1+1)(d_{G_2}(v)+n_1+1)\\
&\times\prod_{u\in V(G_1)}\prod_{v\in V(G_2)}(d_{G_1}(u)+n_2)(d_{G_2}(v)+n_1+1)\Bigg]^2\prod_{v\in V(G_2)}(d_{G_2}(v)+n_1+1)^2\\
=&\Bigg[\prod_{uv\in E(G_1)}(d_{G_1}(u)d_{G_1}(v)+n_2(d_{G_1}(u)+d_{G_1}(v))+n_2^2)\\
&\times\prod_{uv\in E(G_2)}(d_{G_2}(u)d_{G_2}(v)+(n_1+1)(d_{G_2}(u)+d_{G_2}(v))+(n_1+1)^2)\\
&\times\prod_{u\in V(G_1)}\prod_{v\in V(G_2)}(d_{G_1}(u)d_{G_2}(v)+(n_1+1)d_{G_1}(u)+n_2d_{G_2}(v)+n_2(n_1+1))\Bigg]^2\\
&\times\prod_{v\in V(G_2)}(d^2_{G_2}(v)+(n_1+1)^2+2d_{G_2}(v)(n_1+1)).
\end{aligned}$$

By Lemma 1.1, we have

$$\begin{aligned}
\leq\;&\left[\frac{\sum_{uv\in E(G_1)}(d_{G_1}(u)d_{G_1}(v)+n_2(d_{G_1}(u)+d_{G_1}(v))+n_2^2)}{m_1}\right]^{2m_1}\\
&\times\left[\frac{\sum_{uv\in E(G_2)}(d_{G_2}(u)d_{G_2}(v)+(n_1+1)(d_{G_2}(u)+d_{G_2}(v))+(n_1+1)^2)}{m_2}\right]^{2m_2}\\
&\times\left[\frac{\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}(d_{G_1}(u)d_{G_2}(v)+(n_1+1)d_{G_1}(u)+n_2d_{G_2}(v)+n_2(n_1+1))}{n_1n_2}\right]^{2n_1n_2}\\
&\times\left[\frac{\sum_{v\in V(G_2)}(d^2_{G_2}(v)+(n_1+1)^2+2d_{G_2}(v)(n_1+1))}{n_2}\right]^{n_2}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{M_2(G_1)+n_2M_1(G_1)+m_1n_2^2}{m_1}\right]^{2m_1}\\
&\quad\times\left[\frac{M_2(G_2)+(n_1+1)M_1(G_2)+m_2(n_1+1)^2}{m_2}\right]^{2m_1}\\
&\quad\times\left[\frac{4m_1m_2+2n_1n_2m_2+2m_1n_2(n_1+1)+n_1n_2^2(n_1+1)}{n_1n_2}\right]^{2n_1n_2}\\
&\quad\times\left[\frac{M_1(G_2)+4m_2(n_1+1)+n_2(n_1+1)^2}{n_2}\right]^{n_2}.
\end{aligned}
$$

Furthermore, the above equality holds if and only if $d_{G_1}(u)d_{G_1}(v)+n_2(d_{G_1}(u)+d_{G_1}(v))+n_2^2 = d_{G_1}(r)d_{G_1}(s)+n_2(d_{G_1}(r)+d_{G_1}(s))+n_2^2$ for any $uv, rs \in E(G_1)$, $d_{G_2}(u)d_{G_2}(v)+(n_1+1)(d_{G_2}(u)+d_{G_2}(v))+(n_1+1)^2 = d_{G_2}(r)d_{G_2}(s)+(n_1+1)(d_{G_2}(r)+d_{G_2}(s))+(n_1+1)^2$ for any $uv, rs \in E(G_2)$, $d_{G_1}(u)d_{G_2}(v)+(n_1+1)d_{G_1}(u)+n_2d_{G_2}(v)+n_2(n_1+1) = d_{G_1}(r)d_{G_2}(s)+(n_1+1)d_{G_1}(r)+n_2d_{G_2}(s)+n_2(n_1+1)$ for any $u, r \in V(G_1), v, s \in V(G_2)$ and $d_{G_2}^2(u)+(n_1+1)^2+2d_{G_2}(u)(n_1+1) = d_{G_2}^2(v)+(n_1+1)^2+2d_{G_2}(v)(n_1+1)$ for $u, v \in V(G_2)$ by Lemma 1.1. This implies that the equality holds in (2.2) if and only if G_1 and G_2 must be regular graphs. □

Theorem 2.8. *Let G_1 and G_2 be two graphs. Then*

$$
\begin{aligned}
\text{(2.3)}\quad \prod_1^{*}(G_1\blacktriangledown G_2) &\le \left[\frac{M_1(G_1)+2m_1n_2}{m_1}\right]^{2m_1}\times\left[\frac{M_1(G_2)+2m_2(n_1+1)}{m_2}\right]^{2m_2}\\
&\quad\times\left[\frac{2m_1n_2+2n_1m_2+n_1n_2(n_1+n_2+1)}{n_1n_2}\right]^{2n_1n_2}\\
&\quad\times\left[\frac{2(2m_2+n_2(n_1+1))}{n_2}\right]^{n_2}.
\end{aligned}
$$

The equality holds in (2.3) if and only if G_1 and G_2 are regular graphs.

Proof. Using the definition of modified first multiplicative Zagreb index, we have

$$
\prod_1^{*}(G_1\blacktriangledown G_2) = \prod_{uv\in E(G_1\blacktriangledown G_2)} (d_{G_1\blacktriangledown G_2}(u)+d_{G_1\blacktriangledown G_2}(v))
$$

$$= \Bigg[\prod_{uv\in E(G_1)} (d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2) \prod_{uv\in E(G_2)} (d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1 + 2)$$

$$\times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} (d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1 + 1)\Bigg]^2 \prod_{v\in V(G_2)} 2(d_{G_2}(v) + n_1 + 1).$$

By Lemma 1.1, we have

$$\leq \left[\frac{\sum\limits_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) + 2n_2)}{m_1}\right]^{2m_1}$$

$$\times\left[\frac{\sum\limits_{uv\in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) + 2(n_1 + 1))}{m_2}\right]^{2m_2}$$

$$\times\left[\frac{\sum\limits_{u\in V(G_1)} \sum\limits_{v\in V(G_2)} (d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1 + 1)}{n_1n_2}\right]^{2n_1n_2}$$

$$\times\left[\frac{\sum\limits_{v\in V(G_2)} 2(d_{G_2}(v) + n_1 + 1)}{n_2}\right]^{n_2}$$

$$= \left[\frac{M_1(G_1) + 2m_1n_2}{m_1}\right]^{2m_1} \times \left[\frac{M_1(G_2) + 2m_2(n_1 + 1)}{m_2}\right]^{2m_2}$$

$$\times\left[\frac{2m_1n_2 + 2n_1m_2 + n_1n_2(n_1 + n_2 + 1)}{n_1n_2}\right]^{2n_1n_2} \times \left[\frac{2(2m_2 + n_2(n_1 + 1))}{n_2}\right]^{n_2}.$$

Moreover, the above equality holds if and only if $d_{G_1}(u) + d_{G_1}(v) + 2n_2 = d_{G_1}(r) + d_{G_1}(s) + 2n_2$ for any $uv, rs \in E(G_1)$, $d_{G_2}(u) + d_{G_2}(v) + 2(n_1 + 1) = d_{G_2}(r) + d_{G_2}(s) + 2(n_1+1)$ for any $uv, rs \in E(G_2)$, $d_{G_1}(u)+n_2+d_{G_2}(v)+n_1+1 = d_{G_1}(r)+n_2+d_{G_2}(s)+n_1 + 1$ for any $u, r \in V(G_1), v, s \in V(G_2)$ and $d_{G_2}(u) + n_1 + 1 = d_{G_2}(v) + n_1 + 1$ for any $u, v \in V(G_2)$ by Lemma 1.1. Hence the equality holds in (2.3) if and only if both G_1 and G_2 are regular graphs. □

3. Distance Based Topological Indices

In this section, we need two auxiliary coindices conceived by Došlić [9] namely first and second Zagreb coindices and which are defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \text{ and } \overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u)d_G(v)$$

respectively.

Theorem 3.1. *Let G_1 and G_2 be two graphs. Then* $W(G_1 \blacktriangledown G_2) = 2(n_1+n_2)(2(n_1+n_2)-1) - 2(m_1+m_2+n_1n_2) - n_2 + 3(n_1^2 + n_2(n_2-1) - 2m_2)$.

Proof. Since $diam(G_1 \blacktriangledown G_2) = 3$ and by Lemma 1.2, we have

$$\begin{aligned} W(G_1 \blacktriangledown G_2) &= 2(n_1+n_2)(2(n_1+n_2)-1) - 2(m_1+m_2+n_1n_2) - n_2 \\ &\quad +3[\text{number of unordered pairs of vertices of distance 3 in } G_1 \blacktriangledown G_2] \\ &= 2(n_1+n_2)(2(n_1+n_2)-1) - 2(m_1+m_2+n_1n_2) - n_2 \\ &\quad +3[n_1^2 + 2(\frac{n_2(n_2-1)}{2} - m_2)]. \end{aligned}$$

□

Lemma 3.1. *Let G be a graph of order n and size m with $diam(G) \leq 2$. Then*

$$H^*(G) = \frac{m}{2} + \frac{n(n-1)}{4}.$$

Proof. Suppose $diam(G) \leq 2$. Then adjacent vertices of G are distance one and non adjacent vertices of G are distance two. Therefore,

$$\begin{aligned} H^*(G) &= \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)} \\ &= \sum_{uv \in E(G)} 1 + \sum_{uv \notin E(G)} \frac{1}{2} \\ &= m + \frac{1}{2}\left(\frac{n(n-1)}{2} - m\right) \\ &= \frac{m}{2} + \frac{n(n-1)}{4}. \end{aligned}$$

□

Theorem 3.2. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned} H^*(G_1 \blacktriangledown G_2) &= \frac{(m_1+m_2+n_1n_2)+n_2}{2} + \frac{2(n_1+n_2)(2(n_1+n_2)-1)}{2} \\ &\quad + \frac{1}{3}[n_1^2 + n_2(n_2-1) - 2m_2]. \end{aligned}$$

Proof. Since $diam(G_1 \blacktriangledown G_2) = 3$ and by Lemma 3.1, we have

$$\begin{aligned} H^*(G_1 \blacktriangledown G_2) &= \frac{(m_1+m_2+n_1n_2)+n_2}{2} + \frac{2(n_1+n_2)(2(n_1+n_2)-1)}{2} \\ &\quad + \frac{1}{3}[\text{number of unordered pairs of vertices of distance 3 in } G_1 \blacktriangledown G_2] \\ &= \frac{(m_1+m_2+n_1n_2)+n_2}{2} + \frac{2(n_1+n_2)(2(n_1+n_2)-1)}{2} \\ &\quad + \frac{1}{3}[n_1^2 + \sum_{v \in V(G_2)} (n_2 - 1 - d_{G_2}(u))]. \end{aligned}$$

□

Lemma 3.2. *Let G be a graph of order n and size m with $diam(G) \leq 2$. Then*

$$DD_+(G) = 4m(n-1) - M_1(G).$$

Proof. By definition of sum-degree distance index, we have

$$\begin{aligned} DD_+(G) &= \sum_{u,v \in V(G)} [d_G(u) + d_G(v)] d_G(u,v) \\ &= \sum_{uv \in E(G)} (d_G(u) + d_G(v)) + 2 \sum_{uv \notin E(G)} (d_G(u) + d_G(v)) \text{ as } diam(G) \leq 2. \\ &= M_1(G) + 2\overline{M}_1(G). \end{aligned}$$

We know that from [1], $\overline{M}_1(G) = 2m(n-1) - M_1(G)$. Therefore we get the desired result. □

Theorem 3.3. *Let G_1 and G_2 be two graphs. Then* $DD_+(G_1 \blacktriangledown G_2) = 4(2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1) - 2[M_1(G_1) + n_1n_2^2 + 4n_2m_1 + M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1)] + 3[4m_1n_1 + 2n_1^2n_2 + 2[\overline{M}_1(G_2) + 2(n_1+1)(\frac{n_2(n_2-1)}{2} - m_2)]]$.

Proof. Since $diam(G_1 \blacktriangledown G_2) = 3$ and by Lemma 3.2, we have

$$\begin{aligned} DD_+(G_1 \blacktriangledown G_2) &= 4(2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1) - M_1(G_1 \blacktriangledown G_2) \\ &\quad +3 \sum_{\substack{v,u \in V(G_1 \blacktriangledown G_2) \\ d_{G_1 \blacktriangledown G_2}(u,v)=3}} (d_{G_1 \blacktriangledown G_2}(u) + d_{G_1 \blacktriangledown G_2}(v)) \\ &= 4(2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1) - M_1(G_1 \blacktriangledown G_2) \\ &\quad +3\Big[\sum_{u \in V(G_1)} \sum_{u \in V(G_1)} (d_{G_1}(u) + d_{G_1}(v) + 2n_2) \\ &\quad +2 \sum_{uv \notin E(G_2)} (d_{G_2}(u) + d_{G_2}(v) + 2(n_1+1)) \Big]. \end{aligned}$$

Using Corollary 2.1 we get the required result. □

Lemma 3.3. *Let G be a graph of order n and size m with $diam(G) \leq 2$. Then*

$$DD_*(G) = 4m^2 - M_1(G) - M_2(G).$$

Proof. Using the definition of product-degree distance index, we have

$$\begin{aligned} DD_*(G) &= \sum_{u,v \in V(G)} d_G(u)d_G(v)d_G(u,v) \\ &= \sum_{uv \in E(G)} d_G(u)d_G(v) + 2 \sum_{uv \notin E(G)} d_G(u)d_G(v) \ as \ diam(G) \leq 2. \\ &= M_2(G) + 2\overline{M}_2(G). \end{aligned}$$

From [1], we have $\overline{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)$. Therefore we get the desired result. □

Theorem 3.4. *Let G_1 and G_2 be two graphs. Then* $DD_*(G_1 \blacktriangledown G_2) = 4(2(m_1+m_2+n_1n_2)+n_2^2)^2 - 2[M_1(G_1) + n_1n_2^2 + 4n_2m_1 + M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1)] - 2[M_2(G_1) + n_2M_1(G_1) + m_1n_2^2 + M_2(G_2) + (n_1+1)M_1(G_2) + m_2(n_1+1)^2 + 4m_1m_2 + 2m_1n_2(n_1+1) + 2m_2n_1n_2 + n_2^2n_1(n_1+1)] - M_1(G_2) - n_2(n_1+1)^2 - 4m_2(n_1+1) + 3[4m_1^2 + 4n_2m_1 + n_1^2n_2^2 + 2(\overline{M}_2(G_2) + (n_2+1)\overline{M}_1(G_2) + (n_1+1)^2(\frac{n_2(n_2-1)}{2} - m_2))]$.

Proof. Since $diam(G_1 \blacktriangledown G_2) = 3$ and by Lemma 3.3, we have

$$\begin{aligned}
DD_*(G_1 \blacktriangledown G_2) &= 4(2(m_1 + m_2 + n_1 n_2) + n_2^2)^2 - M_1(G_1 \blacktriangledown G_2) - M_2(G_1 \blacktriangledown G_2) \\
&\quad +3 \sum_{\substack{v,u \in V(G_1 \blacktriangledown G_2) \\ d_{G_1 \blacktriangledown G_2}(u,v)=3}} (d_{G_1 \blacktriangledown G_2}(u) d_{G_1 \blacktriangledown G_2}(v)) \\
&= 4(2(m_1 + m_2 + n_1 n_2) + n_2^2)^2 - M_1(G_1 \blacktriangledown G_2) - M_2(G_1 \blacktriangledown G_2) \\
&\quad +3 \Big[\sum_{u \in V(G_1)} \sum_{u \in V(G_1)} (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2) \\
&\quad +2 \sum_{uv \notin E(G_2)} (d_{G_2}(u) + n_1 + 1)(d_{G_2}(v) + n_1 + 1) \Big] \\
&= 4(2(m_1 + m_2 + n_1 n_2) + n_2^2)^2 - M_1(G_1 \blacktriangledown G_2) - M_2(G_1 \blacktriangledown G_2) \\
&\quad +3[4m_1^2 + 4n_2 m_1 + n_1^2 n_2^2 + 2(\overline{M}_2(G_2) + (n_2+1)\overline{M}_1(G_2) \\
&\quad +(n_1+1)^2(\frac{n_2(n_2-1)}{2} - m_2))].
\end{aligned}$$

By substituting results of the Corollary 2.1 and Theorem 2.2 in above, we get the required result. □

Lemma 3.4. *Let G be a graph of order n and size m with $diam(G) \leq 2$. Then*

$$RDD_+(G) = m(n-1) + \tfrac{1}{2} M_1(G).$$

Proof. By definition of $RDD_+(G)$ and since $diam(G) \leq 2$, we have

$$\begin{aligned}
RDD_+(G) &= \sum_{uv \in E(G)} (d_G(u) + d_G(v)) + \frac{1}{2} \sum_{uv \notin E(G)} (d_G(u) + d_G(v)) \\
&= M_1(G) + \frac{1}{2} \overline{M}_1(G).
\end{aligned}$$

We know that from [1], $\overline{M}_1(G) = 2m(n-1) - M_1(G)$. Therefore we get the desired result. □

Theorem 3.5. *Let G_1 and G_2 be two graphs. Then* $RDD_+(G_1 \blacktriangledown G_2) = (2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1)+M_1(G_1)+n_1n_2^2+4n_2m_1+M_1(G_2)+n_2(n_1+1)^2+4m_2(n_1+1)+\frac{1}{3}[4n_1m_1+2n_1^2n_2+2[\overline{M}_1(G_2)+2(n_1+1)(\frac{n_2(n_2-1)}{2}-m_2)]]$.

Proof. Since $diam(G_1 \blacktriangledown G_2) = 3$ and by Lemma 3.4, we have

$$\begin{aligned}
RDD_+(G_1 \blacktriangledown G_2) &= (2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1)+\frac{1}{2}M_1(G_1 \blacktriangledown G_2) \\
&\quad +\frac{1}{3}\sum_{\substack{v,u\in V(G_1 \blacktriangledown G_2)\\ d_{G_1 \blacktriangledown G_2}(u,v)=3}} (d_{G_1 \blacktriangledown G_2}(u)+d_{G_1 \blacktriangledown G_2}(v)) \\
&= (2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1)+\frac{1}{2}M_1(G_1 \blacktriangledown G_2) \\
&\quad +\frac{1}{3}\Big[\sum_{u\in V(G_1)}\sum_{u\in V(G_1)} (d_{G_1}(u)+d_{G_1}(v)+2n_2) \\
&\quad +2\sum_{uv\notin E(G_2)} (d_{G_2}(u)+d_{G_2}(v)+2(n_1+1))\Big] \\
&= (2(m_1+m_2+n_1n_2)+n_2)(2(n_1+n_2)-1)+\frac{1}{2}M_1(G_1 \blacktriangledown G_2) \\
&\quad +\frac{1}{3}[4n_1m_1+2n_1^2n_2+2[\overline{M}_1(G_2)+2(n_1+1)(\frac{n_2(n_2-1)}{2}-m_2)]].
\end{aligned}$$

Using Corollary 2.1 in above, we get the required result. □

Lemma 3.5. *Let G be a graph of order n and size m with $diam(G) \leq 2$. Then*

$$RDD_*(G) = m^2 + \tfrac{1}{2}M_2(G) - \tfrac{1}{4}M_1(G).$$

Proof. Using the definition of $RDD_*(G)$ and since $diam(G) \leq 2$, we have

$$\begin{aligned}
RDD_*(G) &= \sum_{uv\in E(G)} d_G(u)d_G(v) + \frac{1}{2}\sum_{uv\notin E(G)} d_G(u)d_G(v) \\
&= M_2(G) + \frac{1}{2}\overline{M}_2(G).
\end{aligned}$$

From [1], we have $\overline{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)$. Therefore we get the desired result. □

Theorem 3.6. *Let* G_1 *and* G_2 *be two graphs. Then* $RDD_*(G_1\blacktriangledown G_2) = (2(m_1+m_2+n_1n_2)+n_2)^2 + M_2(G_1) + n_2M_1(G_1) + m_1n_2^2 + M_2(G_2) + (n_1+1)M_1(G_2) + m_2(n_1+1)^2 + 4m_1m_2 + 2m_1n_2(n_1+1) + 2m_2n_1n_2 + n_2^2n_1(n_1+1) + \frac{1}{2}[M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1)] - \frac{1}{2}[M_1(G_1) + n_1n_2^2 + 4n_2m_1 + M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1)] + \frac{1}{3}[4m_1^2 + 4n_2m_1 + n_1^2n_2^2 + 2(\overline{M}_2(G_2) + (n_2+1)\overline{M}_1(G_2) + (n_1+1)^2(\frac{n_2(n_2-1)}{2} - m_2))]$.

Proof. Since $diam(G_1\blacktriangledown G_2) = 3$ and by Lemma 3.5, we have

$$\begin{aligned}
RDD_*(G_1\blacktriangledown G_2) &= (2(m_1+m_2+n_1n_2)+n_2) + \frac{1}{2}M_2(G_1\blacktriangledown G_2) - \frac{1}{4}M_1(G_1\blacktriangledown G_2) \\
&\quad + \frac{1}{3}\sum_{\substack{v,u\in V(G_1\blacktriangledown G_2)\\ d_{G_1\blacktriangledown G_2}(u,v)=3}} (d_{G_1\blacktriangledown G_2}(u)d_{G_1\blacktriangledown G_2}(v)) \\
&= (2(m_1+m_2+n_1n_2)+n_2) + \frac{1}{2}M_2(G_1\blacktriangledown G_2) - \frac{1}{4}M_1(G_1\blacktriangledown G_2) \\
&\quad + \frac{1}{3}\Bigg[\sum_{u\in V(G_1)}\sum_{u\in V(G_1)} (d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2) \\
&\quad + 2\sum_{uv\notin E(G_2)} (d_{G_2}(u)+n_1+1)(d_{G_2}(v)+n_1+1)\Bigg] \\
&= (2(m_1+m_2+n_1n_2)+n_2)^2 + \frac{1}{2}M_2(G_1\blacktriangledown G_2) - \frac{1}{4}M_1(G_1\blacktriangledown G_2) \\
&\quad + \frac{1}{3}[4m_1^2 + 4n_2m_1 + n_1^2n_2^2 + 2(\overline{M}_2(G_2) + (n_2+1)\overline{M}_1(G_2) \\
&\quad + (n_1+1)^2(\frac{n_2(n_2-1)}{2} - m_2))].
\end{aligned}$$

By substituting results of the Corollary 2.1 and Theorem 2.2 in above, we get the required result. □

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(1) Department of Mathematics, BLDEA's S. B. Arts and K. C. P. Science College, Vijayapur - 586 103, Karnataka, India.

Email address: shreekantpatil949@gmail.com

(2) Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India.

Email address: b.basavanagoud@gmail.com