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SOME INTEGRAL INEQUALITIES OF (s, p)-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we introduce the notion of (s, p)-convex functions and establish an integral equality and some Hermite-Hadamard type integral inequalities of (s, p)-convex functions in fractional form. Also give some Hermite-Hadamard type integral inequalities of product of two (s, p)-convex functions in fractional form.

1. Preliminaries

Definition 1.1. Let $\mathcal{K}$ be an interval of real numbers. Then the function $\psi : \mathcal{K} \to \mathbb{R}$ is called convex if the following inequality holds:

$$\psi(ru + (1-r)v) \leq r\psi(u) + (1-r)\psi(v) \tag{1.1}$$

for all $u, v \in \mathcal{K}$ and $r \in [0, 1]$. On the other hand, ψ is called concave if the inequality in (1.1) is reversed.

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The Hermite-Hadamard inequality for convex functions $\psi : \mathcal{K} \to \mathbb{R}$ on an interval of real line is defined as:

$$\psi\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \psi(w)dw \le \frac{\psi(u)+\psi(v)}{2}, \tag{1.2}$$

where $u, v \in \mathcal{K}$ with $u < v$.

Definition 1.2 ([6])**.** Let $\mathcal{K} \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $\psi : \mathcal{K} \to \mathbb{R}$ is said to be harmonically convex if

$$\psi\left(\frac{uv}{ru+(1-r)v}\right) \le r\psi(u) + (1-r)\psi(v) \tag{1.3}$$

for all $u, v \in \mathcal{K}$ and $r \in [0,1]$. While, ψ is harmonically concave if the inequality in (1.3) is reversed.

Theorem 1.1 ([6])**.** *Let $\psi : \mathcal{K} \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $u, v \in \mathcal{K}$ with $u < v$. If $\psi \in L[u,v]$, then the following inequalities hold:*

$$\psi\left(\frac{2uv}{u+v}\right) \le \frac{uv}{v-u}\int_u^v \frac{\psi(w)}{w^2}dw \le \frac{\psi(u)+\psi(v)}{2}. \tag{1.4}$$

Definition 1.3 ([5])**.** Let $s \in (0,1]$. A function $\psi : \mathcal{K} \subset \mathbb{R}_0 \to \mathbb{R}_0$, where $\mathbb{R}_0 = [0,\infty)$, is called s-convex function in second sense if the following inequality holds:

$$\psi(ru+(1-r)v) \le r^s\psi(u) + (1-r)^s\psi(v) \tag{1.5}$$

for all $u, v \in \mathcal{K}$ and $r \in [0,1]$.

Theorem 1.2 ([3])**.** *Suppose that $\psi : \mathbb{R}_0 \to \mathbb{R}_0$ be an s-convex function in second sense, where $s \in (0,1)$ and let $u, v \in [0,\infty)$, $u \le v$. If $\psi \in L[u,v]$, then the following inequalities hold:*

$$2^{s-1}\psi\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \psi(w)dw \le \frac{\psi(u)+\psi(v)}{s+1}. \tag{1.6}$$

Definition 1.4 ([7],[8])**.** Let $\mathcal{K} \subset (0,\infty) = \mathbb{R}_+$ be an interval and $p \in \mathbb{R}\backslash\{0\}$. A function $\psi : \mathcal{K} \to \mathbb{R}$ is said to be p-convex function if

$$\psi\left((ru^p + (1-r)v^p)^{\frac{1}{p}}\right) \leq r\psi(u) + (1-r)\psi(v) \tag{1.7}$$

holds for all $u, v \in \mathcal{K}$ and $r \in [0,1]$. However, ψ is called p-concave if the inequality in (1.7) is reversed.

Theorem 1.3 ([7])**.** *Let $\psi : \mathcal{K} \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $u, v \in \mathcal{K}$ with $u < v$. If $\psi \in L[u,v]$, then we have*

$$\psi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{p}{v^p - u^p}\int_u^v \frac{\psi(w)}{w^{1-p}}dw \leq \frac{\psi(u)+\psi(v)}{2}. \tag{1.8}$$

Definition 1.5 ([9])**.** Let $\psi \in L[u,v]$. The right-hand side and left-hand side Riemann- Liouville fractional integrals $\mathcal{J}^{\alpha}_{u+}\psi$ and $\mathcal{J}^{\alpha}_{v-}\psi$ of order $\alpha > 0$ with $v > u \geq 0$ are defined by

$$\mathcal{J}^{\alpha}_{u+}\psi(w) = \tfrac{1}{\Gamma(\alpha)}\int_u^w (w-t)^{\alpha-1}\psi(t)dt,\ w > u$$

and

$$\mathcal{J}^{\alpha}_{v-}\psi(w) = \tfrac{1}{\Gamma(\alpha)}\int_w^v (t-w)^{\alpha-1}\psi(t)dt,\ w < v$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

Sarikaya et al. [10] and Set et al. [11] proved the following Hermite-Hadamard type integral inequalities of convex functions and s-convex functions, respectively, in fractional form.

Theorem 1.4 ([10])**.** *Let $\psi : [u,v] \to \mathbb{R}$ be a positive convex function with $u < v$ and $\psi \in L[u,v]$. Then the following inequalities for fractional integrals hold:*

$$\psi\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[\mathcal{J}^{\alpha}_{u+}\psi(v) + \mathcal{J}^{\alpha}_{v-}\psi(u)\right] \leq \frac{\psi(u)+\psi(v)}{2}, \tag{1.9}$$

with $\alpha > 0$.

Theorem 1.5 ([11])**.** *Let $s \in (0,1]$ and $\alpha > 0$. Let $\psi : [u,v] \to \mathbb{R}$ be a positive s-convex function in the second sense with $0 \leq u < v$ and $\psi \in L[u,v]$. Then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & 2^{s-1}\psi\left(\frac{u+v}{2}\right) \\ \text{(1.10)} \qquad & \leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[\mathcal{J}_{u+}^{\alpha}\psi(v)+\mathcal{J}_{v-}^{\alpha}\psi(u)\right] \\ & \leq \left[\frac{\alpha}{\alpha+s}+\alpha\beta(\alpha,s+1)\right]\frac{\psi(u)+\psi(v)}{2}, \end{aligned}$$

where β is Euler Beta function.

Fang and Shi [4] introduced the notion of (p,h)-convex function as follows:

Definition 1.6 ([4])**.** Let J be a real interval and $\mathcal{K}$ is p-convex set. Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. Then $\psi : \mathcal{K} \to \mathbb{R}$ is called a (p,h)-convex function, if ψ is non-negative and

$$\psi\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right) \leq h(r)\psi(u)+h(1-r)\psi(v) \qquad (1.11)$$

for all $u, v \in \mathcal{K}$ and $r \in (0,1)$.

In next section, we will define (s,p)-convex function which is subclass of (p,h)-convex functions defined by Fang and Shi [4] and establish some Hermite-Hadamard type inequalities for (s,p)-convex functions via Riemann-Liouville fractional integrals.

2. Integral inequalities of (s,p)-convex functions via fractional integrals

Definition 2.1. Let $s \in (0,1]$, $p \in \mathbb{R} \setminus \{0\}$ and $\mathcal{K} \subset (0,\infty)$ be an interval. Then a function $\psi : \mathcal{K} \subset (0,\infty) \to (0,\infty)$ is said to be

(s,p)-convex function, if

$$\psi\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)\leq r^s\psi(u)+(1-r)^s\psi(v) \tag{2.1}$$

for all $u,v\in\mathcal{K}$ with $u<v$ and $r\in[0,1]$.

Remark 1. *In Definition 2.1,*

1. *if we take $p=1$, then we have Definition 1.3,*
2. *if we take $s=1$, then we have Definition 1.4,*
3. *if we take $s=1$, and $p=1$ then we have Definition 1.1.*

We know that the Euler Beta function is define as:

$$\beta(u,v)=\int_0^1 w^{u-1}(1-w)^{v-1}dw,$$

where $u>0$ and $v>0$. Through out the paper, we denote $\mathbb{R}_+=(0,\infty)$.

Theorem 2.1. *Let $s\in(0,1]$, $r\in[0,1]$ and $p\in\mathbb{R}\setminus\{0\}$. Let $\psi:\mathcal{K}\subset\mathbb{R}_+\to\mathbb{R}_+$ be a (s,p)-convex function, where $\mathcal{K}$ is an interval, such that $\psi\in L[u,v]$ for $u,v\in\mathcal{K}^o$, where $\mathcal{K}^o$ is the interior of $\mathcal{K}$, with $u<v$ and $\alpha>0$. Then the following inequalities for fractional integrals hold:*

(i) If $p>0$, then

$$\begin{aligned}2^{s-1}\psi\left(\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}}\right)&\leq\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\\&\leq\left[\frac{\alpha}{\alpha+s}+\alpha\beta(\alpha,s+1)\right]\frac{\psi(u)+\psi(v)}{2},\end{aligned} \tag{2.2}$$

where $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[u^p,v^p]$.

(ii) If $p<0$, then

(2.3)
$$2^{s-1}\psi\left(\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\alpha+1)}{2(u^p-v^p)^{\alpha}}\left[\mathcal{J}^{\alpha}_{v^p_+}(\psi\circ\tau)(u^p)+\mathcal{J}^{\alpha}_{u^p_-}(\psi\circ\tau)(v^p)\right]$$
$$\leq \left[\frac{\alpha}{\alpha+s}+\alpha\beta(\alpha,s+1)\right]\frac{\psi(u)+\psi(v)}{2},$$

where $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[v^p,u^p]$.

Proof. (i) Let $p>0$. Since ψ is (s,p)-convex then for all $x,y\in\mathcal{K}$, we have
$$\psi\left(\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}\right)\leq\frac{\psi(x)+\psi(y)}{2^s}.$$

Taking $x=(ru^p+(1-r)v^p)^{\frac{1}{p}}$ and $y=((1-r)u^p+rv^p)^{\frac{1}{p}}$ with $r\in[0,1]$, we get

(2.4)
$$\psi\left(\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}}\right)\leq\frac{\psi\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)+\psi\left(((1-r)u^p+rv^p)^{\frac{1}{p}}\right)}{2^s}.$$

Multiplying both sides of (2.4) by $r^{\alpha-1}$ and integrating with respect to r over [0,1], implies

(2.5)
$$\frac{1}{\alpha}\psi\left(\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}}\right)\leq\frac{\Gamma(\alpha)}{2^s(v^p-u^p)^{\alpha}}\left[\mathcal{J}^{\alpha}_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^{\alpha}_{v^p_-}(\psi\circ\tau)(a^p)\right].$$

Which is the left hand side of (2.2). Again applying (s,p)-convexity of ψ, for all $r\in[0,1]$, we have

$$\frac{\psi\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)+\psi\left(((1-r)u^p+rv^p)^{\frac{1}{p}}\right)}{2^s} \tag{2.6}$$
$$\leq\frac{(r^s+(1-r)^s)(\psi(u)+\psi(v))}{2^s}.$$

Multiplying both sides of (2.6) by $r^{\alpha-1}$ and integrating with respect to r over [0,1], yields

$$\begin{aligned}(2.7)\qquad &\frac{\Gamma(\alpha)}{2^s(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\\ &\leq\left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right]\frac{\psi(u)+\psi(v)}{2^s}.\end{aligned}$$

Thus from (2.5) and (2.7) we get (2.2).

(ii) The proof is similar with (i). □

Remark 2. *Under the assumptions of Theorem* 2.1 *we have the following.*

1. *If* $p = 1$, *then we get* (1.10),
2. *If* $p = 1$ *and* $s = 1$, *then we get* (1.9),
3. *If* $s = 1$ *and* $\alpha = 1$, *then we get* (1.8),
4. *If* $p = 1$ *and* $\alpha = 1$, *then we get* (1.6),
5. *If* $p = 1$ *and* $\alpha = 1$ *and* $s = 1$, *then we get* (1.2),
6. *If* $p = -1$ *and* $\alpha = 1$ *and* $s = 1$, *then we get* (1.4).

Lemma 2.1. *Let* $p \in \mathbb{R}\setminus\{0\}$. *Let* $\psi : \mathcal{K} \subset \mathbb{R}_+ \to \mathbb{R}_+$, *where* $\mathcal{K}$ *is an interval, be a differentiable function on* $\mathcal{K}^o$, *where* $\mathcal{K}^o$ *is the interior of* $\mathcal{K}$, *such that* $\psi' \in L[u, v]$ *for* $u, v \in \mathcal{K}^o$ *with* $u < v$ *and* $\alpha > 0$. *Then the following equalities for fractional integrals hold:*

(i) *If* $p > 0$, *then*

$$\begin{aligned}&\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\\ (2.8)\qquad &=\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(w-u^p)^\alpha(\psi\circ\tau)'(w)dw\\ &\quad-\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(v^p-w)^\alpha(\psi\circ\tau)'(w)dw,\end{aligned}$$

where $\tau(w) = w^{\frac{1}{p}}$ *for all* $w \in [u^p, v^p]$.

(ii) If $p<0$, then

$$\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(u^p-v^p)^\alpha}\left[\mathcal{J}^\alpha_{v^p_+}(\psi\circ\tau)(u^p)+\mathcal{J}^\alpha_{u^p_-}(\psi\circ\tau)(v^p)\right]$$

$$=\frac{1}{2(u^p-v^p)^\alpha}\int_{v^p}^{u^p}(w-v^p)^\alpha(\psi\circ\tau)'(w)dw \tag{2.9}$$
$$-\frac{1}{2(u^p-v^p)^\alpha}\int_{v^p}^{u^p}(u^p-w)^\alpha(\psi\circ\tau)'(w)dw,$$

where $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[v^p,u^p]$.

Proof. Consider,

$$\begin{aligned}H&=\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(w-u^p)^\alpha(\psi\circ\tau)'(w)dw\\&\quad-\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(v^p-w)^\alpha(\psi\circ\tau)'(w)dw\\&=H_1-H_2.\end{aligned}\tag{2.10}$$

Then by integrating by parts, we get

$$\begin{aligned}H_1&=\frac{1}{2(v^p-u^p)^\alpha}\Big[(w-u^p)^\alpha(\psi\circ\tau)(w)\mid_{u^p}^{v^p}\\&\quad-\alpha\int_{u^p}^{v^p}(w-u^p)^{\alpha-1}(\psi\circ\tau)(w)dw\Big]\\&=\frac{1}{2}(\psi\circ\tau)(v^p)-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p).\end{aligned}\tag{2.11}$$

Similarly, we have

$$\begin{aligned}H_2&=\frac{1}{2(v^p-u^p)^\alpha}\Big[(v^p-w)^\alpha(\psi\circ\tau)(w)\mid_{u^p}^{v^p}\\&\quad+\alpha\int_{u^p}^{v^p}(v^p-w)^{\alpha-1}(\psi\circ\tau)(w)dw\Big]\\&=-\frac{1}{2}(\psi\circ\tau)(u^p)+\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p).\end{aligned}\tag{2.12}$$

Thus from (2.10), (2.11) and (2.12), we achieve (2.8).

(ii) The proof is similar with (i). □

Remark 3. *In the above Lemma* 2.1 *if we take* $p=1$*, then we have*

(2.13)
$$\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}[\mathcal{J}^{\alpha}_{u+}\psi(v)+\mathcal{J}^{\alpha}_{v-}\psi(u)]$$
$$=\frac{1}{2(v-u)^{\alpha}}\int_u^v(w-u)^{\alpha}\psi'(w)dw-\frac{1}{2(v-u)^{\alpha}}\int_u^v(v-w)^{\alpha}\psi'(w)dw.$$

Theorem 2.2. *Let* $s\in(0,1]$*,* $r\in[0,1]$ *and* $p\in\mathbb{R}\setminus\{0\}$*. Let* $\psi:\mathcal{K}\subset\mathbb{R}_+\to\mathbb{R}_+$*, where* $\mathcal{K}$ *is an interval, be a differentiable function on* $\mathcal{K}^o$*, where* $\mathcal{K}^o$ *is the interior of* $\mathcal{K}$*, such that* $\psi'\in L[u,v]$ *for* $u,v\in\mathcal{K}^o$ *with* $u<v$ *and* $\alpha>0$*. Let* $|\psi'|$ *be* (s,p)*-convex function on* $[u,v]$*. Then the following inequalities for fractional integrals hold:*

(*i*) *If* $p>0$*, then*

(2.14)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^{\alpha}}\left[\mathcal{J}^{\alpha}_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^{\alpha}_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|$$
$$\leq\frac{v^p-u^p}{2}\left[B_1(\alpha,s,p)|\psi'(u)|+B_2(\alpha,s,p)|\psi'(v)|\right],$$

where $\tau(w)=w^{\frac{1}{p}}$ *for all* $w\in[u^p,v^p]$*.*

(*ii*) *If* $p<0$*, then*

(2.15)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(u^p-v^p)^{\alpha}}\left[\mathcal{J}^{\alpha}_{v^p_+}(\psi\circ\tau)(u^p)+\mathcal{J}^{\alpha}_{u^p_-}(\psi\circ\tau)(v^p)\right]\right|$$
$$\leq\frac{u^p-v^p}{2}\left[B_1(\alpha,s,p)|\psi'(u)|+B_2(\alpha,s,p)|\psi'(v)|\right],$$

with $\tau(w)=w^{\frac{1}{p}}$ *for all* $w\in[v^p,u^p]$*. Where*

$$B_1=\int_0^1\frac{r^s(1-r)^{\alpha}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr+\int_0^1\frac{r^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr$$

and

$$B_2=\int_0^1\frac{(1-r)^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr+\int_0^1\frac{r^{\alpha}(1-r)^s}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr.$$

Proof. (i) Let $p>0$. Using Lemma 2.1(i), we get

(2.16)
$$\begin{aligned}
&\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|\\
&\le \frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(w-u^p)^\alpha|(\psi\circ\tau)'(w)|dw\\
&\quad+\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(v^p-w)^\alpha|(\psi\circ\tau)'(w)|dw\\
&=\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(w-u^p)^\alpha\frac{1}{pw^{1-\frac{1}{p}}}|\psi'(w^{\frac{1}{p}})|dw\\
&\quad+\frac{1}{2(v^p-u^p)^\alpha}\int_{u^p}^{v^p}(v^p-w)^\alpha\frac{1}{pw^{1-\frac{1}{p}}}|\psi'(w^{\frac{1}{p}})|dw.
\end{aligned}$$

Setting $w=ru^p+(1-r)v^p$, we find

(2.17)
$$\begin{aligned}
&\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|\\
&\le\frac{1}{2(v^p-u^p)^{\alpha-1}}\Big[\int_0^1\frac{(ru^p+(1-r)v^p-u^p)^\alpha}{p(ru^p+(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'((ru^p+(1-r)v^p)^{\frac{1}{p}})\right|dr\\
&\quad+\int_0^1\frac{(v^p-ru^p-(1-r)v^p)^\alpha}{p(ru^p+(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'((ru^p+(1-r)v^p)^{\frac{1}{p}})\right|dr\Big]\\
&=\frac{v^p-u^p}{2}\Big[\int_0^1\frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'((ru^p+(1-r)v^p)^{\frac{1}{p}})\right|dr\\
&\quad+\int_0^1\frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'((ru^p+(1-r)v^p)^{\frac{1}{p}})\right|dr\Big].
\end{aligned}$$

Since $|\psi'|$ is (s,p)-convex function on $[u,v]$, we have

$$\left|\psi'((ru^p+(1-r)v^p)^{\frac{1}{p}})\right|\le r^s\,|\psi'(u)|+(1-r)^s|\psi'(v)|.$$

Then (2.17) gives

(2.18)

$$\begin{aligned}
&\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|\\
&\le\frac{v^p-u^p}{2}\left[\int_0^1\frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}(r^s|\psi'(u)|+(1-r)^s|\psi'(v)|)dr\right.\\
&\quad\left.+\int_0^1\frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}(r^s|\psi'(u)|+(1-r)^s|\psi'(v)|)dr\right]\\
&=\frac{v^p-u^p}{2}\left[\left\{\int_0^1\frac{r^s(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr+\int_0^1\frac{r^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\right\}|\psi'(u)|\right.\\
&\quad\left.+\left\{\int_0^1\frac{(1-r)^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr+\int_0^1\frac{r^\alpha(1-r)^s}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr\right\}|\psi'(v)|\right]\\
&=\frac{v^p-u^p}{2}\left[B_1(\alpha,s,p)|\psi'(u)|+B_2(\alpha,s,p)|\psi'(v)|\right].
\end{aligned}$$

Hence (i) is proved.

(ii) The proof is similar with (i) by using Lemma 2.1(ii). □

Theorem 2.3. *Let $s\in(0,1]$, $r\in[0,1]$ and $p\in\mathbb{R}\setminus\{0\}$. Let $\psi:\mathcal{K}\subset\mathbb{R}_+\to\mathbb{R}_+$, where $\mathcal{K}$ is an interval, be a differentiable function on $\mathcal{K}^o$, where $\mathcal{K}^o$ is the interior of $\mathcal{K}$, such that $\psi'\in L[u,v]$ for $u,v\in\mathcal{K}^o$ with $u<v$ and $\alpha>0$. Let $|\psi'|^q$, $q\ge1$, be (s,p)-convex function on $[u,v]$. Then the following inequalities for fractional integrals hold:*

(i) *If $p>0$, then*

(2.19)

$$\begin{aligned}
&\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|\\
&\le\frac{v^p-u^p}{2}\Big[(B_3(\alpha,s,p))^{1-\frac{1}{q}}\left\{|B_4\psi'(u)|^q+B_5(\alpha,s,p)|\psi'(v)|^q\right\}^{\frac{1}{q}}\\
&\quad+(B_6(\alpha,s,p))^{1-\frac{1}{q}}\left\{B_7|\psi'(u)|^q+B_8(\alpha,s,p)|\psi'(v)|^q\right\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[u^p,v^p]$.

(ii) If $p<0$, then

(2.20)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(u^p-v^p)^\alpha}\left[\mathcal{J}^\alpha_{v^p_+}(\psi\circ\tau)(u^p)+\mathcal{J}^\alpha_{u^p_-}(\psi\circ\tau)(v^p)\right]\right|$$
$$\leq\frac{u^p-v^p}{2}\Big[(B_3(\alpha,s,p))^{1-\frac{1}{q}}\left\{B_4|\psi'(u)|^q+B_5(\alpha,s,p)|\psi'(v)|^q\right\}^{\frac{1}{q}}$$
$$+(B_6(\alpha,s,p))^{1-\frac{1}{q}}\left\{B_7|\psi'(u)|^q+B_8(\alpha,s,p)|\psi'(v)|^q\right\}^{\frac{1}{q}}\Big],$$

with $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[v^p,u^p]$. Where

$$B_3=\int_0^1\frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr,\ B_4=\int_0^1\frac{(1-r)^\alpha r^s}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr,$$
$$B_5=\int_0^1\frac{(1-r)^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr,\ B_6=\int_0^1\frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr,$$

and

$$B_7=\int_0^1\frac{r^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr,\ B_8=\int_0^1\frac{r^\alpha(1-r)^s}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr.$$

Proof. (i) Let $p>0$. Using (2.17), power mean inequality and the (s,p)-convexity of $|\psi'|^q$, we find

(2.21)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|$$
$$\leq\frac{v^p-u^p}{2}\Bigg[\int_0^1\frac{(1-r)^\alpha}{p\left(ru^p-(1-r)v^p\right)^{1-\frac{1}{p}}}\left|\psi'\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)\right|dr$$
$$+\int_0^1\frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)\right|dr\Bigg]$$
$$\leq\frac{v^p-u^p}{2}\Bigg[\left(\int_0^1\frac{(1-r)^\alpha}{p\left(ru^p-(1-r)v^p\right)^{1-\frac{1}{p}}}dr\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)\right|^q dr\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'((ru^p+(1-r)v^p)^{\frac{1}{p}})\right|^q dr\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{v^p-u^p}{2}\Bigg[\left(\left(\int_0^1 \frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1 \frac{(1-r)^\alpha r^s}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}|\psi'(u)|^q dr\right.$$

$$\left.+\int_0^1 \frac{(1-r)^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}|\psi'(v)|^q dr\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}dr\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{r^{\alpha+s}}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}|\psi'(u)|^q dr\right.$$

$$\left.+\int_0^1 \frac{(1-r)^s r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}|\psi'(v)|^q dr\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{v^p-u^p}{2}\Big[(B_3(\alpha,s,p))^{1-\frac{1}{q}}\left\{|B_4 f'(a)|^q+B_5(\alpha,s,p)|\psi'(v)|^q\right\}^{\frac{1}{q}}$$

$$+(B_6(\alpha,s,p))^{1-\frac{1}{q}}\left\{B_7|\psi'(u)|^q+B_8(\alpha,s,p)|\psi'(v)|^q\right\}^{\frac{1}{q}}\Big].$$

This completes the proof of (i).

(ii) The proof is similar with (i). □

Theorem 2.4. *Let $s\in(0,1]$, $r\in[0,1]$ and $p\in\mathbb{R}\setminus\{0\}$. Let $\psi:\mathcal{K}\subset\mathbb{R}_+\to\mathbb{R}_+$, where $\mathcal{K}$ is an interval, be a differentiable function on $\mathcal{K}^o$, where $\mathcal{K}^o$ is the interior of $\mathcal{K}$, such that $\psi'\in L[u,v]$ for $u,v\in\mathcal{K}^o$ with $u<v$ and $\alpha>0$. Let $|\psi'|^q$, where $q,l>1$ such that $\frac{1}{l}+\frac{1}{q}=1$,* *be (s,p)-convex function on $[u,v]$. Then the following inequalities for fractional integrals hold:*

(i) *If $p>0$, then*

(2.22)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|$$
$$\leq\frac{v^p-u^p}{2}\left[(B_9(\alpha,p,l))^{\frac{1}{l}}+(B_{10}(\alpha,p,l))^{\frac{1}{l}}\right]\left[\frac{|\psi'(u)|^q+|\psi'(v)|^q}{s+1}\right]^{\frac{1}{q}}.$$

Where $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[u^p,v^p]$.

(ii) *if $p<0$, then*

(2.23)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(u^p-v^p)^\alpha}\left[\mathcal{J}^\alpha_{v^p_+}(\psi\circ\tau)(u^p)+\mathcal{J}^\alpha_{u^p_-}(\psi\circ\tau)(v^p)\right]\right|$$
$$\leq\frac{u^p-v^p}{2}\left[(B_9(\alpha,p,l))^{\frac{1}{l}}+(B_{10}(\alpha,p,l))^{\frac{1}{l}}\right]\left[\frac{|\psi'(u)|^q+|\psi'(v)|^q}{s+1}\right]^{\frac{1}{q}},$$

with $\tau(w)=w^{\frac{1}{p}}$ for all $w\in[v^p,u^p]$. Where

$$B_9=\int_0^1\left(\frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\right)^l dr,\ B_{10}=\int_0^1\left(\frac{r^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\right)^l dr.$$

Proof. (i) Let $p>0$. Using (2.17), Holder's inequality and the (s,p)-convexity of $|\psi'|^q$ implies,

(2.24)
$$\left|\frac{\psi(u)+\psi(v)}{2}-\frac{\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha}\left[\mathcal{J}^\alpha_{u^p_+}(\psi\circ\tau)(v^p)+\mathcal{J}^\alpha_{v^p_-}(\psi\circ\tau)(u^p)\right]\right|$$
$$\leq\frac{v^p-u^p}{2}\left[\int_0^1\frac{(1-r)^\alpha}{p(ru^p-(1-r)v^p)^{1-\frac{1}{p}}}\left|\psi'\left((ru^p+(1-r)v^p)^{\frac{1}{p}}\right)\right|dr\right.$$

$$
\begin{aligned}
&+ \int_0^1 \frac{r^\alpha}{p(ru^p - (1-r)v^p)^{1-\frac{1}{p}}} \left|\psi'\left((ru^p + (1-r)v^p)^{\frac{1}{p}}\right)\right| dr \Bigg] \\
&\le \frac{v^p - u^p}{2} \Bigg[\left(\int_0^1 \left(\frac{(1-r)^\alpha}{p(ru^p - (1-r)v^p)^{1-\frac{1}{p}}} \right)^l dr \right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 \left|\psi'\left((ru^p + (1-r)v^p)^{\frac{1}{p}}\right)\right|^q dr \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left(\frac{r^\alpha}{p(ru^p - (1-r)v^p)^{1-\frac{1}{p}}} \right)^l dr \right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 \left|\psi'\left((ru^p + (1-r)v^p)^{\frac{1}{p}}\right)\right|^q dr \right)^{\frac{1}{q}} \Bigg] \\
&\le \frac{v^p - u^p}{2} \Bigg[\left(\int_0^1 \left(\frac{(1-r)^\alpha}{p(ru^p - (1-r)v^p)^{1-\frac{1}{p}}} \right)^l dr \right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 r^s |\psi'(u)|^q + \int_0^1 (1-r)^s \, |\psi'(v)|^q \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left(\frac{r^\alpha}{p(ru^p - (1-r)v^p)^{1-\frac{1}{p}}} \right)^l dr \right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 r^s |\psi'(u)|^q + \int_0^1 (1-r)^s |\psi'(v)|^q \right)^{\frac{1}{q}} \Bigg] \\
&= \frac{v^p - u^p}{2} \left[(B_9(\alpha, p, l))^{\frac{1}{l}} + (B_{10}(\alpha, p, l))^{\frac{1}{l}} \right] \left[\frac{|\psi'(u)|^q + |\psi'(v)|^q}{s+1} \right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of (i).

(ii) The proof is similar with (i). □

3. Integral inequalities of product of two (s,p)-convex functions via fractional integrals

Chen and Wu in [1] and [2] investigated the Hermite-Hadamard type inequalities for products of two h-convex functions and products of two s-convex functions in fractional form, respectively. In this section, we will establish some Hermite-Hadamard type inequalities for products of two (s,p)-convex functions in fractional form.

Theorem 3.1. *Let $s_1, s_2 \in (0,1]$, $r, \lambda \in [0,1]$ and $p_1, p_2 \in \mathbb{R} \setminus \{0\}$. Let $\psi, \varphi : \mathcal{K} \subset \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathcal{K}$ is an interval, be (s_1,p_1)-convex function and (s_2,p_2)-convex function, respectively, such that $\psi, \varphi \in L[u,v]$ for $u, v \in \mathcal{K}^o$, where $\mathcal{K}^o$ is the interior of $\mathcal{K}$, with $u < v$ and $\alpha_1, \alpha_2 > 0$. Then the following inequalities for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(v^{p_1}-u^{p_1})^{\alpha_1}(v^{p_2}-u^{p_2})^{\alpha_2}}\big[\mathcal{J}^{\alpha_1}_{u^{p_1}_+}(\psi\circ\tau_1)(v^{p_1})\mathcal{J}^{\alpha_2}_{u^{p_2}_+}(\varphi\circ\tau_2)(v^{p_2}) \\ &\quad + \mathcal{J}^{\alpha_1}_{v^{p_1}_-}(\psi\circ\tau_1)(u^{p_1})\mathcal{J}^{\alpha_2}_{v^{p_2}_-}(\varphi\circ\tau_2)(u^{p_2})\big] \\ &\quad \le \left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)} + \beta(\alpha_1, s_1+1)\beta(\alpha_2, s_2+1)\right] M(u,v) \\ &\quad + \left[\frac{\beta(\alpha_1, s_1+1)}{\alpha_2+s_2} + \frac{\beta(\alpha_2, s_2+1)}{\alpha_1+s_1}\right] N(u,v), \end{aligned} \tag{3.1}$$

where $\tau_1(x) = x^{\frac{1}{p_1}}$ for all $x \in [u^{p_1}, v^{p_1}]$, $\tau_2(y) = y^{\frac{1}{p_2}}$ for all $y \in [u^{p_2}, v^{p_2}]$, and $M(u,v) = \psi(u)\varphi(u) + \psi(v)\varphi(v)$, $N(u,v) = \psi(u)\varphi(v) + \psi(v)\varphi(u)$.

Proof. Since ψ is (s_1,p_1)-convex and φ is (s_2,p_2)-convex, then for $r, \lambda \in [0,1]$, we have

$$\psi\left((ru^{p_1} + (1-r)v^{p_1})^{\frac{1}{p_1}}\right) \le r^{s_1}\psi(u) + (1-r)^{s_1}\psi(v)$$

and

$$\varphi\left((\lambda u^{p_2} + (1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right) \le \lambda^{s_2}\varphi(u) + (1-\lambda)^{s_2}\varphi(v).$$

Then from above, we obtain

$$
\begin{aligned}
&\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)\\
&\leq r^{s_1}\lambda^{s_2}\psi(u)\varphi(u)+(1-r)^{s_1}(1-\lambda)^{s_2}\psi(v)\varphi(v)+r^{s_1}(1-\lambda)^{s_2}\psi(u)\varphi(v)\\
&\quad+(1-r)^{s_1}\lambda^{s_2}\varphi(u)\psi(v).
\end{aligned}
\tag{3.2}
$$

Similarly,

$$
\begin{aligned}
&\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\\
&\leq r^{s_1}\lambda^{s_2}\psi(v)\varphi(v)+(1-r)^{s_1}(1-\lambda)^{s_2}\psi(u)\varphi(u)+r^{s_1}(1-\lambda)^{s_2}\psi(v)\varphi(u)\\
&\quad+(1-r)^{s_1}\lambda^{s_2}\varphi(v)\psi(u).
\end{aligned}
\tag{3.3}
$$

Then by combining the inequalities (3.2) and (3.3), we get

$$
\begin{aligned}
&\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)\\
&+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\\
&\leq\left(r^{s_1}\lambda^{s_2}+(1-r)^{s_1}(1-\lambda)^{s_2}\right)\left[\psi(u)\varphi(u)+\psi(v)\varphi(v)\right]\\
&\quad+\left(r^{s_1}(1-\lambda)^{s_2}+(1-r)^{s_1}\lambda^{s_2}\right)\left[\psi(v)\varphi(u)+\psi(u)\varphi(v)\right].
\end{aligned}
\tag{3.4}
$$

By multiplying (3.4) by $r^{\alpha_1-1}\lambda^{\alpha_2-1}$ and integrating with respect to r and λ over $[0,1]\times[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1 r^{\alpha_1-1}\lambda^{\alpha_2-1}\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)\\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\Big]drd\lambda\\
&\leq\int_0^1\int_0^1 r^{\alpha_1-1}\lambda^{\alpha_2-1}\Big[\left(r^{s_1}\lambda^{s_2}+(1-r)^{s_1}(1-\lambda)^{s_2}\right)\left[\psi(u)\varphi(u)+\psi(v)\varphi(v)\right]\\
&\quad+\left(r^{s_1}(1-\lambda)^{s_2}+(1-r)^{s_1}\lambda^{s_2}\right)\left[\psi(v)\varphi(u)+\psi(u)\varphi(v)\right]\Big]drd\lambda.
\end{aligned}
\tag{3.5}
$$

By letting $x = ru^{p_1} + (1-r)v^{p_1}$ and $y = \lambda u^{p_2} + (1-\lambda)v^{p_2}$, we get

(3.6)
$$\int_0^1 \int_0^1 r^{\alpha_1-1}\lambda^{\alpha_2-1}\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)drd\lambda$$
$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(v^{p_1}-u^{p_1})^{\alpha_1}(v^{p_2}-u^{p_2})^{\alpha_2}}\mathcal{J}^{\alpha_1}_{u^{p_1}_+}(\psi\circ\tau_1)(v^{p_1})\mathcal{J}^{\alpha_2}_{u^{p_2}_+}(\varphi\circ\tau_2)(v^{p_2}).$$

Similarly, by letting $x = rv^{p_1} + (1-r)u^{p_1}$ and $y = \lambda v^{p_2} + (1-\lambda)u^{p_2}$, we find

(3.7)
$$\int_0^1 \int_0^1 r^{\alpha_1-1}\lambda^{\alpha_2-1}\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)drd\lambda$$
$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(v^{p_1}-u^{p_1})^{\alpha_1}(v^{p_2}-u^{p_2})^{\alpha_2}}\mathcal{J}^{\alpha_1}_{v^{p_1}_-}(\psi\circ\tau_1)(u^{p_1})\mathcal{J}^{\alpha_2}_{v^{p_2}_-}(\varphi\circ\tau_2)(u^{p_2}).$$

Also note that,

(3.8)
$$\int_0^1 \int_0^1 r^{\alpha_1-1}\lambda^{\alpha_2-1}\left(r^{s_1}\lambda^{s_2}+(1-r)^{s_1}(1-\lambda)^{s_2}\right)[\psi(u)\varphi(u)+\psi(v)\varphi(v)]\,drd\lambda$$
$$= \left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)} + \beta(\alpha_1, s_1+1)\beta(\alpha_2, s_2+1)\right] M(u,v)$$

and

(3.9)
$$\int_0^1 \int_0^1 r^{\alpha_1-1}\lambda^{\alpha_2-1}(r^{s_1}(1-\lambda)^{s_2}+(1-r)^{s_1}\lambda^{s_2})[\psi(v)\varphi(u)+\psi(u)\varphi(v)]drd\lambda$$
$$= \left[\frac{\beta(\alpha_2, s_2+1)}{\alpha_1+s_1} + \frac{\beta(\alpha_1, s_1+1)}{\alpha_2+s_2}\right] N(u,v).$$

Hence by substituting values of (3.6)-(3.9) in (3.5), we get (3.1). □

Corollary 3.1. *Under the assumptions of Theorem* 3.1 *we have the following.*

1. *If* $p_1 = p_2 = 1$, *then*

$$\begin{aligned}
&\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(v-u)^{\alpha_1+\alpha_2}}\left[\mathcal{J}_{u_+}^{\alpha_1}\psi(v)\mathcal{J}_{u_+}^{\alpha_2}\varphi(v)+\mathcal{J}_{v_-}^{\alpha_1}\psi(u)\mathcal{J}_{v_-}^{\alpha_2}\varphi(u)\right]\\
&\leq\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\beta(\alpha_1,s_1+1)\beta(\alpha_2,s_2+1)\right]M(u,v)\\
&\quad+\left[\frac{\beta(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\beta(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]N(u,v).
\end{aligned}\tag{3.10}$$

2. *If* $s_1 = s_2 = 1$, *then*

$$\begin{aligned}
&\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(v^{p_1}-u^{p_1})^{\alpha_1}(v^{p_2}-u^{p_2})^{\alpha_2}}\left[\mathcal{J}_{u_+^{p_1}}^{\alpha_1}(\psi\circ\tau_1)(v^{p_1})\mathcal{J}_{u_+^{p_2}}^{\alpha_2}(\varphi\circ\tau_2)(v^{p_2})\right.\\
&\quad\left.+\mathcal{J}_{v_-^{p_1}}^{\alpha_1}(\psi\circ\tau_1)(u^{p_1})\mathcal{J}_{v_-^{p_2}}^{\alpha_2}(\varphi\circ\tau_2)(u^{p_2})\right]\\
&\leq\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\beta(\alpha_1,2)\beta(\alpha_2,2)\right]M(u,v)\\
&\quad+\left[\frac{\beta(\alpha_1,2)}{\alpha_2+1}+\frac{\beta(\alpha_2,2)}{\alpha_1+1}\right]N(u,v).
\end{aligned}\tag{3.11}$$

3. *If* $p_1 = p_2 = 1$ *and* $s_1 = s_2 = 1$, *then*

$$\begin{aligned}
&\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(v-u)^{\alpha_1+\alpha_2}}\left[\mathcal{J}_{u_+}^{\alpha_1}\psi(v)\mathcal{J}_{u_+}^{\alpha_2}\varphi(v)+\mathcal{J}_{v_-}^{\alpha_1}\psi(u)J_{v_-}^{\alpha_2}\varphi(u)\right]\\
&\leq\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\beta(\alpha_1,2)\beta(\alpha_2,2)\right]M(u,v)\\
&\quad+\left[\frac{\beta(\alpha_1,2)}{\alpha_2+1}+\frac{\beta(\alpha_2,2)}{\alpha_1+1}\right]N(u,v).
\end{aligned}\tag{3.12}$$

4. *If* $p_1 = p_2 = 1$, $s_1 = s_2 = 1$ *and* $\alpha_1 = \alpha_2 = 1$, *then*

$$\frac{2}{(v-u)^2}\int_u^v\psi(x)dx\int_u^v\varphi(y)dy\leq\frac{M(u,v)+N(u,v)}{2}.\tag{3.13}$$

Theorem 3.2. *Let* $s_1, s_2 \in (0,1]$, $r, \lambda \in [0,1]$ *and* $p_1, p_2 \in \mathbb{R}\setminus\{0\}$. *Let* $\psi, \varphi : \mathcal{K} \subset \mathbb{R}_+ \to \mathbb{R}_+$, *where* $\mathcal{K}$ *is an interval, be* (s_1, p_1)*-convex function and* (s_2, p_2)*-convex function, respectively, such that* $\psi, \varphi \in$

$L[u,v]$ for $u,v\in\mathcal{K}^o$, where $\mathcal{K}^o$ is the interior of $\mathcal{K}$, with $u<v$ and $\alpha_1,\alpha_2>0$. Then the following inequalities for fractional integrals hold:

(3.14)
$$\begin{aligned}
&2^{s_1+s_2}\psi\left(\left(\frac{u^{p_1}+v^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\varphi\left(\left(\frac{u^{p_2}+v^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\le \frac{\Gamma(\alpha_1\alpha_2+1)}{(v^{p_1}-u^{p_1})^{\alpha_1}(v^{p_2}-u^{p_2})^{\alpha_2}}\Big[\mathcal{J}^{\alpha_1}_{u^{p_1}_+}(\psi\circ\tau_1)(v^{p_1})\mathcal{J}^{\alpha_2}_{u^{p_2}_+}(\varphi\circ\tau_2)(v^{p_2})\\
&\quad+\mathcal{J}^{\alpha_1}_{v^{p_1}_-}(\psi\circ\tau_1)(u^{p_1})\mathcal{J}^{\alpha_2}_{v^{p_2}_-}(\varphi\circ\tau_2)(u^{p_2})\Big]\\
&\quad+\alpha_1\alpha_2\left[\frac{\beta(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\beta(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]M(u,v)\\
&\quad+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\beta(\alpha_1,s_1+1)\beta(\alpha_2,s_2+1)\right]N(u,v),
\end{aligned}$$

where $\tau_1(x)=x^{\frac{1}{p_1}}$ for all $x\in[u^{p_1},v^{p_1}]$, $\tau_2(y)=y^{\frac{1}{p_2}}$ for all $y\in[u^{p_2},v^{p_2}]$ and $M(u,v)=\psi(u)\varphi(u)+\psi(v)\varphi(v)$, $N(u,v)=\psi(u)\varphi(v)+\psi(v)\varphi(u)$.

Proof. Since ψ is (s_1,p_1)-convex and φ is (s_2,p_2)-convex, then

$$\begin{aligned}
&\psi\left(\left(\frac{u^{p_1}+v^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\varphi\left(\left(\frac{u^{p_2}+v^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\le\psi\left(\left(\frac{ru^{p_1}+(1-r)v^{p_1}}{2}+\frac{rv^{p_1}+(1-r)u^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\\
&\quad\times\varphi\left(\left(\frac{\lambda u^{p_2}+(1-\lambda)v^{p_2}}{2}+\frac{\lambda v^{p_2}+(1-\lambda)u^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)
\end{aligned}\tag{3.15}$$

$$
\begin{aligned}
&\leq \frac{1}{2^{s_1+s_2}}\left[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\right] \\
&\quad\times\left[\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)+\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\right] \\
&= \frac{1}{2^{s_1+s_2}}\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)\Big] \\
&\leq \frac{1}{2^{s_1+s_2}}\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\Big] \\
&\quad+\frac{1}{2^{s_1+s_2}}\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)\Big] \\
&\leq \frac{1}{2^{s_1+s_2}}\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\Big] \\
&\quad+\frac{1}{2^{s_1+s_2}}\Big[(r^{s_1}\psi(u)+(1-r)^{s_1}\psi(v))((1-\lambda)^{s_2}\varphi(u)+\lambda^{s_2}\varphi(v)) \\
&\quad+(r^{s_1}\psi(v)+(1-r)^{s_1}\psi(u))((1-\lambda)^{s_2}\varphi(v)+\lambda^{s_2}\varphi(u))] \\
&= \frac{1}{2^{s_1+s_2}}\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right) \\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\Big] \\
&\quad+\frac{1}{2^{s_1+s_2}}\Big[\left(r^{s_1}(1-\lambda)^{s_2}+(1-r)^{s_1}\lambda^{s_2}\right)(\psi(u)\varphi(u) \\
&\quad+\psi(v)\varphi(v))+\left((1-r)^{s_1}(1-\lambda)^{s_2}+r^{s_1}\lambda^{s_2}\right)(\psi(u)\varphi(v)+\psi(v)\varphi(u))\Big].
\end{aligned}
$$

Summing up, we have

$$
\begin{aligned}
&2^{s_1+s_2}\psi\left(\left(\frac{u^{p_1}+v^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\varphi\left(\left(\frac{u^{p_2}+v^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\leq\Big[\psi\left((ru^{p_1}+(1-r)v^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda u^{p_2}+(1-\lambda)v^{p_2})^{\frac{1}{p_2}}\right)\\
&\quad+\psi\left((rv^{p_1}+(1-r)u^{p_1})^{\frac{1}{p_1}}\right)\varphi\left((\lambda v^{p_2}+(1-\lambda)u^{p_2})^{\frac{1}{p_2}}\right)\Big]\\
&\quad+\Big[(r^{s_1}(1-\lambda)^{s_2}+(1-r)^{s_1}\lambda^{s_2})(\psi(u)\varphi(u)+\psi(v)\varphi(v))\\
&\quad+((1-r)^{s_1}(1-\lambda)^{s_2}+r^{s_1}\lambda^{s_2})\left(\psi(u)\varphi(v)+\psi(v)\varphi(u)\right)\Big].
\end{aligned}
\tag{3.16}
$$

Multiplying the above inequality (3.16) by $r^{\alpha_1-1}\lambda^{\alpha_2-1}$ and integrating with respect to r and λ over $[0,1]\times[0,1]$, we get (3.14). Hence the proof is completed. □

Corollary 3.2. *Under the assumptions of Theorem* 3.2 *we have the following.*

1. *If* $p_1=p_2=1$, *then*

(3.17)

$$
\begin{aligned}
&2^{s_1+s_2}\psi\left(\left(\frac{u+v}{2}\right)\right)\varphi\left(\left(\frac{u+v}{2}\right)\right)\\
&\leq\frac{\Gamma(\alpha_1\alpha_2+1)}{(v-u)^{\alpha_1+\alpha_2}}\left[\mathcal{J}^{\alpha_1}_{u+}\psi(v)\mathcal{J}^{\alpha_2}_{u+}\varphi(v)+\mathcal{J}^{\alpha_1}_{v-}\psi(u)\mathcal{J}^{\alpha_2}_{v-}\varphi(u)\right]\\
&\quad+\alpha_1\alpha_2\left[\frac{\beta(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\beta(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]M(u,v)\\
&\quad+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\beta(\alpha_1,s_1+1)\beta(\alpha_2,s_2+1)\right]N(u,v).
\end{aligned}
$$

2. *If* $s_1 = s_2 = 1$, *then*

(3.18)
$$2^{s_1+s_2}\psi\left(\left(\frac{u^{p_1}+v^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\varphi\left(\left(\frac{u^{p_2}+v^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)$$
$$\leq \frac{\Gamma(\alpha_1\alpha_2+1)}{(v^{p_1}-u^{p_1})^{\alpha_1}(v^{p_2}-u^{p_2})^{\alpha_2}}\Big[\mathcal{J}^{\alpha_1}_{u_+^{p_1}}(\psi\circ\tau_1)(v^{p_1})\mathcal{J}^{\alpha_2}_{u_+^{p_2}}(\varphi\circ\tau_2)(v^{p_2})$$
$$+\mathcal{J}^{\alpha_1}_{v_-^{p_1}}(\psi\circ\tau_1)(u^{p_1})\mathcal{J}^{\alpha_2}_{v_-^{p_2}}(\varphi\circ\tau_2)(u^{p_2})\Big]$$
$$+\alpha_1\alpha_2\left[\frac{\beta(\alpha_1,2)}{\alpha_2+1}+\frac{\beta(\alpha_2,2)}{\alpha_1+1}\right]M(u,v)$$
$$+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\beta(\alpha_1,2)\beta(\alpha_2,2)\right]N(u,v).$$

3. *If* $p_1 = p_2 = 1$ *and* $s_1 = s_2 = 1$, *then*

$$\begin{aligned}
&4\psi\left(\left(\frac{u+v}{2}\right)\right)\varphi\left(\left(\frac{u+v}{2}\right)\right)\\
&\leq \frac{\Gamma(\alpha_1\alpha_2+1)}{(v-u)^{\alpha_1+\alpha_2}}\left[\mathcal{J}^{\alpha_1}_{u_+}\psi(v)\mathcal{J}^{\alpha_2}_{u_+}\varphi(v)+\mathcal{J}^{\alpha_1}_{v_-}\psi(u)\mathcal{J}^{\alpha_2}_{v_-}\varphi(u)\right]\\
&+\alpha_1\alpha_2\left[\frac{\beta(\alpha_1,2)}{\alpha_2+1}+\frac{\beta(\alpha_2,2)}{\alpha_1+1}\right]M(u,v)\\
&+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\beta(\alpha_1,2)\beta(\alpha_2,2)\right]N(u,v).
\end{aligned}\tag{3.19}$$

4. *If* $p_1 = p_2 = 1$, $s_1 = s_2 = 1$ *and* $\alpha_1 = \alpha_2 = 1$, *then*

$$\begin{aligned}
&4\psi\left(\left(\frac{u+v}{2}\right)\right)\varphi\left(\left(\frac{u+v}{2}\right)\right)\\
&\leq \frac{2}{(v-u)^2}\int_u^v\psi(x)dx\int_u^v\varphi(y)dy+\frac{M(u,v)+N(u,v)}{2}.
\end{aligned}\tag{3.20}$$

Conclusion

In Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4, some Hermite-Hadamard type inequalities for (s,p)-convex functions in fractional form are obtained. Remark 2 provides some previous results for convex functions, p-convex functions, s-convex functions and harmonically convex functions. Similary, in section 3, we gave some Hermite-Hadamard type inequalities for product of two (s, p)-convex functions in fractional form and also gave some Hermite-Hadamard type inequalities for product of two p-convex functions, two s-convex functions and two convex functions. All the results given in this paper can be extended for (p, h)-convex functions as well.

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