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COMMON FIXED POINT FOR A SEQUENCE OF MULTIVALUED (G,θ)-PREŠIĆ TYPE MAPS IN SYMMETRIC SPACES ENDOWED WITH A GRAPH

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Abstract. We have obtained some new common fixed point results for a sequence of multivalued (G,θ)-Prešić type mappings in a symmetric space equipped with a graph. An example of application is provided. Some results from the literature are extended or improved.

1. Introduction and preliminaries

Since its proof by Banach in 1922, the fixed point contraction principle has been the subject of intensive research for the quest of generalization and/or improvements. The extensions obtained so far concern either the structure of the metric space in consideration or the involved self-mapping. We first quote Wilson [26] who introduced in 1931 the concept of a symmetric space (or semi-metric) as a generalization of a metric space. In a symmetric space, the triangular inequality is missing. Yet, several fixed point results have been obtained in the setting of such spaces. The recent paper [5] gives a unified approach to the theory. We also refer to [13], [15], [19], [23] and references therein.

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In 2004, Ran and Reurings [22] proved an existence and uniqueness result for some continuous order preserving mappings that satisfy the contraction condition only for comparable ordered pairs in a complete metric space.

In 2008, another important direction of research was initiated by Jachymski [12] who replaced the order contraction condition by another one on the edges of the graph. For this purpose, he employed the concept of G-contraction. Then, many authors extended the Banach G-contraction in different ways. We refer to [1]-[4], [6], [9], and references therein.

In connection with the recent results in the literature, we are interested in Prešić's results [20, 21, 25] for mappings defined from the cartesian product X^k, for some positive integer k into the metric space X. Prešić proved the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space, k a positive integer, and $T : X^k \to X$ a contraction on a product of metric spaces:*

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{i=k} \alpha_i d(x_i, x_{i+1}),$$

for every $x_1, x_2, \ldots, x_{k+1} \in X$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are nonnegative constants such that $\alpha_1 + \alpha_2 + \ldots + \alpha_k < 1$. Then there exists a unique point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $(x_n)_n$ is convergent and $\lim_{n\to\infty} x_n = T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_n)$.

We also mention Jleli and Samet results in [14], where the definition of a θ-contraction is introduced and where a generalization of the Banach contraction principle is proved. They denoted by Θ the set of all functions $\theta : [0, \infty) \to [1, \infty)$ which satisfy the following conditions:

(Θ_1) θ is non-decreasing,

(Θ_2) for each sequence $(t_n)_n \subset [0, \infty)$, $\lim\limits_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim\limits_{n\to\infty} t_n = 0^+$,

(Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim\limits_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Our aim in this paper is to prove a common fixed point theorem for a sequence of multivalued (G, θ)-Prešić type mappings in a symmetric space endowed with a graph. The main existence theorem is presented in Section 2 together with an example of application. Three corollaries are then given in Section 3. We first collect some basic notions and primary results we need to develop subsequent results. $\mathbb{N}$ will refer to the set of positive integers.

Definition 1.1. [26] For a nonempty set X, a function $D : X \times X \to [0, \infty)$ is said to be semi-metric on X if for any $x_1, x_2 \in X$, the following conditions are satisfied:

(W_1) $D(x_1, x_2) = 0$ if and only if $x_1 = x_2$,

(W_2) $D(x_1, x_2) = D(x_2, x_1)$.

(X, D) is known as a symmetric space. For a semi-metric D on a set X with $r^* > 0$ and $x_0 \in X$, we set

$$B(x_0, r^*) = \{x \in X : D(x_0, x) < r^*\}.$$

The topology $\tau_D = \{U\}$ on (X, D) is defined as follows: for every $x_0 \in U$, there is a $r^* > 0$ with $B(x_0, r^*) \subset U$. Then U is called an open neighborhood of x_0. Note that a symmetric space need not be a Hausdorff space [7]. However the notion of D-convergence of sequences can be defined as in metric spaces:

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} D(x_n, x) = 0.$$

For a symmetric space (X, D), some changes are used with regards to the missing triangle inequality. Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be given sequences in a symmetric space (X, D) and x_0, y_0 elements of X. Consider the following properties:

(W_3) $\lim\limits_{n\to\infty} D(x_n, x_0) = 0$ and $\lim\limits_{n\to\infty} D(x_n, y_0) = 0$ imply $x_0 = y_0$,

(W_4) $\lim\limits_{n\to\infty} D(x_n, x_0) = 0$ and $\lim\limits_{n\to\infty} D(x_n, y_n) = 0$ imply $\lim\limits_{n\to\infty} D(y_n, x_0) = 0$,

(W) $\lim\limits_{n\to\infty} D(x_n, y_n) = 0$ and $\lim\limits_{n\to\infty} D(y_n, z_n) = 0$ imply $\lim\limits_{n\to\infty} D(x_n, z_n) = 0$,

(JMS) $\lim\limits_{n\to\infty} D(x_n, y_n) = 0$ and $\lim\limits_{n\to\infty} D(y_n, z_n) = 0$ implies $\lim\limits_{n\to\infty} D(x_n, z_n) \neq \infty$,

(CC) $\lim\limits_{n\to\infty} D(x_n, x_0) = 0$ imply $\lim\limits_{n\to\infty} D(x_n, y_0) = D(x_0, y_0)$,

(SC) $\lim\limits_{n\to\infty} D(x_n, x_0) = 0$ imply $\overline{\lim\limits_{n\to\infty}} D(x_n, y_0) \leq D(x_0, y_0)$.

Remark 1. *Properties (W_3) and (W_4) were given in Wilson* [26], *(W) in Miheţ* [16], *(JMS) was introduced by Jachymski et al.* [13], *(CC) in Cho et al.* [8] *(see, also* [7]*), and (SC) was suggested by Aranelovic and Keckic in* [5]. *If the topology τ_D is a Hausdorff space induced by the semi-metric, then (W_3) is satisfied.*

Definition 1.2. [11, 13] In a symmetric space (X, D), a given sequence (x_n) is said to be a D-Cauchy sequence whenever for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $D(x_n, x_m) < \varepsilon$, for $m, n \geq n_\varepsilon$. The symmetric space (X, D) is called D-Cauchy complete if every D-Cauchy sequence $(x_n)_n$ in X is D-convergent.

Definition 1.3. [10] A symmetric space (X, D) is called D-complete if $\sum\limits_{n=1}^{\infty} D(x_{n+1}, x_n) < +\infty$ implies the D-convergence of (x_n).

Definition 1.4. [17] Assume that (X, D) is a symmetric space and Y is a nonempty subset of X. We say that

(i) Y is D-closed if $Y = \overline{Y}$, where

$$\overline{Y} = \{x \in X : D(x, Y) = 0\} \text{ and } D(x, Y) = \inf\{D(x, y) : y \in Y\}.$$

(ii) Y is D-bounded if $\delta_D(Y) < +\infty$, where

$$\delta_D(Y) = \sup\{D(y_1, y_2) : y_1, y_2 \in Y\}.$$

The definition given below is that of a generalized Hausdorff distance.

Definition 1.5. [17] Assume that (X, D) is a given symmetric space and $C_D(X)$ is a nonempty collection of closed subsets of X and $CB_D(X)$ is a nonempty collection of closed bounded subsets of X. For X_1, X_2 in $CB_D(X)$, define

$$\delta_D(X_1, X_2) = \sup\{D(x_1, X_2) : x_1 \in X_1\}$$

and

$$H_D(X_1, X_2) = \max\{\delta_D(X_1, X_2), \delta_D(X_2, X_1)\}.$$

H_D is known as the Pompeiu-Hausdorff semi-metric on $CB_D(X)$.

The second part of this introduction concerns graph and fixed point theories. A graph G is a pair (V, E), where V is a nonempty set and E is a subset of a given binary relation on V. Elements of E are called edges and are denoted $E(G)$ and elements of V, denoted $V(G)$, are called vertices. If the direction is imposed on E, that is when the edges are directed, we get a directed graph, shortened as a digraph. Suppose that any two vertices of G cannot be connected by more than one edge. Then G is denoted by the pair $(V(G), E(G))$. If there exists an edge between each pair of vertices, then the graph G is said to be complete. For any two vertices x and y, we say that there is a path in G between x to y if there exists a finite sequence $(x_n)_n$, $n \in \{1, 2, \ldots, k\}$ of vertices such that $x = x_1, x_2, \ldots, x_k = y$ and $(x_{n-1}, x_n) \in E(G)$, where $n \in \{1, 2, \ldots, k\}$. G is said to be connected if there is a path connecting every two vertices. $\widetilde{G}$ stands for the undirected graph of G when the direction of edges is ignored. G is called weakly connected when $\widetilde{G}$ is connected. G^{-1} refers to the reverse direction graph of G. We have

$$E(G^{-1}) = \{(x_1, x_2) \in X \times X : (x_2, x_1) \in E(G)\}.$$

Let G be a directed graph with symmetric edges and $\widetilde{G}$ its undirected graph. Then

$$E(\widetilde{G}) = E(G^{-1}) \cup E(G).$$

The following definition of G-H_d-Prešić contraction is given by Shahzad and Shukla [24].

Definition 1.6. Let (X, d) be a metric space endowed with a graph, k a positive integer, and $T : X^k \to CB(X)$ a mapping. Suppose that for every path $(x_i)_{i=1}^{i=k+1}$ of $k+1$ vertices in G, the following conditions are verified:
(GP1) There exist nonnegative constants α_i's such that $\sum_{i=1}^{i=k} \alpha_i < 1$ and

$$H(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \sum_{i=1}^{i=k} \alpha_i d(x_i, x_{i+1}).$$

(GP2) If $x_{k+1} \in T(x_1, x_2, \dots, x_k)$ and $x_{k+2} \in T(x_2, x_3, \dots, x_{k+1})$ are such that $d(x_{k+1}, x_{k+2}) < \max\{d(x_i, x_{i+1}) : i = 1, 2, \dots, k\}$, then $(x_{k+1}, x_{k+2}) \in E(G)$.
Then the mapping T is called a set-valued G-Prešić operator.

Throughout this paper, we assume that (X, D) is a symmetric space. Let $G = (V(G), E(G))$ be a directed graph without parallel edges such that $V(G) = X$ and let the diagonal of $X \times X$ be contained in $E(G)$.

2. Main result

The authors of [2, 3] discussed Definition 1.6 and found that condition (GP2) is not appropriate. Here is a counter-example inspired from [4, Example 1.6].

Example 2.1. *Let $X = \mathbb{R}^2$ endowed with the Euclidean distance d and $k = 2$. The graph $G = (V(G), E(G))$ is defined by $V(G) = X$ and*

$$E(G) = \{((u_1, u_2), (v_1, v_2)) \in \mathbb{R}^2 \times \mathbb{R}^2, u_1 + u_2 \leq v_1 + v_2\}.$$

Let $T : X^2 \to CB(X)$ be a multivalued mappings defined by

$$T(x, y) = \{(u, v) \in \mathbb{R}^2, \sqrt{u^2 + v^2} \leq 5\}, \forall x, y \in \mathbb{R}^2.$$

Since T *is a constant multivalued mapping, then it is a set-valued Prešić type contraction. Therefore,* T *must be a set-valued* G*-Prešić operator but the condition (GP2) fails. Indeed, if* $(x_1, x_2, x_3, x_4) = ((0,3),(3,3),(4,2),(3,1))$*, then* $(x_i)_{i=1}^{i=3}$ *is a path of* 3 *vertices in* G*,* $x_3 \in T(x_1, x_2)$*,* $x_4 \in T(x_2, x_3)$ *and* $\sqrt{2} = d(x_3, x_4) < \max\{d(x_1, x_2), d(x_2, x_3)\} = \max\{3, \sqrt{2}\} = 3$ *while* $(x_3, x_4) \notin E(G)$*.*

Therefore we suggest the following definition as an alternative for Definition 1.6.

Definition 2.1. Let (X, D) be a symmetric space endowed with a graph G, k a positive integer, and $(T_n)_n$ a sequence of multivalued mappings of X^k into $C_D(X)$. Then the sequence $(T_n)_n$ is called (G, θ)-Prešić sequence if for every path $(x_i)_{i=1}^{i=k+1}$ of $k+1$ vertices in G, the following conditions hold:
(i) if $x_{k+1} \in T_p(x_1, x_2, \ldots, x_k)$, there exists $x_{k+2} \in T_q(x_2, x_3, \ldots, x_{k+1})$ such that $(x_{k+1}, x_{k+2}) \in E(G)$,
(ii) $\theta(D(x_{k+1}, x_{k+2})) \leq \theta(\max\{D(x_i, x_{i+1}) : 1 \leq i \leq k\})^{\lambda}$,
for $p, q = 1, 2, \ldots$, where $\lambda \in (0, 1)$ and $\theta \in \Theta$.

Remark 2. *If* $(T_n)_n$ *is a* (G, θ)*-Prešić sequence, then* $(T_n)_n$ *is both a* (G^{-1}, θ)*-Prešić sequence and a* $(\tilde{G}, \theta)$*-Prešić sequence.*

Let us recall the property $(A^{\star})$ which is similar to the property that was given in [12].

Definition 2.2. Let (X, D) be a symmetric space endowed with a directed graph G. We say that the triplet (X, D, G) has property $(A^{\star})$ if for any sequence $(x_n)_n$ in X, if $x_n \to x$ in τ_D and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

We state and prove the existence results of common fixed points for a (G, θ)-Prešić sequence of multivalued mappings.

Theorem 2.1. *Let (X,D) be a D-complete symmetric space endowed with a directed graph G satisfying (W_4) and such that the triple (X,D,G) has the property $(A^\star)$. Let k be a positive integer and $(T_n)_n$ a (G,θ)-Prešić sequence of multivalued mappings of X^k into $C_D(X)$. Suppose that there exists a path $(x_i)_{i=1}^{i=k+1}$ of $k+1$ vertices in G such that $x_{k+1}\in T_1(x_1,x_2,\dots,x_k)$. Then $(T_n)_n$ has a common fixed point, i.e., there exists $x\in X$ such that $x\in \bigcap\limits_{n\in\mathbb{N}} T_n(x,\dots,x)$.*

Proof.

(a) Construction of a convergent sequence in (X,D) . Suppose that there is a path $(x_i)_{i=1}^{i=k+1}$ of $k+1$ vertices in G such that $x_{k+1}\in T_1(x_1,x_2,\dots,x_k)$. Since $(T_n)_n$ is a (G,θ)-Prešić sequence of multivalued mappings, there exists $x_{k+2}\in T_2(x_2,x_3,\dots,x_{k+1})$ such that $(x_{k+1},x_{k+2})\in E(G)$ and

$$\theta(D(x_{k+1},x_{k+2}))\le \theta(\max\{D(x_i,x_{i+1}):\ 1\le i\le k\})^{\lambda}.$$

Since $(x_i)_{i=2}^{k+2}$ is a path of $k+1$ vertices in G, $x_{k+2}\in T_2(x_2,x_3,\dots,x_{k+1})$, and $(T_n)_{n\in\mathbb{N}}$ is a (G,θ)-Prešić sequence of multivalued mappings, there exists $x_{k+3}\in T_3(x_3,x_4,\dots,x_{k+2})$ such that $(x_{k+2},x_{k+3})\in E(G)$ and

$$\theta(D(x_{k+2},x_{k+3}))\le \theta(\max\{D(x_{i+1},x_{i+2}):\ 1\le i\le k\})^{\lambda}.$$

By induction, we construct a sequence $(x_n)_n$ such that $(x_n,x_{n+1})\in E(G)$, $x_{n+k}\in T_n(x_n,x_{n+1},\dots,x_{n+k-1})$ for all $n\in\mathbb{N}$ and

$$\theta(D(x_{n+k},x_{n+k+1}))\le \theta(\max\{D(x_{n+i-1},x_{n+i}):\ 1\le i\le k\})^{\lambda},\ \forall\, n\in\mathbb{N}. \tag{2.1}$$

Let $\nu=\max\{\theta(D(x_i,x_{i+1}))^{\lambda^{\frac{-i}{k}}}\ :\ 1\le i\le k\}$. We show that

$$\theta(D(x_n,x_{n+1}))\le \nu^{\lambda^{\frac{n}{k}}},\ \forall\, n\in\mathbb{N}. \tag{2.2}$$

By the definition of ν, it is clear that inequality (2.2) holds for $n = 1, \ldots, k$. Let the k inequalities

$$\theta(D(x_n, x_{n+1})) \leq \nu^{\lambda^{\frac{n}{k}}},\ \theta(D(x_{n+1}, x_{n+2})) \leq \nu^{\lambda^{\frac{n+1}{k}}},\ \ldots,\ \theta(D(x_{n+k-1}, x_{n+k})) \leq \nu^{\lambda^{\frac{n+k-1}{k}}}$$

be the induction hypothesis. By (2.1) and the definition of θ, we obtain that for all $n \in \mathbb{N}$

$$\begin{aligned}
\theta(D(x_{n+k}, x_{n+k+1}))) &\leq \theta(\max\{D(x_{n+i-1}, x_{n+i}) :\ 1 \leq i \leq k\})^{\lambda} \\
&= \max\{\theta(D(x_{n+i-1}, x_{n+i})) :\ 1 \leq i \leq k\}^{\lambda} \\
&\leq \max\{\nu^{\lambda^{\frac{n+i-1}{k}}} :\ 1 \leq i \leq k\}^{\lambda} \\
&= (\nu^{\lambda^{\frac{n}{k}}})^{\lambda} \\
&= \nu^{\lambda^{\frac{n+k}{k}}},
\end{aligned}$$

which completes the inductive proof of (2.2). Taking the limit as $n \to \infty$ in (2.2), we get $\theta(D(x_n, x_{n+1})) \to 1$. By definition of θ, $D(x_n, x_{n+1}) \to 0$, as $n \to \infty$. By (Θ_3), there exist $r \in (0, 1)$ and $l \in (0, +\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(D(x_n, x_{n+1})) - 1}{[D(x_n, x_{n+1})]^r} = l.$$

• Let $l < \infty$ and $B = \frac{l}{2}$. By the definition of the limit, there exists a positive integer n_0 such that for all $n \geq n_0$

$$\Big|\frac{\theta(D(x_n, x_{n+1})) - 1}{[D(x_n, x_{n+1})]^r} - l\Big| \leq B.$$

This implies

$$\frac{\theta(D(x_n, x_{n+1})) - 1}{[D(x_n, x_{n+1})]^r} \geq B.$$

Then

$$n[D(x_n, x_{n+1})]^r \leq An[\theta(D(x_n, x_{n+1})) - 1],$$

where $A = \frac{1}{B}$.

• Let $l = \infty$ and $B > 0$ be an arbitrary positive number. By the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\frac{\theta(D(x_n, x_{n+1})) - 1}{[D(x_n, x_{n+1})]^r} \geq B.$$

Then for all $n \geq n_0$

$$n[D(x_n, x_{n+1})]^r \leq An[\theta(D(x_n, x_{n+1})) - 1],$$

where $A = \frac{1}{B}$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[D(x_n, x_{n+1})]^r \leq An[\theta(D(x_n, x_{n+1})) - 1].$$

By (2.2), we obtain

$$n[D(x_n, x_{n+1})]^r \leq An\Big[[\nu^{\lambda^{\frac{n}{k}}} - 1\Big],$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$, we get

$$\lim_{n \to +\infty} n[D(x_n, x_{n+1})]^r = 0.$$

From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$

$$n[D(x_n, x_{n+1})]^r \leq 1.$$

Therefore, for all $n \geq n_1$

$$D(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}.$$

Hence,

$$\sum_{n=1}^{\infty} D(x_n, x_{n+1}) \quad \leq \quad \sum_{n=1}^{n_1 - 1} D(x_n, x_{n+1}) + \sum_{n=n_1}^{\infty} \frac{1}{n^{\frac{1}{r}}} < \infty.$$

Since (X, D) is a D-complete symmetric space, there exists $x \in X$ such that $\lim_{n \to \infty} D(x_n, x) = 0$ in τ_D.

(c) x **is a common fixed point of** $(T_n)_{n\in\mathbb{N}}$. We show that $x \in \bigcap_{n\in\mathbb{N}} T_n(x,\ldots,x)$. Suppose that $x \notin \bigcap_{n\in\mathbb{N}} T_n(x,\ldots,x)$, i.e., there exists $m \in \mathbb{N}$ such that $x \notin T_m(x,\ldots,x)$. By Property $(A^\star)$, $(x_n,x) \in E(G)$, for each $n \in \mathbb{N}$. Since $(T_n)_n$ is a (G,θ)-Prešić sequence of multivalued mappings and $x_{n+k} \in T_{n+k}(x_n, x_{n+1},\ldots,x_{n+k-1})$, there exists $y_n^1 \in T_m(x_{n+1},x_{n+2},\ldots,x_{n+k-1},x)$ such that $(x_{n+k},y_n^1) \in E(G)$ and

$$\theta(D(x_{n+k},y_n^1)) \le \theta(\max\{D(x_n,x_{n+1}),D(x_{n+1},x_{n+2}),\ldots,D(x_{n+k-1},x)\})^\lambda.$$

Taking the limit as $n \to \infty$, we obtain $\lim_{n\to+\infty} \theta(D(x_{n+k},y_n^1)) = 1$. By definition of θ, $D(x_{n+k},y_n^1) \to 0$, as $n \to \infty$. By (W_4), $D(y_n^1,x) \to 0$, as $n \to \infty$.
Since $(T_n)_n$ is a (G,θ)-Prešić sequence of multivalued mappings and

$$y_n^1 \in T_m(x_{n+1},x_{n+2},\ldots,x_{n+k-1},x),$$

there exists $y_n^2 \in T_m(x_{n+2},x_{n+3},\ldots,x_{n+k-1},x,x)$ such that $(y_n^2,y_n^1) \in E(G)$ and

$$\theta(D(y_n^2,y_n^1)) \le \theta(\max\{D(x_{n+1},x_{n+2}),D(x_{n+2},x_{n+3}),\ldots,D(x_{n+k-1},x),D(x,x)\})^\lambda.$$

As $n \to \infty$, we obtain $\lim_{n\to+\infty} \theta(D(y_n^2,y_n^1)) = 1$. By definition of θ, $D(y_n^2,y_n^1) \to 0$, as $n \to \infty$. By (W_4), $D(y_n^2,x) \to 0$, as $n \to \infty$. We repeat the process until we arrive at $y_n^{k-1} \in T_m(x_{n+k-1},x,\ldots,x)$ and $D(y_n^{k-1},x) \to 0$, as $n \to \infty$. Since $(T_n)_n$ is a (G,θ)-Prešić sequence of multivalued mappings and $y_n^{k-1} \in T_m(x_{n+k-1},x,\ldots,x)$, there exists $y_n^k \in T_m(x,\ldots,x)$ such that $(y_n^{k-1},y_n^k) \in E(G)$ and

$$\theta(D(y_n^{k-1},y_n^k)) \le \theta(\max\{D(x_{n+k-1},x),D(x,x),\ldots,D(x,x)\})^\lambda.$$

As $n \to \infty$, we get $\lim_{n\to+\infty} \theta(D(y_n^{k-1},y_n^k)) = 1$. By definition of θ, $D(y_n^{k-1},y_n^k) \to 0$ as $n \to \infty$. Using (W_4), we get $D(y_n^k,x) \to 0$, as $n \to \infty$ which in turn implies $D(x,T_m(x,\ldots,x)) = 0$. Since $T(x,\ldots,x)$ is closed in τ_D, we deduce that $x \in T_m(x,\ldots,x)$ which is a contradiction. Hence $x \in \bigcap_{n\in\mathbb{N}} T_n(x,\ldots,x)$ which shows that x is a common fixed point of $(T_n)_n$. □

Example 2.2. *Let* $X = \{\frac{1}{2^i} : \ i \in \mathbb{N}\} \cup \{0\}$ *be the set with the semi-metric* $D : X \times X \to [0, +\infty)$ *defined by*

$$D(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{1}{2^i}, & \text{if } x = \frac{1}{2^i}, y = 0 \ \text{ or } \ x = 0, y = \frac{1}{2^i}, \\ \frac{1}{2^{i+j}}, & \text{if } x = \frac{1}{2^i}, y = \frac{1}{2^j} \ \text{ or } \ x = \frac{1}{2^j}, y = \frac{1}{2^i}\cdot \end{cases}$$

Note that the function D *is not a metric. Indeed, for* $j > i+1$*, we have*

$$\frac{1}{2^i} = D\left(\frac{1}{2^i}, 0\right) > D\left(0, \frac{1}{2^j}\right) + D\left(\frac{1}{2^j}, \frac{1}{2^i}\right) = \frac{2^i+1}{2^{i+j}}\cdot$$

The graph $G = (V(G), E(G))$ *is defined by* $V(G) = X$ *and*

$$E(G) = \{(x,x), x \in X\} \cup \left\{ \left(\frac{1}{2^i}, \frac{1}{2^{i+1}}\right), \left(\frac{1}{2^i}, 0\right) : \ i \in \mathbb{N} \right\}.$$

Let $\theta(t) = e^{t\sqrt{t}}$ *and* $\lambda = \frac{1}{2}$*, and* $T_n : X^2 \to C_D(X)$ *be a sequence of multivalued mappings defined by*

$$T_n(x,y) = \begin{cases} \{x\}, & \text{if } \ x = y \in \{0, \frac{1}{2}\}, \\ \left\{\frac{1}{2^{i+2}}, \frac{1}{2^{i+4}}, \dots, \frac{1}{2^{i+4+n}}\right\}, & \text{if } \ x = \frac{1}{2^i}, y = \frac{1}{2^{i+1}}, \\ \left\{\frac{1}{2^2}\right\}, & \text{if } \ \text{otherwise.} \end{cases}$$

We claim that $(T_n)_n$ *is* (G, θ)*-Prešić sequence. We distinguish four cases:*

- *Case 1. If* $(x_1, x_2, x_3) = \left(\frac{1}{2^i}, \frac{1}{2^{i+1}}, \frac{1}{2^{i+2}}\right)$*, then*

(i) $x_3 = \frac{1}{2^{i+2}} \in T_p(x_1, x_2) = \{\frac{1}{2^{i+2}}, \frac{1}{2^{i+4}}, \dots, \frac{1}{2^{i+4+p}}\}$ *and*
$T_q(x_2, x_3) = \{\frac{1}{2^{i+3}}, \frac{1}{2^{i+4}}, \dots, \frac{1}{2^{i+5+q}}\}$*. Let* $x_4 = \frac{1}{2^{i+3}}$*, then* $(x_3, x_4) \in E(G)$ *and*
(ii) $e^{\frac{1}{2^{2i+5}}\sqrt{\frac{1}{2^{2i+5}}}} \leq e^{\frac{1}{2^{2i+2}}\sqrt{\frac{1}{2^{2i+1}}}}$.

- *Case 2. If* $(x_1, x_2, x_3) = \left(\frac{1}{2^i}, \frac{1}{2^i}, \frac{1}{2^2}\right)$*,* $i \neq 1$*, then*

(i) $x_3 = \frac{1}{2^2} \in T_p(x_1, x_2) = \{\frac{1}{2^2}\}$ *and* $T_q(x_2, x_3) = \{\frac{1}{2^2}\}$*. Let* $x_4 = \frac{1}{2^2}$*, then* $(x_3, x_4) \in E(G)$ *and*
(ii) $1 \leq e^{\frac{1}{2^{i+3}}\sqrt{\frac{1}{2^{i+2}}}}$.

- *Case 3. If* $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$*, then*

(i) $x_3 = \frac{1}{2} \in T_p(x_1, x_2) = \{\frac{1}{2}\}$ *and* $T_q(x_2, x_3) = \{\frac{1}{2}\}$*. Let* $x_4 = \frac{1}{2}$*, then* $(x_3, x_4) \in E(G)$

and

(ii) $1 \leq 1$.

• *Case 4. If* $(x_1, x_2, x_3) = (0, 0, 0)$, *then*

(i) $x_3 = 0 \in T_p(x_1, x_2) = \{0\}$ *and* $T_q(x_2, x_3) = \{0\}$. *Let* $x_4 = 0$, *then* $(x_3, x_4) \in E(G)$ *and*

(ii) $1 \leq 1$.

Appealing to Theorem 2.1, we conclude that $(T_n)_n$ *has a common fixed point. Actually, we have* $0 \in \bigcap_{n\in\mathbb{N}} T_n(0,0)$ *and* $\frac{1}{2} \in \bigcap_{n\in\mathbb{N}} T_n(\frac{1}{2}, \frac{1}{2})$.

3. Consequences

Taking $T_n = T$ for all $n \in \mathbb{N}$ in Theorem 2.1, we obtain the following fixed point result for set-valued (G, θ)-Prešić type which generalizes the results of Prešić [21], Nadler [18], as well as the recent results of [14, 24] and several known results in metric spaces.

Corollary 3.1. *Let* (X, D) *be a* D*-complete symmetric space endowed with a directed graph* G *satisfying* (W_4) *and suppose that the triple* (X, D, G) *has the property* $(A^\star)$. *Let* k *be a positive integer and suppose that the mapping* $T : X^k \to C_D(X)$ *satisfies the following conditions:*
for every path $(x_i)_{i=1}^{i=k+1}$ *of* $k+1$ *vertices in* G, *we have*
(i) if $x_{k+1} \in T(x_1, x_2, \ldots, x_k)$, *then there exists* $x_{k+2} \in T(x_2, x_3, \ldots, x_{k+1})$ *such that* $(x_{k+1}, x_{k+2}) \in E(G)$ *and*
(ii) $\theta(D(x_{k+1}, x_{k+2})) \leq \theta(\max\{D(x_i, x_{i+1}) : 1 \leq i \leq k\})^\lambda$, *where* $\lambda \in (0, 1)$ *and* $\theta \in \Theta$.
If there exists a path $(x_i)_{i=1}^{i=k+1}$ *of* $k+1$ *vertices in* G *such that* $x_{k+1} \in T(x_1, x_2, \ldots, x_k)$, *then* T *has a fixed point, i.e., there exists* $x \in X$ *such that* $x \in T(x, x, \ldots, x)$.

Let $\theta(t) = e^{\sqrt{t}}$ and $\lambda = \sqrt{\alpha}$ in Corollary 3.1. Then we obtain

Corollary 3.2. *Let (X, D) be a D-complete symmetric space endowed with a directed graph G satisfying (W_4) and suppose that the triple (X, D, G) has the property $(A^{\star})$. Let k be a positive integer and suppose that the mapping $T : X^k \to C(X)$ satisfies the following conditions:*
for every path $(x_i)_{i=1}^{i=k+1}$ of $k+1$ vertices in G, we have
(i) if $x_{k+1} \in T(x_1, x_2, \ldots, x_k)$ there exists $x_{k+2} \in T(x_2, x_3, \ldots, x_{k+1})$ such that $(x_{k+1}, x_{k+2}) \in E(G)$ and
(ii) $D(x_{k+1}, x_{k+2}) \leq \alpha \max\{D(x_i, x_{i+1}) : 1 \leq i \leq k\}$, where $\alpha \in (0, 1)$.
If there exists a path $(x_i)_{i=1}^{i=k+1}$ of $k+1$ vertices in G such that $x_{k+1} \in T(x_1, x_2, \ldots, x_k)$, then T has a fixed point.

The following result is an immediate consequence of Corollary 3.2.

Corollary 3.3. *Let (X, D) be a D-complete symmetric space satisfying (W_4), k a positive integer, and $T : X^k \to CB_D(X)$ a mapping satisfying the following condition:*

$$H_D(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \alpha \max\{D(x_i, x_{i+1}) : 1 \leq i \leq k\},$$

for every $x_1, x_2, \ldots, x_{k+1} \in X$, where $\alpha < 1$. Then T has a fixed point.

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