Jordan Journal of Mathematics and Statistics (JJMS), 14(3), 2021, pp 505 - 516
DOI: https://doi.org/10.47013/14.3.8

ON COMMUTING GRAPHS ASSOCIATED TO BCI-ALGEBRAS

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Abstract. In this paper, first, the graph $\Gamma(X)$ associated to a BCI-algebra X is studied and some related properties are established. Especially, a necessary and sufficient condition for $\Gamma(X)$ to be a complete graph is given. After that, the commuting graph associated to a BCI-algebra X, denoted by G(X), is defined and some related properties are investigated. The paper provides a necessary and sufficient condition for the p-semisimple part of X to be an ideal. Moreover, a condition for an element of a BCI-algebra X to be minimal is given. Finally, it is proved that a BCI-algebra X is p-semisimple if and only if G(X) is a complete graph.

1. Introduction

Many authors applied graph theory in connection with some algebraic structures and obtained some interesting results. For example, Beck, I. [2] associated to any commutative ring R its zero-divisor graph G(R) ; and F.R. DeMeyer, T. McKenzie and K. Schneideretc [3] associated to any commutative semigroup S its zero-divisor graph $\Gamma(S)$. For first time, Y.B. Jun and K.J. Lee [7] introduced the concept of associative graph of a BCK/BCI algebra and provided several examples. They gave conditions for a proper (quasi-)ideal of a BCK/BCI-algebra to be l-prime.

2000 *Mathematics Subject Classification.* 06F35, 03G25, 68R10.
Key words and phrases. BCI-algebra, commutative BCK-algebra, ideal, p-semisimple, graph associated to a BCI-algebra, commuting graph.

Received: April 10 , 2020 Accepted: July 12, 2021 .

The notion of BCK-algebras was introduced by Y. Imai and K. Iséki [4] in 1966 as a generalization of set-theoretic difference and propositional calculi. In the same year, K. Iséki introduced the notion of BCI-algebras which is a generalization of BCK-algebras [5]. These algebras are two important classes of logical algebras. In this paper, following [7], we study the graph $\Gamma(X)$ associated to the BCI-algebra X, defined in [7], and establish some related properties. We give a necessary and sufficient condition for $\Gamma(X)$ to be a complete graph. It is well known that p-semisimple algebras are equivalent to abelian groups. According to this, for any BCI-algebra X, we define a new commuting graph associated to the BCI-algebra X, denoted by G(X), and investigate some related properties. Also, we provide a necessary and sufficient condition for the p-semisimple part of a BCI-algebra to be an ideal. Finally, we prove that a BCI-algebra X is p-semisimple if and only if G(X) is a complete graph.

Now, we recall some definitions and results on BCI-algebras and graph theory. The reader is referred to [4, 5, 6, 8, 9, 11] for more details.

Definition 1.1. By a *BCI*-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:
BCI-1: $((x * y) * (x * z)) * (z * y) = 0$,
BCI-2: $(x * (x * y)) * y = 0$,
BCI-3: $x * x = 0$,
BCI-4: $x * y = 0$ and $y * x = 0$ imply $x = y$.

A *BCI*-algebra X satisfying $0 * x = 0$ for all $x \in X$ is called a *BCK*-algebra. In any *BCI*/*BCK*-algebra X one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

A non-empty subset A of X is called a subalgebra of X if $x * y \in A$ for all $x, y \in A$. The set $B_X := \{x \in X \mid 0 * x = 0\}$ is called the *BCK*-part of X. The element a of X is called a minimal element if $x \leq a$ implies $x = a$ for all $x \in X$. The set of all minimal elements of X, denoted by P_X, is called p-semisimple part of X. It is proved that $P_X = \{x \in X \mid 0 * (0 * x) = x\}$. A BCI-algebra X is said to be a p-semisimple if $P_X = X$. It is well known that (i) $B_X \cap P_X = \{0\}$; (ii) $x \in X$ is a minimal element if and only if $x = 0 * u$ for some $u \in X$.

In any BCI-algebra X, the following hold: for any $x, y, z \in X$,

(a_1) $x * 0 = x$,

(a_2) $(x * y) * z = (x * z) * y$,

(a_3) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$,

(a_4) $(x * z) * (y * z) \leq x * y$,

(a_5) $x * (x * (x * y)) = x * y$,

(a_6) $0 * (x * y) = (0 * x) * (0 * y)$,

(a_7) $x * (x * y) \leq y$.

A BCK-algebra X is called commutative if it satisfies the condition: $x * (x * y) = y * (y * x)$ for all $x \in X$. In this case, $x * (x * y)$ (and $y * (y * x)$) is the greatest lower bound of x and y with respect to BCK-order $\leq$, and we denote it by $x \wedge y$.

A subset A of a BCI/BCK-algebra X is called an *ideal* of X if it satisfies (i) $0 \in A$ and (ii) $x, y * x \in A$ imply $y \in A$ for all $x, y \in X$.

Proposition 1.1. [11] *Let X be a BCI-algebra. Then the p-semisimple part P of X is an ideal if and only if $x * a \in P_X$ implies $x = 0$ for any $a \in P_X$ and $x \in B_X$.*

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ be the set of all vertices of G, and $E(G)$ be the set of all edges of G. A graph G is called complete if every two vertices of G are connected. The complete graph with n vertices is denoted by K_n. A graph G is called null if the set $E(G)$ is empty. The null graph with n vertices is denoted by N_n. Let G and H be two graphs with $V(G) \cap V(H) = \phi$. The union $G \cup H$ is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

2. GRAPH BASED ON BCI-ALGEBRAS

To investigate the properties of a graph associated to a BCI-algebra, defined by Y.B. Jun [7], we recall some definitions.

Definition 2.1. [7] Let X be a BCI-algebra. Then

(i) For any subset A of X, we will use the notation $l(A)$ to denote the set

$$l(A) := \{x \in X \mid x * a = 0, \forall a \in A\};$$

(ii) For any $x \in X$, we will use the notation Z_x to denote the set of all elements $y \in X$ such that $l(\{x, y\}) = \{0\}$, that is,

$$Z_x := \{y \in X \mid l(\{x, y\}) = \{0\}\}.$$

Definition 2.2. [7] By the associated graph of a BCK/BCI-algebra X, denoted $\Gamma(X)$, we mean the graph whose vertices are just the elements of X, and for distinct vertices $x, y \in V(\Gamma(X))$, there is an edge connecting x and y if and only if $l(\{x, y\}) = \{0\}$.

Lemma 2.1. *Let X be a BCI-algebra and $a \in X$. Then*

$$a \in Z_a \text{ if and only if } a = 0.$$

Proof. Let $a \in Z_a$. Then $l(\{a\}) = \{0\}$. By axiom BCI-3, $a \in l(\{a\})$. Hence $a = 0$.

Conversely, let $a = 0$. We show that $0 \in Z_0$, or equivalently, $l(\{0\}) = \{0\}$. Obviously, $0 \in l(\{0\})$; and for any $t \in l(\{0\})$, we get $t * 0 = 0$, which yield t=0. This completes the proof. □

Lemma 2.2. *Let X be a BCI-algebra and $a, b \in X$ with $a \leq b$. Then the following hold:*

(i) $l(\{a\}) \subseteq l(\{b\})$;

(ii) $Z_b \subseteq Z_a$.

Proof. (i) It follows from $a \leq b$ that $a * b = 0$. Let $x \in l(\{a\})$. Then $x * a = 0$ and so by (a_4), we have $x * b = (x * b) * (x * a) \leq a * b = 0$. From this we obtain $x * b = 0$ and consequently, $x \in l(\{b\})$.

(ii) Let $x \in Z_b$. Then $l(\{x, b\}) = \{0\}$. From this follows that

$$0 * x = 0 \text{ and } 0 * b = 0. \tag{2.1}$$

Now, assume that $t \in l(\{x, a\})$. Hence $t * x = 0$ and $t * a = 0$. From $t * a = 0$, we get $t \in l(\{a\})$ and so by (i), we have $t \in l(\{b\})$, that is, $t * b = 0$. Moreover, $t * x = 0$. Thus $t \in l(\{x, b\})$. But $l\{x, b\}) = \{0\}$, hence $t = 0$. Next we show that $0 \in l(\{x, b\})$. By (a_3), from $a \leq b$, we obtain $0 * b \leq 0 * a$ and so by (1), we conclude $0 \leq 0 * a$, that is, $0 * a \in B_X$. But $0 * a \in P_X$. Thus $0 * a \in B_X \cap P_X = \{0\}$ and so $0 * a = 0$. Hence, by (2.1), we get $0 \in l(\{x, a\})$ and consequently, $l(\{x, a\} = \{0\}$. Therefore $x \in Z_a$, which completes the proof. □

We provide a condition for Z_a to be an ideal.

Theorem 2.1. *Let X be a BCI-algebra. Then the following are equivalent:*

(i) *Z_a is an ideal of X;*

(ii) *$a \in B_X$.*

Proof. $(i) \Rightarrow (ii)$ Let Z_a be an ideal of X for some $a \in X$. Then $0 \in Z_a$ and so there is $x \in X$ such that $l(\{a, x\}) = \{0\}$. From this follows that $0 * a = 0$, that is, $a \in B_X$.

$(ii) \Rightarrow (i)$ Let $a \in B_X$. Then it is easy to see that $l(\{a, 0\}) = \{0\}$, which yield $0 \in Z_a$. Now, let $x, y * x \in Z_a$. Then $l(\{a, x\}) = \{0\}$ and $l(\{a, y * x\}) = \{0\}$ and so from $0 \in l(\{a, y * x\})$, we get $0 * (y * x) = 0$. Let $t \in l(\{a, y\})$. Then, we have

$$t * a = 0 \textit{ and } t * y = 0. \tag{2.2}$$

Now, we show that $t * (y * x) \in l(\{a, x\})$. For this, we have

$$\begin{aligned} \text{by } (a_2) \qquad (t * (y * x)) * a &= (t * a) * (y * x) \\ \text{by } (2.4) \qquad &= 0 * (y * x) \\ &= 0. \end{aligned}$$

Therefore

$$(t * (y * x)) * a = 0 \tag{2.3}$$

Also, using (a_2), (a_4) and (2), we get

$$(t * (y * x)) * x = (t * x) * (y * x) \leq t * y = 0. \tag{2.4}$$

From (2.3) and (2.4), we conclude $t * (y * x) \in l(\{x, a\}) = \{0\}$ and so $t * (y * x) = 0$. Moreover by (2.2), $t * a = 0$. Thus $t \in l(\{y * x, a\}) = \{0\}$ and so $t = 0$. Therefore $l(\{y, a\}) = \{0\}$, and consequently $y \in Z_a$. Hence Z_a is an ideal of X. □

The following example shows that Z_a is not necessary be an ideal.

Example 2.1. [11] *Let* $(X = \{0, 1, a, b\}; *, 0)$ *be a* BCI*-algebra in which the operation "* $*$ *" is given by the following table:*

$*$	0	1	a	b
0	0	0	b	a
1	1	0	b	a
a	a	a	0	b
b	b	b	a	0

By some routine calculations, one can check that $Z_a = \{a\}$ *which is not ideal, since* $0 \notin Z_a$.

The following lemma determines the degree of the vertex in graph $\Gamma(X)$.

Lemma 2.3. *Let X be a BCI-algebra. Then for any vertex $a \in V(\Gamma(X))$,*

$$deg_{\Gamma(X)}a = \begin{cases} |Z_a| - 1 & \text{if } a = 0, \\ |Z_a| & \text{otherwise.} \end{cases}$$

Proof. Let $a \neq 0$ be a vertex of $V(\Gamma(X))$. Then by Lemma 2.1, $a \notin Z_a$. Thus for any vertex $x \in V(\Gamma(X))$, it is easy to see that

$$\begin{aligned} x \neq a \text{ and } x \text{ is connected to vertex } a &\Leftrightarrow x \neq a \text{ and } l(\{x, a\} = \{0\} \\ &\Leftrightarrow x \in Z_a. \end{aligned}$$

This implies that $deg_{\Gamma(X)}a = |Z_a|$ whenever $a \neq 0$. If $a = 0$, then by Lemma 2.1, $a \in Z_a$ and so similar to the previous argument, we conclude $deg_{\Gamma(X)}a = |Z_a| - 1$ □

Lemma 2.4. *Let X be a BCI-algebra. Then the following conditions hold:*

(i) *For any $a \in X$, $l(\{0, a\}) = \{0\}$ if and only $a \in B_X$;*

(ii) $deg_{\Gamma(X)}0 = |B_X| - 1$*;*

(iii) *For any $a \neq 0$, $a \in B_X$ if and only if $deg_{\Gamma(X)}a \geq 1$;*

(iv) *For any $a \in P_X$ with $a \neq 0$, $deg_{\Gamma(X)}a = 0$.*

Proof. (i) Let $l(\{0, a\}) = \{0\}$ for some $a \in X$. It follows from $0 \in l(\{0, a\})$ that $0 * a = 0$, which yield $a \in B$.

Conversely, assume that $a \in B_X$. Then $0 * a = 0$ and so $0 \in l(\{0, a\})$. Now, let $t \in l(\{0, a\})$. Then $t * 0 = 0$ and so $t = 0$. Therefore $l(\{0, a\}) = \{0\}$.

(ii) By (i), the result is obvious.

(iii) By (i), the vertex $0 \neq a \in V(B_X)$ is connected to the vertex 0. Hence $deg_{\Gamma(X)}a \geq 1$.

Conversely, let $deg_{\Gamma(X)}a \geq 1$. Then there exists a vertex x connected to vertex a, that is, $l(\{x, a\}) = \{0\}$. Hence $0 * a = 0$ and so $a \in B_X$.

(iv) Let $0 \neq a \in P_X$. Then from $B_X \cap P_X = \{0\}$, we get $a \notin B_X$ and so by (iii), $deg_{\Gamma(X)}a = 0$. □

In the following, we provide a relationship between a complete graph and a commutative BCK-algebra.

Theorem 2.2. *Let X be a BCK-algebra. Then the following conditions are equivalent:*

(i) $\Gamma(X)$ is a complete graph*;*

(ii) X is commutative and for any $a, b \in X$*,* $a \wedge b = 0$*.*

Proof. $(i) \Rightarrow (ii)$ Let $a, b \in X$. Then by (i), the vertex a is connected to the vertex b and so $l(\{a, b\}) = \{0\}$. Since $a * (a * b) \leq a, b$, we get $a * (a * b) \in l(\{a, b\})$, which yield $a * (a * b) = 0$. Similarly, from $b * (b * a) \leq a, b$ we obtain $b * (b * a) = 0$ and so $a * (a * b) = b * (b * a) = 0$. This implies that X is commutative and $a \wedge b = 0$.

$(ii) \Rightarrow (i)$ Let $a, b \in X$ and let $t \in l(\{a, b\})$. Then $t * a = 0$ and $t * b = 0$ and so $t \leq a, b$. This implies that $t \leq a \wedge b = 0$, and so $t = 0$. Moreover, obviously, $0 \in l(\{a, b\})$. Hence $l(\{a, b\}) = \{0\}$, that is, the vertex a is connected to the vertex b. Therefore $\Gamma(X)$ is a complete graph. □

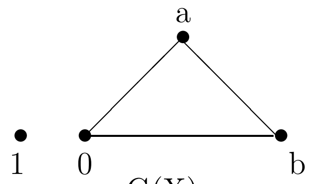
3. On Commuting graphs associated to BCI-algebras

It is known that the commuting graph $G = (V(G), E(G))$ associated to a group G is defined by: " two distance vertices x, y in $V(G)$ are adjacent $\Leftrightarrow xy = yx$."

We note that if $(X; *, 0)$ is a p-semisimple algebra, then the group $(G, .)$ defined by $x.y = x * (0 * y)$ is abelian, which is called the adjoint abelian group. According this, for any BCI-algebra X, we define a commuting graph associated to X, denoted by G(X), and investigate some related properties.

Definition 3.1. For any BCI-algebra X, the commuting graph associated to X, denoted by $G(X)$, is the graph whose vertices are just elements of X and two distinct vertices u and v are connected by edge (u, v) if and only if $u * (0 * v) = v * (0 * u)$.

Example 3.1. *Consider the BCI-algebra* $(X = \{0, 1, a, b\}; *, 0)$ *as in Example 2.1. By routine calculations, one can check that* $B_X = \{0, 1\}$ *and* $P_X = \{0, a, b\}$ *and so the associated commuting graph to* X *is as follows:*



G(X)

Lemma 3.1. *Let* X *be a BCI-algebra. Then, a vertex* $u \in V(G(X))$ *is connected to vertex* $0 * u$ *if and only if* u *is a minimal element of* X*.*

Proof. Let u be connected to $0*u$. Then we have $u*(0*(0*u) = (0*u)*(0*u) = 0$. This implies $u \leq 0*(0*u)$ and so by the minimality of $0*(0*u)$, we get $u = 0*(0*u)$, that is, u is a minimal element of X.

Conversely, let u be a minimal element of X. Then it easy to see that $u*(0*(0*u) = 0 = (0*u)*(0*u)$, which implies that vertex u is connected to vertex $0*u$. □

Theorem 3.1. *Let* X *be a BCI-algebra. Then the following conditions are equivalent:*

(i) X *is p-semisimple;*

(ii) $G(X)$ *is a complete graph.*

Proof. *(i)* ⇒ *(ii)* Let $u, v \in V(G(X))$. Since X is p-semisimple, $u = 0*(0*u)$ and $v = 0*(0*v)$. Thus we have

$$u*(0*v) = (0*(0*u)*(0*v)$$

by (a_2)

$$= (0*(0*v)*(0*u)$$

$$= v*(0*u).$$

Therefore u is connected to v and so $G(X)$ is a complete graph.

$(ii) \Rightarrow (i)$ Let $u \in X$. If $0 * u = u$, then obviously, u is a minimal element. Otherwise, by (ii), u is connected to $0 * u$ and so by Lemma 3.1, u is a minimal element of X. This completes the proof. □

Corollary 3.1. *Let X be a BCI-algebra. Then the subgraph of $G(X)$, induced by the vertices $V(P_X)$, is a complete graph.*

Theorem 3.2. *Let X be a BCI-algebra. X is a BCK-algebra if and only if the graph $G(X)$ is a null graph.*

Proof. Let X be a BCK-algebra and $x, y \in X$ with $x \neq y$. If x is connected to y, then $x = x * (0 * y) = y * (0 * x) = y$, which is a contradiction. Therefore $G(X)$ is a null graph.

Conversely, assume that $G(X)$ is a null graph. If $B_X \neq X$, then there is $x \in X$ such that $0*x \neq 0$. If $0*x = x$, then x is a minimal element of X and so $0*(0*x) = x$. From this, we get $x * (0 * 0) = x = 0 * (0 * x)$. This implies that 0 is connected to x, which contradict to null graph. Hence $0 * x$ and x are distinct, and so by Lemma 3.1, they are connected together, which is a contradiction. Therefore $B_X = X$, that is, X is a BCK-algebra. □

Corollary 3.2. *Let X be a BCI-algebra. Then the subgraph of $G(X)$ induced by the vertices $V(B_X)$ is a null graph.*

By combining Corollaries 3.1 and 3.2, we have the following result.

Proposition 3.1. *Let X be a BCI-algebra. If $X = B_X \cup P_X$, then $G(X)$ is union of a complete graph and a null graph, that is, $G(X) = K_{|B_X|} \cup N_{|P_X|-1}$.*

In general, it is not necessary that P_X be an ideal. The following theorem provides a condition for P_X to be an ideal of X.

Theorem 3.3. *Let X be a BCI-algebra. Then P_X is an ideal of X if and only if the following implication is satisfied.*

$$(\forall x \in B_X)(\forall a \in P_X) \; if \; x \; is \; connected \; to \; a, \; then \; x = 0. \tag{3.1}$$

Proof. Let P_X be an ideal of X, and let x be connected to a for some $x \in B_X$ and $a \in P_X$. Then $x * (0 * a) = a * (0 * x) = a * 0 = a$, and so $x * (0 * a) \in P$. From this and the fact that $0 * a \in P_X$ and P_X is an ideal, we get $x \in P_X$. But $B_X \cap P_X = \{0\}$. Therefore $x = 0$.

Conversely, let the implication (3.1) holds, and let $y * a, a \in P_X$. By Proposition 1.1, it can be assumed that $y \in B_X$; and consequently it suffices to show that $y = 0$. First, we prove that y is connected to $0 * a$. Since P_X is closed under the operation $*$, it follows from $0, y * a \in P_X$ that $0 * (0 * (y * a)) \in P_X$. Now, we have

by the minimality of $y * a$	$y * a = 0 * (0 * (y * a))$
by (a_6)	$= 0 * ((0 * y) * (0 * a))$
since $y \in B_X$	$= 0 * (0 * (0 * a))$
by (a_5)	$= 0 * a$

By the minimality of a and the above result, we get

$$y * (0 * (0 * a)) = y * a = 0 * a = (0 * a) * (0 * y).$$

This implies that y is connected to $0 * a$. Therefore, by (3.1), we conclude $y = 0$, which completes the proof. □

Acknowledgement

The author would like to thank the reviewers for their comments which have improved the paper.

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