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PATTERNS OF TIME SCALE DYNAMIC INEQUALITIES SETTLED BY KANTOROVICH'S RATIO

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Abstract. In this research article, we present an interesting generalization of dynamic Kantorovich's inequality and investigate the additive versions of some dynamic inequalities on time scales. The time scale dynamic inequalities extend and unify some continuous inequalities and their corresponding discrete versions.

1. Introduction

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.1}$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, +\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

Further, we note that

$$\frac{(\kappa_1+1)^2}{4\kappa_1} \leq \frac{(\kappa_2+1)^2}{4\kappa_2}, \tag{1.2}$$

where $1 \leq \kappa_1 \leq \kappa_2$.

The following multiplicative refinement and reverse of Young's inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right) a^{1-v}b^v \leq (1-v)a + vb \leq K^R\left(\frac{a}{b}\right) a^{1-v}b^v, \tag{1.3}$$

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where $a, b > 0$, $v \in [0,1]$, $r = \min\{1-v, v\}$ and $R = \max\{1-v, v\}$.

The first inequality in (1.3) was obtained by Zou et al. in [19] while the second by Liao et al. [10].

An interesting generalization of Kantorovich's inequality was given by Hao [12, p. 122], which follows:

Let $x_k \in (0,+\infty)$, $y_k \in (0,+\infty)$ and $w_k \in [0,+\infty)$ for $k = 1,2,\ldots,n$. Suppose that $0 < m_1 \le x_k \le M_1$ and $0 < m_2 \le y_k \le M_2$, $k = 1,2,\ldots,n$ for some constants m_1, m_2, M_1 and M_2. Further, let $0 < \frac{1}{q} \le \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left(\sum_{k=1}^{n} w_k x_k\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \frac{w_k}{x_k}\right)^{\frac{1}{q}} \le \frac{\frac{1}{p}M_1 + \frac{1}{q}m_1}{(M_1 m_1)^{\frac{1}{q}}} \left(\sum_{k=1}^{n} w_k\right), \tag{1.4}$$

and

$$\left(\sum_{k=1}^{n} w_k x_k^2\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} w_k y_k^2\right)^{\frac{1}{q}} \le \frac{\frac{1}{p}(M_1 M_2) + \frac{1}{q}(m_1 m_2)}{(M_1 m_1)^{\frac{1}{q}} (M_2 m_2)^{\frac{1}{p}}} \left(\sum_{k=1}^{n} w_k x_k y_k\right). \tag{1.5}$$

Now, we consider the following two additive versions of Cassels' inequality as given in [6].

Let $x_k \in (0,+\infty)$, $y_k \in (0,+\infty)$ and $w_k \in [0,+\infty)$ for $k = 1,2,\ldots,n$. Suppose that $m = \min\limits_{1\le k\le n}\left\{\frac{x_k}{y_k}\right\}$ and $M = \max\limits_{1\le k\le n}\left\{\frac{x_k}{y_k}\right\}$. Then

$$0 \le \left(\sum_{k=1}^{n} w_k x_k^2 \sum_{k=1}^{n} w_k y_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^{n} w_k x_k y_k \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \sum_{k=1}^{n} w_k x_k y_k, \tag{1.6}$$

and

$$0 \le \sum_{k=1}^{n} w_k x_k^2 \sum_{k=1}^{n} w_k y_k^2 - \left(\sum_{k=1}^{n} w_k x_k y_k\right)^2 \le \frac{(M-m)^2}{4Mm} \left(\sum_{k=1}^{n} w_k x_k y_k\right)^2. \tag{1.7}$$

We will prove these results from (1.4) to (1.7) on time scales. The calculus of time scales was initiated by Hilger as given in [8]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. In time scales calculus, results are unified and extended. The theory of time scales is applied to unify discrete and continuous analysis and to combine them in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus and diamond–α calculus. This hybrid theory is also widely applied on dynamic inequalities. Basic dynamic inequalities on time scales are given in [1]. Basic work on dynamic inequalities is done by Agarwal, Anastassiou, Bohner, Peterson, O'Regan, Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [4, 5].

For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right–scattered*, while if $\rho(t) < t$, we say that t is *left–scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right–dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left–dense*. If $\mathbb{T}$ has a left–scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be *delta differentiable* at t, and $f^{\Delta}(t)$ is called the *delta derivative* of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous* (*rd–continuous*), if it is continuous at each right–dense point and there exists a finite left–sided limit at every left–dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4, 5].

Definition 2.1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [3, 4, 5].

If $\mathbb{T}$ has a right–scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left–dense continuous* (*ld–continuous*), provided it is continuous at all left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [3, 4, 5].

Definition 2.2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present short introduction of diamond–α derivative as given in [1, 18].

Definition 2.3. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, the diamond–$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t),\ 0 \leq \alpha \leq 1.$$

Thus f is diamond–α differentiable if and only if f is Δ and ∇ differentiable.

The diamond–α derivative reduces to the standard Δ–derivative for $\alpha = 1$, or the standard ∇–derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 2.1 ([18])**.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond–α differentiable at $t \in \mathbb{T}$ and we write $f^{\sigma}(t) = f(\sigma(t))$, $g^{\sigma}(t) = g(\sigma(t))$, $f^{\rho}(t) = f(\rho(t))$ and $g^{\rho}(t) = g(\rho(t))$. Then*

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

(iii) For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1-\alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

Definition 2.4 ([18])**.** Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond–$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha) \int_a^t h(s)\nabla s, \ 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.2 ([18])**.** *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$–integrable functions on $[a, b]_\mathbb{T}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;
(ii) $\int_a^t cf(s) \diamond_\alpha s = c \int_a^t f(s) \diamond_\alpha s$;
(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;
(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;
(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

Lemma 2.1 ([2])**.** *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. Assume that $f(x)$ and $g(x)$ are $\diamond_\alpha$–integrable functions on $[a, b]_\mathbb{T}$.*

(i) *If $f(x) \ge 0$ for all $x \in [a, b]_\mathbb{T}$, then $\int_a^b f(x) \diamond_\alpha x \ge 0$.*
(ii) *If $f(x) \le g(x)$ for all $x \in [a, b]_\mathbb{T}$, then $\int_a^b f(x) \diamond_\alpha x \le \int_a^b g(x) \diamond_\alpha x$.*
(iii) *If $f(x) \ge 0$ for all $x \in [a, b]_\mathbb{T}$, then $f(x) = 0$ if and only if $\int_a^b f(x) \diamond_\alpha x = 0$.*

3. Main Results

In this section, we give an extension of dynamic Kantorovich's inequality.

Theorem 3.1. *Let $w, f, g \in C([a, b]_\mathbb{T}, \mathbb{R})$ be $\diamond_\alpha$–integrable functions and neither $f \equiv 0$ nor $g \equiv 0$. If $p, q \in \mathbb{R}$, $0 < \frac{1}{q} \le \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)^{\frac{1}{q}} \le \left[\frac{\frac{1}{p}(M_1M_2) + \frac{1}{q}(m_1m_2)}{(M_1m_1)^{\frac{1}{q}}(M_2m_2)^{\frac{1}{p}}}\right] \left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right), \tag{3.1}$$

with some positive constants m_1*,* m_2*,* M_1 *and* M_2 *such that* $\frac{m_1}{M_2} \leq \frac{|f(x)|}{|g(x)|} \leq \frac{M_1}{m_2}$ *on the set* $[a,b]_{\mathbb{T}}$*.*

Proof. We have

$$\begin{aligned}
&\left(\frac{1}{p}\left|\frac{f(x)}{g(x)}\right| - \frac{1}{q}\frac{m_1}{M_2}\right)\left(\left|\frac{f(x)}{g(x)}\right| - \frac{M_1}{m_2}\right) \leq 0 \\
\Rightarrow\quad &\frac{1}{p}\left|\frac{f(x)}{g(x)}\right|^2 - \left(\frac{1}{p}\frac{M_1}{m_2} + \frac{1}{q}\frac{m_1}{M_2}\right)\left|\frac{f(x)}{g(x)}\right| + \frac{1}{q}\frac{M_1 m_1}{M_2 m_2} \leq 0 \\
\Rightarrow\quad &\frac{1}{p}\left|\frac{f(x)}{g(x)}\right|^2 + \frac{1}{q}\frac{M_1 m_1}{M_2 m_2} \leq \left(\frac{1}{p}\frac{M_1}{m_2} + \frac{1}{q}\frac{m_1}{M_2}\right)\left|\frac{f(x)}{g(x)}\right|.
\end{aligned}$$

Multiplying by $|w(x)||g(x)|^2$, we get

$$\frac{1}{p}|w(x)||f(x)|^2 + \frac{1}{q}\frac{M_1 m_1}{M_2 m_2}|w(x)||g(x)|^2 \leq \left(\frac{1}{p}\frac{M_1}{m_2} + \frac{1}{q}\frac{m_1}{M_2}\right)|w(x)||f(x)g(x)|. \tag{3.2}$$

Now, we note that

$$\begin{aligned}
&\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)^{\frac{1}{q}} \\
= &\left(\frac{M_2 m_2}{M_1 m_1}\right)^{\frac{1}{q}}\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b \left(\frac{M_1 m_1}{M_2 m_2}\right)|w(x)||g(x)|^2 \diamond_\alpha x\right)^{\frac{1}{q}} \\
\leq &\left(\frac{M_2 m_2}{M_1 m_1}\right)^{\frac{1}{q}}\left(\int_a^b \left(\frac{1}{p}|w(x)||f(x)|^2 + \frac{1}{q}\frac{M_1 m_1}{M_2 m_2}|w(x)||g(x)|^2\right) \diamond_\alpha x\right) \\
\leq &\left(\frac{M_2 m_2}{M_1 m_1}\right)^{\frac{1}{q}}\int_a^b \left(\frac{1}{p}\frac{M_1}{m_2} + \frac{1}{q}\frac{m_1}{M_2}\right)|w(x)||f(x)g(x)| \diamond_\alpha x,
\end{aligned}$$

where we have used the well-known Young's inequality $\zeta^{\frac{1}{p}}\eta^{\frac{1}{q}} \leq \frac{\zeta}{p} + \frac{\eta}{q}$, valid for nonnegative real numbers ζ and η, and then (3.2). Thus,

$$\begin{aligned}
&\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)^{\frac{1}{q}} \\
&\qquad \leq \left(\frac{M_2 m_2}{M_1 m_1}\right)^{\frac{1}{q}}\left(\frac{1}{p}\frac{M_1}{m_2} + \frac{1}{q}\frac{m_1}{M_2}\right)\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x.
\end{aligned} \tag{3.3}$$

Thus, (3.1) follows from (3.3). □

Remark 1. *If we set* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w(k) = w_k \in [0, +\infty)$, $f(k) = x_k \in (0, +\infty)$ *and* $g(k) = y_k \in (0, +\infty)$ *for* $k = 1, 2, \dots, n$, *then* (3.1) *reduces to* (1.5).

Further, if we replace w_k *by* $\frac{w_k}{x_k}$ *and set* $m_2 = y_k = M_2 = 1$ *for* $k = 1, 2, \dots, n$, *then* (1.5) *reduces to* (1.4).

Remark 2. *We obtain the following results.*

(1) *If we set* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $x_k > 0$, $w(k) = w_k = \frac{1}{x_k}$, $f(k) = x_k$ *for* $k = 1, 2, \dots, n$, $m_2 = g = M_2 = 1$ *and* $p = q = 2$, *then* (3.1) *reduces to the inequality given by P. Schweitzer* [17] *such that*

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\right) \leq \frac{(M_1 + m_1)^2}{4M_1 m_1}. \tag{3.4}$$

(2) *If we set* $\mathbb{T} = \mathbb{R}$, $0 < m_1 \leq f(x) \leq M_1$, $w(x) = \frac{1}{f(x)}$, $m_2 = g = M_2 = 1$ *and* $p = q = 2$, *then* (3.1) *reduces to the inequality given by P. Schweitzer* [17] *such that*

$$\int_a^b f(x)dx \int_a^b \frac{1}{f(x)}dx \leq \frac{(M_1 + m_1)^2}{4M_1 m_1}(b - a)^2, \tag{3.5}$$

where $f(x)$ *and* $\frac{1}{f(x)}$ *are integrable functions on* $[a, b]$.

(3) *If we set* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$, $g(k) = y_k \in (0, +\infty)$ *for* $k = 1, 2, \dots, n$ *and* $p = q = 2$, *then* (3.1) *reduces to the inequality given by Pólya–Szegö* [13] *such that*

$$\frac{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)}{\left(\sum_{k=1}^{n} x_k y_k\right)^2} \leq \left(\frac{\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}}{2}\right)^2, \tag{3.6}$$

where $0 < m_1 \leq x_k \leq M_1$ *and* $0 < m_2 \leq y_k \leq M_2$ *for* $k = 1, 2, \dots, n$.

(4) *If we set* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $x_k > 0$, $y_k \in \mathbb{R}$, $w(k) = w_k = \frac{1}{x_k} y_k^2$, $f(k) = x_k$ *for* $k = 1, 2, \dots, n$, $m_2 = g = M_2 = 1$ *and* $p = q = 2$, *then* (3.1)

reduces to the inequality given by L. V. Kantorovich [9] *such that*

$$\left(\sum_{k=1}^{n} x_k y_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{x_k} y_k^2\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1}{m_1}}+\sqrt{\frac{m_1}{M_1}}\right)^2 \left(\sum_{k=1}^{n} y_k^2\right)^2, \tag{3.7}$$

and he pointed out that inequality (3.7) *is a particular case of* (3.6).

(5) *If we set* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $z_k \in \mathbb{R}$, $w(k) = w_k = z_k^2$, $f(k) = x_k \in (0,+\infty)$, $g(k) = y_k \in (0,+\infty)$ *for* $k = 1,2,\ldots,n$ *and* $p = q = 2$, *then* (3.1) *reduces to the inequality given by Greub–Rheinboldt* [7] *such that*

$$\left(\sum_{k=1}^{n} x_k^2 z_k^2\right)\left(\sum_{k=1}^{n} y_k^2 z_k^2\right) \le \frac{(M_1M_2+m_1m_2)^2}{4M_1M_2m_1m_2}\left(\sum_{k=1}^{n} x_k y_k z_k^2\right)^2, \tag{3.8}$$

where $0 < m_1 \le x_k \le M_1 < \infty$ *and* $0 < m_2 \le y_k \le M_2 < \infty$ *for* $k = 1,2,\ldots,n$.

Now, we give here the following two additive versions of dynamic Cassels' inequality.

Corollary 3.1. *Let* $w,f,g \in C([a,b]_{\mathbb{T}},\mathbb{R})$ *be* $\diamond_\alpha$*–integrable functions and neither* $f \equiv 0$ *nor* $g \equiv 0$. *Then*

$$\begin{aligned} 0 \le \Bigg\{&\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)\left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)\Bigg\}^{\frac{1}{2}} \\ &- \int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \le \frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{Mm}}\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} 0 \le &\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)\left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right) \\ &- \left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^2 \le \frac{(M-m)^2}{4Mm}\left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^2, \end{aligned} \tag{3.10}$$

where $0 < m \le \frac{|f(x)|}{|g(x)|} \le M$ *on the set* $[a,b]_{\mathbb{T}}$.

Proof. Let $p = q = 2$, $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. Subtracting $\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x$ on both sides of the inequality (3.1), we get the desired inequality (3.9).

Further, if $p = q = 2$, $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$, then inequality (3.1) reduces to

$$1 \leq \frac{\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)^{\frac{1}{2}} \left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)^{\frac{1}{2}}}{\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x} \leq \frac{M+m}{2\sqrt{Mm}}. \tag{3.11}$$

By taking the square and subtracting 1 on both sides of the inequality (3.11), respectively, we get the desired inequality (3.10). □

Remark 3. *If we set* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w(k) = w_k \in [0, +\infty)$, $f(k) = x_k \in (0, +\infty)$ *and* $g(k) = y_k \in (0, +\infty)$ *for* $k = 1, 2, \ldots, n$, *then* (3.9) *reduces to* (1.6) *and* (3.10) *reduces to* (1.7).

Next, we give the following two additive versions of the dynamic Pólya–Szegö inequality.

Corollary 3.2. *Let* $w, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*–integrable functions and neither* $f \equiv 0$ *nor* $g \equiv 0$. *Then*

$$0 \leq \left\{\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)\left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)\right\}^{\frac{1}{2}} - \int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \leq \frac{\left(\sqrt{M_1M_2} - \sqrt{m_1m_2}\right)^2}{2\sqrt{M_1M_2m_1m_2}} \int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \tag{3.12}$$

and

$$0 \leq \left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)\left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right) - \left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^2 \leq \frac{(M_1M_2 - m_1m_2)^2}{4M_1M_2m_1m_2} \left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^2, \tag{3.13}$$

where $0 < m_1 \leq |f(x)| \leq M_1 < \infty$ *and* $0 < m_2 \leq |g(x)| \leq M_2 < \infty$ *on the set* $[a, b]_{\mathbb{T}}$.

Proof. Setting $p = q = 2$ and subtracting $\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x$ on both sides of the inequality (3.1), we get the desired inequality (3.12).

Further, if $p = q = 2$, then inequality (3.1) reduces to

$$1 \leq \frac{\left(\int_a^b |w(x)||f(x)|^2 \diamond_\alpha x\right)^{\frac{1}{2}} \left(\int_a^b |w(x)||g(x)|^2 \diamond_\alpha x\right)^{\frac{1}{2}}}{\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x} \leq \frac{M_1M_2 + m_1m_2}{2\sqrt{M_1M_2m_1m_2}}. \tag{3.14}$$

By taking the square and subtracting 1 on both sides of the inequality (3.14), respectively, we get the desired inequality (3.13). □

Remark 4. *Let* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ *and* $g(k) = y_k \in (0, +\infty)$ *for* $k = 1, 2, \ldots, n$. *Then inequality* (3.12) *reduces to*

$$0 \leq \left(\sum_{k=1}^n x_k^2 \sum_{k=1}^n y_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^n x_k y_k \leq \frac{\left(\sqrt{M_1M_2} - \sqrt{m_1m_2}\right)^2}{2\sqrt{M_1M_2m_1m_2}} \sum_{k=1}^n x_k y_k \tag{3.15}$$

and inequality (3.13) *reduces to*

$$0 \leq \sum_{k=1}^n x_k^2 \sum_{k=1}^n y_k^2 - \left(\sum_{k=1}^n x_k y_k\right)^2 \leq \frac{(M_1M_2 - m_1m_2)^2}{4M_1M_2m_1m_2} \left(\sum_{k=1}^n x_k y_k\right)^2, \tag{3.16}$$

where $0 < m_1 \leq x_k \leq M_1 < \infty$ *and* $0 < m_2 \leq y_k \leq M_2 < \infty$ *for* $k = 1, 2, \ldots, n$. *Inequalities* (3.15) *and* (3.16) *are given in* [6].

Remark 5. *Let* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w(k) = w_k \in [0, +\infty)$, $f(k) = x_k \in (0, +\infty)$ *and* $g(k) = y_k \in (0, +\infty)$ *for* $k = 1, 2, \ldots, n$. *Then inequality* (3.12) *reduces to*

$$\begin{aligned} 0 \leq \left(\sum_{k=1}^n w_k x_k^2 \sum_{k=1}^n w_k y_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^n w_k x_k y_k \\ \leq \frac{\left(\sqrt{M_1M_2} - \sqrt{m_1m_2}\right)^2}{2\sqrt{M_1M_2m_1m_2}} \sum_{k=1}^n w_k x_k y_k \end{aligned} \tag{3.17}$$

and inequality (3.13) *reduces to*

$$0 \leq \sum_{k=1}^{n} w_k x_k^2 \sum_{k=1}^{n} w_k y_k^2 - \left(\sum_{k=1}^{n} w_k x_k y_k \right)^2 \leq \frac{(M_1 M_2 - m_1 m_2)^2}{4 M_1 M_2 m_1 m_2} \left(\sum_{k=1}^{n} w_k x_k y_k \right)^2, \tag{3.18}$$

where $0 < m_1 \leq x_k \leq M_1 < \infty$ *and* $0 < m_2 \leq y_k \leq M_2 < \infty$ *for* $k = 1, 2, \ldots, n$. *Inequalities* (3.17) *and* (3.18) *are two additive versions of the Greub–Rheinboldt inequality as given in* [6].

Remark 6. *If we set* $\alpha = 1$*, then we get delta versions and if we set* $\alpha = 0$*, then we get nabla versions of diamond–*α *integral operator inequalities presented in this article.*

Also, if we set $\mathbb{T} = \mathbb{Z}$*, then we get discrete versions and if we set* $\mathbb{T} = \mathbb{R}$*, then we get continuous versions of diamond–*α *integral operator inequalities presented in this article.*

4. Conclusion and Future Work

In this research article, we have presented many well–known dynamic inequalities on time scales via the diamond–α integral, which is defined as a linear combination of the delta and nabla integrals.

In the future research, we will continue to investigate generalizations of dynamic inequalities on time scales. Using this technique, we can also present higher dimensional inequalities and in the fractional setting by using Riemann–Liouville type fractional integral and fractional derivatives. A functional generalization is another technique which is used to generalize inequalities. Quantum calculus and α, β-symmetric quantum calculus are also applied to yield inequalities.

The first and second inequalities given in (1.3) may be used to find many dynamic inequalities such as Rogers–Hölder's inequality, Lyapunov's inequality, Radon's inequality, Bergström's inequality, Schlömilch's inequality, the weighted power mean inequality and Bernoulli's inequality on time scales. Motivated by the works of [11, 14, 15, 16], we can explore further results in harmonized and reconciled form.

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