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THE CONNECTED RESTRAINED EDGE MONOPHONIC NUMBER OF A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a *connected restrained edge monophonic set* of a graph G is a restrained edge monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected restrained edge monophonic set of G is the *connected restrained edge monophonic number* of G and is denoted by $em_{cr}(G)$. We determine bounds for it and some general properties satisfied by this parameter are studied. For every pair a, b of positive integers with $4 \leq a \leq b$, there is a connected graph G such that $em_r(G) = a$ and $em_{cr}(G) = b$, where $em_r(G)$ is the restrained edge monophonic number of G. Also, if n, d and k are positive integers such that $4 \leq d \leq n - 2$, $k \geq 4$ and $n - d - k + 2 \geq 0$, then there exists a connected graph G of order n, monophonic diameter d and $em_{cr}(G) = k$.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph theoretic terminology, we refer to Harary [1, 4]. The *neighborhood*

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of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. A vertex v is a *semi-extreme vertex* of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex.

A *chord* of a path $u_1, u_2, \ldots, u_k$ in G is an edge u_iu_j with $j \geq i+2$. A u-v path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* if every vertex of G lies on a monophonic path joining some pair of vertices in S, and the minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by $m(G)$. The monophonic number of a graph and its related concepts have been studied by several authors in [2, 3, 5, 8]. A set S of vertices of a graph G is an *edge monophonic set* if every edge of G lies on a $x-y$ monophonic path for some elements x and y in S. The minimum cardinality of an edge monophonic set of G is the *edge monophonic number* of G, denoted by $em(G)$.

A set S of vertices of a graph G is a *restrained monophonic set* if either $S = V$ or S is a monophonic set with the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the *restrained monophonic number* of G, and is denoted by $m_r(G)$. The restrained monophonic number of a graph was introduced and studied in [9]. A set S of vertices of a graph G is a *restrained edge monophonic set* if either $S = V$ or S is an edge monophonic set with the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertices. The minimum cardinality of a restrained edge monophonic set of G is the *restrained edge monophonic number* of G and is denoted by $em_r(G)$. The restrained edge monophonic number of a graph was introduced and studied in [10].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\ \{d_m(v,u) :$

$u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [6, 7].

The following theorems will be used in the sequel.

Theorem 1.1. [10] Each semi-extreme vertex of a graph G belongs to every restrained edge monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is a restrained edge monophonic set, then S is the unique minimum restrained edge monophonic set of G.

Theorem 1.2. [10] If T is a tree of order n with k end-vertices such that $n - k \geq 2$, then $em_r(T) = k$.

Theorem 1.3. [9] Let G be a connected graph with cut-vertices and let S be a restrained monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Main Results

Definition 2.1. A *connected restrained edge monophonic set* of a graph G is a restrained edge monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected restrained edge monophonic set of G is the *connected restrained edge monophonic number* of G and is denoted by $em_{cr}(G)$.

The next theorem follows from the fact that every connected restrained edge monophonic set of G is a restrained edge monophonic set of G and by Theorem 1.1.

Theorem 2.2. Every semi-extreme vertex of a connected graph G belongs to each connected restrained edge monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is a connected restrained edge monophonic set of G, then S is the unique minimum connected restrained edge monophonic set of G.

Corollary 2.3. For the complete graph K_n $(n \geq 2)$, $em_{cr}(K_n) = n$.

The converse of Corollary 2.3 need not be true. For the path P_3, $em_{cr}(P_3) = 3$ and P_3 is not a complete graph.

Example 2.4. For the graph G given in Figure 2.1, it is easy to check that the set of all semi-extreme vertices of G is $S = \{v_1, x\}$. Thus any restrained monophonic set of G contains the vertices v_1 and x. But any path from x to v_1 that contains v_3 is not monophonic. Thus a restrained monophonic set of G contains at least one vertex other than x and v_1. Clearly, $S_1 = \{v_1, x, v_4\}$ is a minimum restrained monophonic set of G and so $m_r(G) = 3$. Since the edge v_2v_6 does not lie on any $u-v$ monophonic path for some $u, v \in S_1$, S_1 is not a restrained edge monophonic set of G. It is easily seen that $S_2 = S_1 \cup \{v_6\}$ is a minimum restrained edge monophonic set of G and so $em_r(G) = 4$. Since the subgraph induced by S_2 is not connected, S_2 is not a connected restrained edge monophonic set of G. It is easy to verify that $S_2 \cup \{v_5\}$ is a minimum connected restrained edge monophonic set of G so that $em_{cr}(G) = 5$. Thus the restrained monophonic number, the restrained edge monophonic number and the connected restrained edge monophonic number of a graph need not be equal.

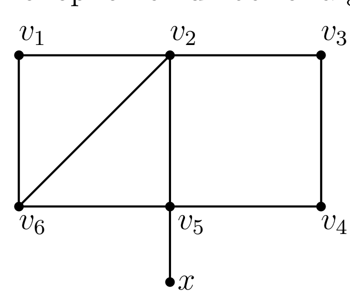

Figure 2.1: G

Theorem 2.5. For any connected graph G of order n, $2 \leq m_r(G) \leq em_r(G) \leq em_{cr}(G) \leq n$, $m_r(G) \neq n-1$, $em_r(G) \neq n-1$ and $em_{cr}(G) \neq n-1$.

Proof. Any restrained monophonic set of G needs at least two vertices and so $m_r(G) \geq 2$. Since every restrained edge monophonic set of G is a restrained monophonic set and every connected restrained edge monophonic set of G is a restrained edge monophonic set, it follows that $m_r(G) \leq em_r(G) \leq em_{cr}(G)$. Also, since the set of all vertices of G is a connected restrained edge monophonic set of G and $V(G) - \{z\}$ is not a connected restrained edge monophonic set of G for any vertex z in G. Hence $em_{cr}(G) \leq n$, $em_{cr}(G) \neq n - 1$. From the definition of restrained monophonic set and restrained edge monophonic set, we have $m_r(G) \neq n-1$ and $em_r(G) \neq n-1$. □

For the path $P_n (n \geq 4)$, $m_r(P_n) = 2$ and for any complete graph $K_n (n \geq 2)$, $m_r(K_n) = em_r(K_n) = em_{cr}(K_n) = n$. Thus the bounds in Theorem 2.5 are sharp. Also, there are examples where all the inequalities in Theorem 2.5 are strict. For the graph G given in Figure 2.1, $m_r(G) = 3$, $em_r(G) = 4$, $em_{cr}(G) = 5$ and $n = 7$. Thus, we have $2 < m_r(G) < em_r(G) < em_{cr}(G) < n$.

Since every connected restrained edge monophonic set of G is a restrained monophonic set, the next theorem follows from Theorem 1.3.

Theorem 2.6. Let G be a connected graph with cut-vertices and S a connected restrained edge monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.

Theorem 2.7. Every cut-vertex of a connected graph G belongs to every connected restrained edge monophonic set of G.

Proof. Let v be any cut-vertex of G and let $G_1, G_2, \cdots, G_r (r \geq 2)$ be the components of $G - v$. Let S be any connected restrained edge monophonic set of G. Then by Theorem 2.6, S contains at least one element from each $G_i (1 \leq i \leq r)$. Since $G[S]$ is connected, it follows that $v \in S$. □

As a consequences of Theorems 2.2 and 2.7, we have the following theorem.

Theorem 2.8. For any connected graph G of order n with k semi-extreme vertices and l cut-vertices, $max\{2, k+l\} \leq em_{cr}(G) \leq n$.

Theorem 2.9. For any tree T of order n, $em_{cr}(T) = n$.

Corollary 2.10. For the star $K_{1,n-1} (n \geq 2)$, $em_{cr}(K_{1,n-1}) = n$.

Theorem 2.11. For any cycle C_n, $em_{cr}(C_n) = \begin{cases} n & \text{if } n = 3, 4 \\ 3 & \text{if } n \geq 5. \end{cases}$

Proof. If $n = 3$, then $G = C_3$ is a complete graph, by Corollary 2.3, we have $em_{cr}(C_3) = 3$.

If $n = 4$, no 2-element or 3-element subset of $V(C_4)$ forms a connected restrained edge monophonic set of C_4 and so $em_{cr}(C_4) = 4$.

If $n \geq 5$, it is easily observed that, any set of three consecutive vertices of C_n forms a minimum connected restrained edge monophonic set of C_n and so $em_{cr}(C_n) = 3$. □

Theorem 2.12. For any connected graph G of order n, $em_{cr}(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $em_{cr}(G) = 2$. Conversely, let $em_{cr}(G) = 2$. Let $S = \{u, v\}$ be a minimum connected restrained edge monophonic set of G. Then uv is an edge. If $G \neq K_2$, then there exists an edge xy different from uv and the edge xy does not lie on any $u - v$ monophonic path. Hence S is not a connected restrained edge monophonic set, which is a contradiction. Thus $G = K_2$. □

Theorem 2.13. If G is a connected graph of order $n \geq 2$ with every vertex of G is either a cut-vertex or a semi-extreme vertex, then $em_{cr}(G) = n$.

Proof. It follows from Theorems 2.2 and 2.7. □

The converse of Theorem 2.13 need not be true. For the cycle C_4, $V(C_4)$ is the unique minimum connected restrained edge monophonic set so that $em_{cr}(C_4) = 4 = n$ and any vertex of C_4 is neither a cut-vertex nor a semi-extreme vertex.

Thus there are a number of classes of graphs G (complete and non-complete) of order n with $em_{cr}(G) = n$. This leads to the following open problem.

Problem 2.14. Characterize the classes of graphs G of order n for which $em_{cr}(G) = n$.

3. Realisation Results for $em_{cr}(G)$

Theorem 3.1. For every pair of positive integers a, b with $4 \leq a \leq b$, there is a connected graph G with $em_r(G) = a$ and $em_{cr}(G) = b$.

Proof. We prove this theorem by considering two cases.

Case 1. $a = b$. By Theorem 1.1 and Corollary 2.3, the complete graph K_a has the desired properties.

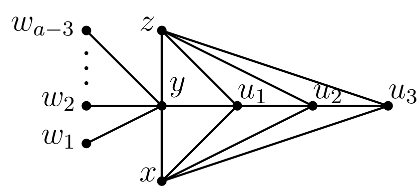

Figure 3.1: G

Case 2. $a < b$ and $b = a + 1$. Let G be the graph obtained from the path $P_3 : x, y, z$ of order 3 by adding a new vertices $w_1, w_2, \cdots, w_{a-3}, u_1, u_2, u_3$ and joining each $w_i (1 \leq i \leq a-3)$ to the vertex y; and joining each $u_i (1 \leq i \leq 3)$ to the vertices x and z; and joining the vertex y with u_1, the vertex u_1 with u_2, the vertex u_2 with u_3. The graph G is shown in Figure 3.1. It is easy to check that $S = \{w_1, w_2, \cdots w_{a-3}, u_3\}$ is the set of all semi-extreme vertices of G. By Theorems 1.1 and 2.2, every restrained edge monophonic set and every connected restrained edge monophonic set of G contain S. Clearly, S is not a restrained edge monophonic set of

G. For any vertex $u \in V - S$, let $S_1 = S \cup \{u\}$ where $u \in \{x, y, z, u_2\}$. Then the edge zu_1 does not lie on any monophonic path joining two vertices of S_1 so that S_1 is not a restrained edge monophonic set of G. Also, if $u = u_1$, then the edge zu_2 does not lie on any monophonic path joining two vertices of S_1 so that S_1 is not a restrained edge monophonic set of G. Now, it is easily verified that $S' = S \cup \{x, z\}$ is a minimum restrained edge monophonic set of G and so $em_r(G) = a$. Since the subgraph induced by S' is not connected, S' is not a connected restrained edge monophonic set of G. It follows from Theorems 2.2 and 2.7 that $S' \cup \{y\}$ is a minimum connected restrained edge monophonic set of G and so $em_{cr}(G) = a + 1$.

If $b - a \geq 2$, then by Theorems 1.2 and 2.9, any tree of order b with a end-vertices has the desired properties. $\square$

We leave the following problem as an open question.

Problem 3.2. Characterize graphs G for which $em_r(G) = em_{cr}(G)$.

Theorem 3.3. If n, d and k are positive integers such that $4 \leq d \leq n - 2$ and $k \geq 4$ and $n - d - k + 2 \geq 0$, then there exists a connected graph G of order n, monophonic diameter d and $em_{cr}(G) = k$.

Proof. Let $C_{d+1} : v_1, v_2, ..., v_{d+1}, v_1$ be the cycle of order $d + 1$. Add $n - d - 1$ new vertices $u_1, u_2, \cdots, u_{k-3}, w_1, w_2, ..., w_{n-d-k+2}$ to C_{d+1} and join each vertex $w_i (1 \leq i \leq n - d - k + 2)$ to both vertices v_2 and v_4; and join each $u_i (1 \leq u_i \leq k - 3)$ to the vertex v_1. This produces a graph G of order n and this graph is shown in Figure 3.2. For any vertex x in $V(G)$, $2 \leq e_m(x) \leq d$, and $e_m(u_i) = e_m(v_d) = d (1 \leq i \leq k - 3)$. Thus the monophonic diameter of G is d. It is easy to check that $S = \{u_1, u_2, \cdots, u_{k-3}, v_1\}$ is the set of all extreme vertices and cut-vertices of G. By Theorems 2.2 and 2.7, every connected restrained edge monophonic set of G contains S. It is clear that S is not a connected restrained edge monophonic set of G. Also, for any vertex $x \in V - S$, $S \cup \{x\}$ is not a connected restrained edge monophonic set of G. It is easy to verify that $S \cup \{v_2, v_{d+1}\}$ is a minimum connected restrained edge monophonic set of G and so $em_{cr}(G) = k$. □

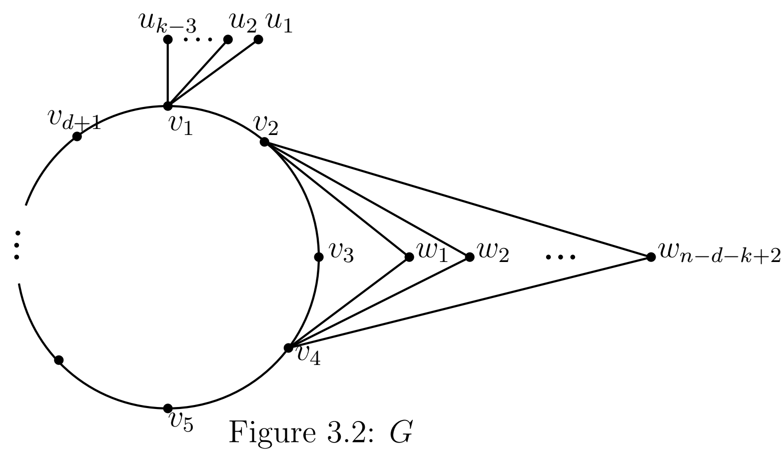

Figure 3.2: G

4. Conclusion

In this paper, a new graph parameter known as connected restrained edge monophonic number of a graph is introduced and several interesting results are discussed. The open problems given in this paper are challenging, and this concept can be extended to several conditional parameters and left for future research.

References

[1] F. Buckley and F. Harary. *Distance in Graphs*. Addison-Wesley, Redwood City, CA, 1990.

[2] M.C. Dourado, F. Protti and J.L.Szwarcfiter, Algorithmic Aspects of Monophonic Convexity, Electronic Notes in Discrete Mathematics, **30**(2008), 177-182.

[3] M.C. Dourado, F. Protti and J.L.Szwarcfiter, Complexity results related to monophonic convexity, Discrete Applied Mathematics, **158**(2010), 1268-1274.

[4] F. Harary, *Graph Theory*, Addison-Wesley Pub. Co., 1969.

[5] E.M Palugaa and S.R. Canoy, Monophonic numbers of the join and composition of connected graphs, Discrete Mathematics, **307**(2007), 1146-1154.

[6] A.P. Santhakumaran and P. Titus, Monophonic distance in graphs, Discrete Mathematics, Algorithms and Applications, **3(2)**(2011), 159-169.

[7] A.P. Santhakumaran and P. Titus, A Note on “Monophonic Distance in Graphs”, Discrete Mathematics, Algorithms and Applications, **4(2)**(2012), DOI: 10.1142/S1793830912500188.

[8] A.P. Santhakumaran, P. Titus and K. Ganesamoorthy. On the Monophonic Number of a Graph. *J. Appl. Math. & Informatics*, **32**(1-2)(2014), 255 - 266.

[9] A. P. Santhakumaran, P. Titus and K. Ganesamoorthy. The Restrained Monophonic Number of a Graph, Communicated.

[10] P. Titus, A.P. Santhakumaran and K. Ganesamoorthy, The Restrained Edge Monophonic Number of a Graph, Bulletin of the International Mathematical Virtual Institute, **7(1)**(2017), 23-30.

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