SOLVING THE OPTIMAL CONTROL OF VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATION VIA MÜNTZ POLYNOMIALS

NEDA NEGARCHI [(1)] AND SAYYED YAGHOUB ZOLFEGHARIFAR [(2)]

ABSTRACT. The main goal of the current paper is to present a direct numerical method for solving optimal control problem for systems governed by Volterra-Fredholm integro-differential equation. This method is based upon a new form of orthogonal Müntz-Legendre polynomials, and collocation method to transform the optimal control problem to a nonlinear programming problem with finite-dimensional. The accuracy and efficiency of the proposed method are examined with illustrative examples.

1. INTRODUCTION

One of the most radical theoretical methods in mathematics is optimal control. Its history dates back to the middle of the last century. By introducing optimal control in the 1960s, various approaches have been proposed for solving optimal control problem (OCP). The aim of all these techniques is to provide efficient algorithms for calculating the exact solution. Belbas studied several numerical methods for solving the OCP governed by Volterra integral equations (VIE) [2]. Schmidt has used direct and indirect approaches to calculate the OCP governed by VIE and differential

equations [29] Also, another numerical technique in [24, 8] based on the discretization of the OCP and its transformation into an optimization problem. Among all of the techniques for solving OCPs, orthogonal functions and polynomials have been considered by a large number of researchers. Tohidi and Samadi [33] used Legendre polynomials to solve OCP governed by VIE and integro-differential equations. More research can be found in Razzaghi and Elnagar [25] and Ross and Fahroo [27] and other researchers [11, 15, 17, 31, 34, 16]. This method is based on the approximation of the state and control functions in terms of the basic functions and orthogonal polynomials. In here, new form of the Müntz-Legendre polynomial is introduced, which are a family of orthogonal polynomials. The orthogonal Müntz systems were introduced by the Armenian mathematicians Badalyan [3] and Taslakyan [32]. Next, they were assessed by Mc Carthy, Sayre and Shawye [18] and then reassessed by Borwein and Erdelyi [6]. For more details see [19, 30, 20]. In this work, we consider a class of OCP governed by Volterra-Fredholm integro-differential equation (VFIDE) which is described by the following minimization problem.

Problem 1:

$$Min \quad Q(y, u) = \int_0^T G(t, y(t), u(t))dt,$$

(1.1) $s.t \quad x'(t) = f(t) + \vartheta_1(t)x(t) + \vartheta_2(t)u(t) + \lambda_1 \int_0^t k_1(t, \tau, x(\tau), u(\tau))d\tau +$

$\lambda_2 \int_0^T k_2(t, \tau, x(\tau), u(\tau))d\tau,$

$x(0) = x_0, \quad t \in [0, T],$

where $f(x)$ is continuous and known, λ_1 and λ_2 are real constants. $x(t)$ and $u(t)$ are real-valued function and continuous. Also both belong to Sobolev space $W^{r,\infty}$ with $r \geq 2$ (see [1, 9]).

The OCP is usually expressed by two types of function, namely the state and control functions $(x(t), u(t))$. The control function directs the evolution of the system from

one step to the next, and the state function describes the behavior of the system. In optimal control, the state and control functions are both unknown. Due to the simplicity and efficiency of the orthogonal polynomials, we use the Müntz-Legender polynomials to determine $x(t)$ and $u(t)$.

The rest of work is organized as follows: In section 2, respectively, Müntz-Legendre polynomial, and some of their properties are expressed. In section 3, the description of the method is performed for solving control problem 1. Then, in the next section, the convergence of the method is expressed under several lemma and theorem. Also, the proposed method is applied to some examples to show the accuracy and efficiency of the method in section 5. The conclusion of this work is given in section 6.

2. Orthogonal Müntz-Legendre polynomials

In this research, the Müntz polynomials and their basic properties recalled. Initially, the orthogonal Müntz systems were introduced by the Armenian mathematicians Badalyan [3] and Taslakyan [32]. Next, they were assessed by Mc Carthy, Sayre and Shawyer [18], and then reassessed by Borwein and Erdélyi [6]. For more details see [19, 20, 21, 30].

Let $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct non negative real with $0 \leq \lambda_0 < \lambda_1 < ... \to \infty$ for which the Müntz space $M_n(\Lambda) = span\left\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\right\}$ is dense in $C[0,1]$. The celebrated Müntz theorem asserts that for the sequence $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$, the elements of the form $\sum_{i=0}^{n} a_i x^{\lambda_i}$ as the Müntz polynomials are dense in $L^2[0,1]$ if and only if $\sum_{i=0}^{\infty} \lambda_i^{-1} = \infty$. (see[5, 7]) In the following, the orthogonal Müntz-Legendre polynomials introduced on $[0,1]$ with respect to weight function $w(x) = 1$. In the case $\lambda_i > -1/2$ for all i, and $\lambda_i \neq \lambda_v$ for all $i \neq v$,

$$(2.1) \qquad L_n(x) = \sum_{i=0}^{n} c_{n,i} x^{\lambda_i}, \quad c_{n,i} = \frac{\prod_{v=0}^{n-1} (\lambda_i + \overline{\lambda_v} + 1)}{\prod_{v=0, v \neq i}^{n} (\lambda_i - \lambda_v)}.$$

The Müntz-Legendre polynomials satisfy the following relations: (see [21, 22, 23])

$$(2.2) \qquad xL_j'(x) - xL_{j-1}'(x) = \alpha_j L_j(x) + (1 + \bar{\alpha}_{j-1})L_{j-1}(x), \quad j = 1, 2, 3, \ldots,$$

$$(2.3) \qquad xL_j''(x) = (\alpha_j - 1)L_j'(x) + \sum_{k=0}^{j-1}(\alpha_k + \overline{\alpha}_k + 1)L_k'(x), \quad j = 0, 1, 2, \ldots.$$

3. DESCRIPTION OF METHOD

In this section, we present the discretization process of Problem (1).

Consider the dynamic system of the Problem (1) as follows:

$$x'(t) = f(t) + \vartheta_1(t)x(t) + \vartheta_2(t)u(t)+$$
$$(3.1) \qquad \lambda_1 \int_0^t k_1(t, \tau, x(\tau), u(\tau))d\tau + \lambda_2 \int_0^T k_2(t, \tau, x(\tau), u(\tau))d\tau,$$
$$t \in [0, T].$$

The transformation of an integral differential equation to the integral equation is done by integrating. So, integrating from Eq. (3.1) on $[0, T]$ leads to:

(3.2)

$$x(t) = x_0 + \int_0^t (\vartheta_1(\tau)u(\tau) + \vartheta_2(\tau)u(\tau) + p(\tau))d\tau,$$

$$p(t) = f(t) + \lambda_1 \int_0^t k_1(t, \tau, x(\tau), u(\tau))d\tau + \lambda_2 \int_0^T k_2(t, \tau, x(\tau), u(\tau))d\tau, \qquad t \in [0, T],$$

In the following, the dynamic system is discretized using the set of shifted LGL points $(\xi_i, \ i = 0, 1, .., N)$ as:

(3.3)

$$x(\xi_i) = x_0 + \int_0^{\xi_i} (\vartheta_1(\tau)x(\tau) + \vartheta_2(\tau)u(\tau) + p(\tau))d\tau,$$

$$p(\xi_i) = f(\xi_i) + \lambda_1 \int_0^{\xi_i} k_1(\xi_i, \tau, x(\tau), u(\tau))d\tau + \lambda_2 \int_0^T k_2(\xi_i, \tau, x(\tau), u(\tau))d\tau.$$

Now, using linear transformation $\tau = \hat{\tau}_i(\eta) = \frac{\xi_i}{2}(\eta + 1)$ and $\tau = \tilde{\tau}_i(\eta) = \frac{T}{2}(\eta + 1)$ transform the interavels $[0, T]$ and $[0, \xi_i]$ into $[-1, 1]$.

Then, the Legendre-Gauss-Lobatto (LGL) quadrature applied for approximating Eq. (3.4) as:

$$x(\xi_i) \approx x_0 + \left(\tfrac{\xi_i}{2}\right) \sum_{j=0}^{N} \omega_j (\vartheta_1(\hat{\tau}_i(\eta_j))\, x(\hat{\tau}_i(\eta_j)) + \vartheta_2(\hat{\tau}_i(\eta_j))\, u(\hat{\tau}_i(\eta_j)) + p(\hat{\tau}_i(\eta_j))),$$

(3.4)
$$p(\xi_i) = f(\xi_i) + \lambda_1\left(\tfrac{\xi_i}{2}\right) \sum_{j=0}^{N} \omega_j k_1(\xi_i, \hat{\tau}_i(\tau_j), x(\hat{\tau}_i(\tau_j)), u(\hat{\tau}_i(\tau_j))) +$$
$$\lambda_2\left(\tfrac{T}{2}\right) \sum_{j=0}^{N} \omega_j k_2(\xi_i, \tilde{\tau}_i(\tau_j), x(\tilde{\tau}_i(\tau_j)), u(\tilde{\tau}_i(\tau_j))),$$

where $\eta_0 = -1$, $\eta_N = 1$ and τ_j, $j = 1, 2, ..., N - 1$ are the LGL nodes and the LGL weights are

$\omega_j = 2/(N(N + 1)(p_N(\xi_j))^2)$, $j = 0, 1, ..., N$. Notice that $p_N(x)$ is the Legendre polynomial of Nth degree.

Also,

(3.5)
$$\int_0^T G(t, x(t), u(t))dt = \left(\tfrac{T}{2}\right) \int_{-1}^{1} G(\tfrac{T}{2}(\xi + 1)), x_N(\tfrac{T}{2}(\xi + 1)), u_N(\tfrac{T}{2}(\xi + 1))d\xi$$
$$\approx \left(\tfrac{T}{2}\right) \sum_{j=0}^{N} \omega_j G(\tfrac{T}{2}(\xi_j + 1), x_N(\tfrac{T}{2})(\xi_j + 1)), u_N(\tfrac{T}{2}(\xi_j + 1)) \quad = \sum_{j=0}^{N} \gamma_j G(\zeta_j, x_N \zeta_j, u_N \zeta_j).$$

Then, nonlinear programming (NLP) problem is given by:

Problem 2:

$$Min \quad J_N(X, U)$$

(3.6)
$$s.t \quad R_i(X, U) + W_i(P) = -x_0,$$
$$H_i(X, U) - \hat{p}_i = -f(\xi_i), \quad i = 0, 1, ..., N,$$

where $X = (\hat{x}_0, \hat{x}_1, ..., \hat{x}_N)$, $U = (\hat{u}_0, \hat{u}_1, ..., \hat{u}_N)$ and $P = (\hat{p}_0, \hat{p}_1, ..., \hat{p}_N)$ are the unknown parameters, and

$$J_N(X,U) := \frac{T}{2} \sum_{j=0}^{N} \omega_j G(\frac{T}{2}(\xi_j+1), x_N(\frac{T}{2}(\xi_j+1)), u_N(\frac{T}{2}(\xi_j+1))) = \sum_{j=0}^{N} \gamma_j G(\zeta_j, \hat{x}_j, \hat{u}_j),$$

$$R_i(X,U) := -x(\xi_i) + (\frac{\xi_i}{2}) \sum_{j=0}^{N} \omega_j(\vartheta_1(\hat{\tau}_i(\eta_j)) \, x(\hat{\tau}_i(\eta_j)) + \vartheta_2(\hat{\tau}_i(\eta_j))u(\hat{\tau}_i(\eta_j)),$$

$$W_i(P) := (\frac{\xi_i}{2}) \sum_{j=0}^{N} \omega_j p(\hat{\tau}_i(\eta_j)),$$

$$H_i(X,U) := \lambda_1(\frac{\xi_i}{2}) \sum_{j=0}^{N} \omega_j k_1(\xi_i, \hat{\tau}_i(\eta_j), x(\hat{\tau}_i(\eta_j)), u(\hat{\tau}_i(\eta_j)))+$$

$$\lambda_2(\frac{T}{2}) \sum_{j=0}^{N} \omega_j k_2(\xi_i, \tilde{\tau}_i(\eta_j), x(\tilde{\tau}_i(\eta_j)), u(\tilde{\tau}_i(\eta_j))).$$

By using the following relaxation the feasibility of NLP problem can be guaranteed.

(3.7)

$$R_i(X,U) + W_i(P) + x_0 \leq (N-1)^{\frac{3}{2}-r}, \qquad H_i(X,U) - \hat{p}_i + f(\xi_i) \leq (N-1)^{\frac{3}{2}-r}.$$

When N tends to infinity and $r \geq 2$, the difference between Eq. (3.8) and the constrains of the NLP problem vanishes [13].

For convenience, $x(t)$ and $u(t)$ considered as follows:

$$(3.8) \qquad x(t) \approx x_N(t) = \sum_{m=0}^{N} x_N(t_m)g_m(t), \quad u(t) \approx u_N(t) = \sum_{m=0}^{N} u_N(t_m)g_m(t).$$

4. Theoretical analysis

In this section, convergence of the proposed method provide for problem (1).

Definition 4.1. [9] A function $\psi : [0,T] \to R$ belong to Sobolev space $W^{z,l}$, if its jth weak derivative $\psi^{(j)}$, lies in $L^l[0,T]$ for all $0 \leq j \leq z$ with the norm $\|\psi\|_{W^{z,l}} = \sum_{j=0}^{z} \|, \psi^{(j)}\|_{L^l}$

where $\|\psi\|_{L^l}$ denotes the usual Lebesgue norm defined for $1 \leq l < \infty$ as follows

$$\|\psi\|_{L^l} = (\int_0^T |\psi(t)|^l dt)^{\frac{1}{l}}$$

Lemma4.1 Given any function $\psi \in W^{z,\infty}$, $t \in [0, T]$ there is a polynomial $g_N(t)$ of degree N or less, such that

$$(4.1) \qquad |\psi(t) - g_N(t)| \leq CC_0 N^{-z}, \quad \forall t \in [0, T],$$

where C is a constant independent of N, z is the order of smoothness of ψ and $C_0 = \|\psi\|_{W^{z,\infty}}$ ($g_N(t)$ with the smallest norm $\|\psi(t) - g_N(t)\|_{L^\infty}$ is called the N th order best polynomial approximation of $\psi(t)$ in the norm of L^∞.)

Proof. Proof. see [9]. □

Theorem 4.1. *Given any feasible solution $(x(t), u(t))$ for problem (1), suppose $x(t), u(t)$ belong to $W^{z,\infty}$ with $z \geq 2$. Then, there is a positive integer N_1 such that for any $N > N_1$, the problem (2) has a feasible solution $(\hat{x}_i, \hat{u}_i)$ such that, the feasible solution satisfies*

$$(4.2) \qquad |x(t_i) - \hat{x}_i| \leq d_1 (N-1)^{1-z}, \; |u(t_i) - \hat{u}_i| \leq d_2 (N-1)^{1-z}, \quad i = 0, ..., N.$$

Note that, t_i are the shifted LGL nodes and $d_1, d_2 > 0$ are constant and independent of N.

Proof. Proof. see [14, 22, 23]. □

In the next theorem, the convergence of the following sequence is discussed.

$$(4.3) \qquad \{(x_N^*(t_i), u_N^*(t_i)), \; 0 \leq i \leq N\}_{N=N\bullet}^{\infty}$$

Theorem 4.2. *Assume that $\{(x_N^*(t_i), u_N^*(t_i)), \; 0 \leq i \leq N\}_{N=N\bullet}^{\infty}$ be a sequence of optimal solutions to the problem (2) . If the function sequence has a subsequence that uniformly converges to the continuous function $\{(p_1(t), p_2(t))\}$ on interval $[0, T]$ then, $\hat{x}(t) = \int_{t_0}^{t} p_1(\nu)d\nu + \hat{x}_0$ and $\hat{u}(t) = \int_{t_0}^{t} p_2(\nu)d\nu + \hat{u}_0$ are the optimal solution to the problem (1).*

Proof. Proof. see [21]. □

5. NUMERICAL EXAMPLES

In this section, three examples are given to demonstrate the applicability efficiency of the Müntz-Legendre collocation method (MLCM). All computations are carried out in Mathematica version 10 software. In order to analyze the errors of the present method, the following notations are introduced

$$(5.1) \quad E^J = |J^* - \bar{J}^*|, \quad \|E_N^x\|_\infty = \max |E_N^x(t_i)| \, and \, \|E_N^u\|_\infty = \max |E_N^u(t_i)|, \quad for$$

$i = 0, 1, ..., N$, where $\{t_i\}_{i=0}^N$ are the shifted LGL nodes.

Example 5.1. *Consider the following OCP governed by VFIDE* [16, 22]

$Min \ J(x, u) = \int_0^1 ((x(t) - e^t)^2 + (u(t) - e^{3t})^2)dt$

$s.t \quad x'(t) - \frac{3}{2}x(t) + \frac{1}{2}u(t) - \int_0^t (e^{t-\tau}x^3(\tau))d\tau = 0,$

$x(0) = 1, \quad 0 \le t \le 1.$

Trivially, the optimal value of the cost function is $J^* = 0$. The exact solutions of state and control function are $x^*(t) = e^t$ and $u^*(t) = e^{3t}$. Table 1. shows the numerical results of the proposed method for this example for $N = 4, 8, 13, 20$ with $\alpha = 0.75, 0.9$. Figure 1. (a) and (b) show respectively the approximate and exact solutions of state and control function using mentioned method for $N = 20$, $\alpha = 0.75$.

TABLE 1. Numerical results for Example 1.

	MLCM for $\alpha = 0.9$			MLCM for $\alpha = 0.75$		
N	$\|E_N^x\|_\infty$	$\|E_N^u\|_\infty$	E^J	$\|E_N^x\|_\infty$	$\|E_N^u\|_\infty$	E^J
4	1.23633E-3	4.36113E-2	1.42109E-14	1.34949E-3	1.13054E-1	3.55271E-14
8	1.85662E-4	5.10768E-4	1.33227E-14	1.33029E-4	2.09748E-4	1.50990E-14
13	2.84700E-5	8.30347E-5	1.14908E-14	9.66820E-6	2.85761E-5	1.06581E-14
20	8.21672E-7	2.04315E-6	2.19581E-15	6.62876E-8	1.97616E-7	7.94850E-15
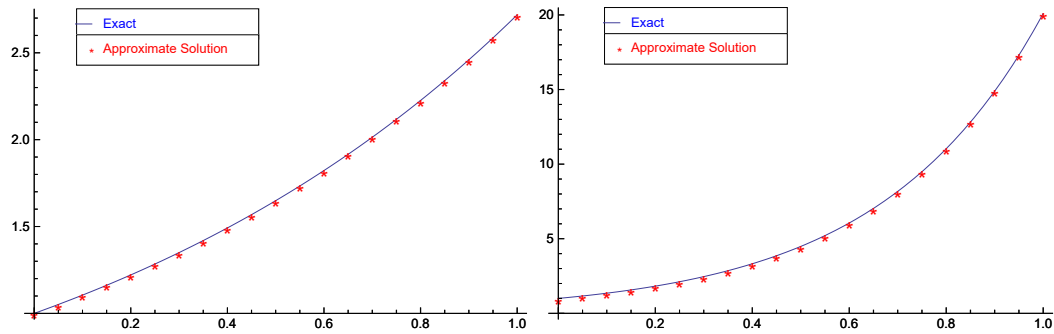
(a) Exact and Approximate solutions $x(t)$.   (b) Exact and Approximate solutions $u(t)$.

FIGURE 1. Numerical results of Example 1 for $N = 20, \alpha = 0.75$.

**Example 5.2.** *Consider the following OCP governed by VFIDE* [16, 22]

$$Min\ J(x,u) = \int_0^1 \left(x(t) - e^{t^2}\right)^2 + (u(t) - (1+2t))^2 dt$$

$$s.t \quad x'(t) + x(t) - u(t) = \int_0^t \left(t(1+2t)e^{\tau(t-\tau)}x(\tau)\right)d\tau,$$

$$x(0) = 1, \quad 0 \le t \le 1.$$

Where $x^*(t) = e^{t^2}$ and $u^*(t) = 1 + 2t$ are the optimal state and control functions and the optimal value of the cost function is $J^* = 0$.

Table 2. shows the numerical results of the proposed method for this example for $N = 4, 8, 13, 20$ with $\alpha = 0.75, 0.9$. Figure 2. (a) and (b) show respectively the approximate and exact solutions of state and control function using mentioned method
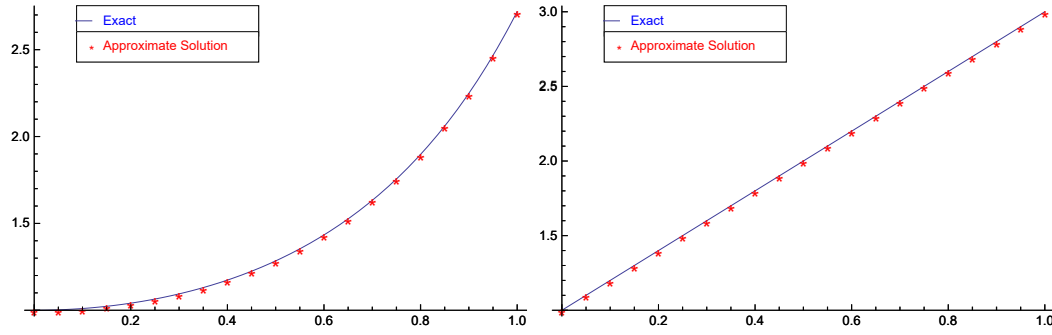
(a) Exact and Approximate solutions of $x(t)$.(b) Exact and Approximate solutions of $u(t)$.

FIGURE 2. Numerical results of Example 2 for $N = 20, \alpha = 0.75$.

for $N = 20$, $\alpha = 0.75$.

TABLE 2. Numerical results for Example 2.

	MLCM for $\alpha = 0.9$			MLCM for $\alpha = 0.75$		
N	$\|E_N^x\|_\infty$	$\|E_N^u\|_\infty$	E^J	$\|E_N^x\|_\infty$	$\|E_N^u\|_\infty$	E^J
4	4.67859E-03	2.92454E-03	4.99600E-15	1.13277E-02	3.92490E-03	5.55112E-15
8	1.76410E-05	3.84116E-04	9.12736E-16	4.47961E-05	2.71483E-04	1.96067E-15
13	8.54377E-07	5.77768E-05	5.60322E-16	1.35063E-07	1.94819E-05	1.18438E-15
20	1.15468E-08	1.65478E-06	1.83958E-16	4.15699E-10	1.32991E-07	1.86017E-16

6. CONCLUSION

In this study, a robust numerical technique was used for solving a class of OCPs governed by VFIDE. The optimal solution of the OCP is determined by a direct method based upon orthogonal function set. The method reduces the OCP to NLP. Illustrative examples have been presented to demonstrate the validity and effectiveness of MLCM, although it should be noted that the choice of coefficient α plays an important role in the results of the MLCM but with increasing N, approximate solutions with different choices of α approach each other. Numerical results given in the

tables show the high precision of the proposed method, with increasing the N, errors are decreased more rapidly. In future research, the method can extend for solving the OCP governed by a system partial differential equation. Also, the method mentioned can develop for solving the OCP with control and state functions of the vector.

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(1) Department of Mathematics , NAJAFABAD BRANCH, Islamic Azad University, NAJAFABAD, IRAN.

Email address: N.negarchi@gmail.com

(2) Department of Building Constructions and Structures, South Ural State University, Chelyabinsk, Russia.

Or

Department of Civil Engineering, Faculty of Engineering , YASOOJ Branch, Islamic Azad University, YASOOJ, IRAN.

Email address: zolfegarifars@susu.ru; syzoalfeghary@gmail.com