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THE THETA-COMPLETE GRAPH RAMSEY NUMBER $R(\theta_n, K_7); n = 7; n \geq 14$.

A. BANIABEDALRUHMAN

ABSTRACT. The Ramsey theory is an important branch in graph Theory. Finding the Ramsey number is an important topic in the Ramsey theory. The Ramsey number $R(G, H)$ is the smallest positive integer n such that any graph of order n contains the graph G or its complement contains the graph H. In this paper, we prove that $R(\theta_n, K_7) = 6(n-1)+1$, $n = 7; n \geq 14$, where θ_n is a theta graph of order n and K_7 is the complete graph of order 7.

1. INTRODUCTION

All graphs in this paper are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order of a graph G is the size of the vertex set of G and is denoted by $|G|$. The size of the largest independent set for a graph H is denoted by $\alpha(H)$. The set of vertices adjacent to a vertex v is denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. For a sub-graph H in G, the neighborhood of H is defined by $N(H) = \bigcup_{u \in H} N(u)$ and $N[H] = N(H) \cup H$. The minimum degree in a graph G is denoted by $\delta(G)$. The cycle and path of order s are denoted by C_s and P_s , respectively. The theta graph of order s is a cycle C_s and an edge joining two non-adjacent vertices in C_s. The complement of the graph G, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$ and two vertices in $\overline{G}$ are adjacent if and only

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if they are not adjacent in G. The Ramsey number $R(G,H)$ is the smallest positive integer n such that any graph of order n contains G or its complement contains H.

Erdös et al. [6] conjectured that $R(C_n,K_m)=(n-1)(m-1)+1$, for all $n\geq m\geq 3$ except $R(C_3,K_3)=6$. Rosta [12] proved the conjecture for $m=3$. Cheng et al. [4] confirmed the conjecture for $m=7$ and proved that $R(C_7,K_8)=43$. Radziszowski and KK. Tse [10] found the exact Ramsey number for a cycle of order 4 versus a complete graph of order 7, $R(C_4,K_7)=22$. For more results on Ramsey numbers, see [11].

Here are some results on Ramsey number of the theta graphs versus the complete graphs. Chvátal and Harary [5], proved that $R(\theta_4,K_4)=11$. Bolze and Harborth [3], determined that $R(\theta_4,K_5)=16$. McNamara [9], proved that $R(\theta_4,K_6)=21$. Bataineh et. al. [2], determined that $R(\theta_n,K_m)=(n-1)(m-1)+1$ for $m=3,4$ and $n>m$. Recently, Jaradat, et. al. [8] confirmed that $R(\theta_n,K_5)=4n-3$ for $n=6$ and $n\geq 10$. More recently, Jaradat et al. [7] proved that $R(\theta_n,K_5)=4n-3$, $n=7,8,9$, and Baniabedalruhman et. al. [1] showed that $R(\theta_n,K_6)=5n-4$, $n\geq 6$.

2. Main Results

In this section, we find the exact Ramsey number $R(\theta_n,K_7)=6(n-1)+1$, $n=7$ and $n\geq 14$. Since the graph $6K_{n-1}$ contains neither θ_n nor 7 independent vertices, then $R(\theta_n,K_7)$ is greater than $6(n-1)$. Therefore, we have to prove that $R(\theta_n,K_7)$ is less than or equal $6(n-1)+1$, $n=7$ and $n\geq 14$.

To achieve our goal, we first prove a sequence of lemmas.

Lemma 2.1. *Let G be a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$. Then $\delta(G)\geq 6$.*

Proof. Let G be a graph that contains a vertex u with $|N(u)| \leq 5$. Then, $|G - N[u]| \geq 31 = R(\theta_7, K_6)$. Thus, $G - N[u]$ contains 6 independent vertices. Therefore, those vertices with u is a 7 independent vertices, a contradiction. The proof is complete. □

Lemma 2.2. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$ and $\{u_1, u_2, \cdots, u_t\}$, $2 \leq t \leq 6$, is an independent set of vertices, then $\mid N(u_1) \cup \cdots \cup N(u_t) \mid \geq 5t + 1$.*

Proof. Suppose that $\mid N(u_1) \cup \cdots \cup N(u_t) \mid < 5t + 1$. Then $\mid G - N[u_1, \cdots, u_t] \mid \geq 37 - 6t = 6(6 - t) + 1$. Now, since $R(\theta_7, K_{7-t}) = 6(6 - t) + 1$, then $G - N[u_1, \cdots, u_t]$ contains a $7 - t$ independent vertices. Those vertices with the vertices u_1, $\cdots$, u_t are a 7 independent vertices, a contradiction. The proof is complete. □

Lemma 2.3. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain K_6.*

Proof. Suppose that G contains K_6. Let $U = \{v_1, v_2, \cdots, v_6\}$ be the vertices of K_6 and let $W = G - U$. Since $\delta(G) \geq 6$ and G does not contain θ_7, then $N(v_i) \cap W \neq \phi$, $i = 1, \cdots, 6$, $N(v_i) \cap N(v_j) \cap W = \phi$ and $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $1 \leq i < j \leq 6$. Since $\delta(G) \geq 6$, then $|N[N(v_1) \cap W] \cap W| \geq 7$. Moreover, since G does not contain θ_7, then $\alpha(N[N(v_1) \cap W] \cap W) \geq 2$ and $xy \notin E(G)$ for any $x \in N[N(v_1) \cap W] \cap W$ and $y \in N(v_i) \cap W$, $i = 2, \cdots, 6$. Therefore, a two independent vertices in $N[N(v_1) \cap W] \cap W$ and $\{w_2, \cdots, w_6\}$ where $w_i \in N(v_i) \cap W$, $i = 2, \cdots, 6$, are a 7 independent vertices, a contradiction. The proof is complete. □

Lemma 2.4. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain $K_1 + C_5$.*

Proof. Suppose that G contains $K_1 + C_5$. Let $U = \{v_1, v_2, \cdots, v_5\}$ and v be the vertices of C_5 and K_1, respectively. Also, let $W = G - (U \cup \{v\})$. Since G does not contain θ_7 , then $N(v_i) \cap N(v_j) \cap W = \phi$ and $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $1 \leq i < j \leq 5$. Since $\delta(G) \geq 6$, then $|N[N(v_1) \cap W] \cap W| \geq 7$. Moreover, since G does not contain θ_7, then $\alpha(N[N(v_1) \cap W] \cap W) \geq 2$ and $xy \notin E(G)$ for any $x \in (N(v_i) \cap W) \cup \{v\}$ and $y \in N[N(v_1) \cap W] \cap W$, $i = 2, \cdots, 5$. Therefore, a two independent vertices in $N[N(v_1) \cap W] \cap W$ and $\{w_2, \cdots, w_5, v\}$ where $w_i \in N(v_i) \cap W$, $i = 2, \cdots, 5$, are a 7 independent vertices, a contradiction. The proof is complete. □

Lemma 2.5. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain $K_1 + P_5$.*

Proof. Suppose that G contains $K_1 + P_5$. Let $U = \{v_1, v_2, \cdots, v_5\}$ and v be the vertices of P_5 and K_1, respectively. Also, let $W = G - (U \cup \{v\})$. Since G does not contain θ_7, then $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $1 \leq i < j \leq 5$, except when $i = 2$ and $j = 4$. Also, $N(v_i) \cap N(v_j) \cap W = \phi$, $1 \leq i < j \leq 5$, except when $i = 2$ and $j = 4$. To complete the proof, we consider two cases.

Case 2.5.1. *$N(v_2) \cap N(v_4) \cap W = \phi$.*

Proof. Since G does not contain θ_7, then $xy \notin E(G)$ for any $x \in N(v_2) \cap W$ and $y \in N(v_4) \cap W$. Since $\delta(G) \geq 6$, then $|N[N(v_1) \cap W] \cap W| \geq 7$. Moreover, since G does not contain θ_7, then $\alpha(N(N(v_1) \cap W) \cap W) \geq 2$ and $xy \notin E(G)$ for any $x \in (N(v_i) \cap W) \cup \{v\}$ and $y \in N(N(v_1) \cap W) \cap W$, $i = 2, \cdots, 5$. Therefore, a two independent vertices in $N(N(v_1) \cap W) \cap W$ and $\{w_2, \cdots, w_5, v_1\}$ where $w_i \in N(v_i) \cap W$, $i = 2, \cdots, 5$, are a 7 independent vertices, a contradiction. □

Case 2.5.2. *$N(v_2) \cap N(v_4) \cap W = w_2$.*

Proof. Since G does not contain θ_7, then $v_i v_3 \notin E(G)$, $i = 1, 5$. Thus, $\{v_1, v_3, v_5\}$ is an independent set. By Lemma 2.2, $|N(\{v_1, v_3, v_5\}) - \{v, v_2, v_4\}| \geq 13$ and hence $|N(v_i) - \{v, v_2, v_4\}| \geq 5$ for some $i = 1, 3$ or 5, say $i = 1$. Therefore, by Lemma

2.3 $\alpha(N(v_1) \cap W) \geq 2$. Since $\delta(G) \geq 6$, then $|N[N(v_5) \cap W] \cap W| \geq 7$. Moreover, since G does not contain θ_7, then $\alpha(N(N(v_5) \cap W) \cap W) \geq 2$. Since G does not contain θ_7, then a two independent vertices in $N(v_1) \cap W$, a two independent vertices in $N(N(v_5) \cap W) \cap W$ and $\{w_2, w_3, v_5\}$ where $w_i \in N(v_i) \cap W$, $i = 2, 3$, are a 7 independent vertices, a contradiction. □

The proof is complete. □

Lemma 2.6. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain K_5.*

Proof. Suppose that G contains K_5. Let $U = \{v_1, v_2, \cdots, v_5\}$ be the vertices of K_5 and let $W = G - U$. By Lemma 2.5, $N(v_i) \cap N(v_j) \cap W = \phi$, $1 \leq i < j \leq 5$. Since G does not contain θ_7, then $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $1 \leq i < j \leq 5$. Since $\delta(G) \geq 6$, then $|N[N(v_i) \cap W] \cap W| \geq 7$, $i = 1, \cdots, 5$. Moreover, since G does not contain θ_7, then $\alpha(N[N(v_i) \cap W] \cap W) \geq 2$, $i = 1, \cdots, 5$. Therefore, since G does not contain θ_7, then $\alpha(G) \geq 2 \times 5 = 10$, a contradiction. The proof is complete. □

Lemma 2.7. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain $K_1 + C_4$.*

Proof. Suppose that G contains $K_1 + C_4$. Let $U = \{v_1, v_2, v_3, v_4\}$ and v be the vertices of C_4 and K_1, respectively. Also, let $W = G - (U \cup \{v\})$. Since $\delta(G) \geq 6$, then $|N(v_i) \cap W| \geq 2$ and there is a vertex $v_5 \in N(v) \cap W$. Now, we have the following observations:

(1) By Lemma 2.5, $v_i v_5 \notin E(G)$, $i = 1, 2, 3, 4$.

(2) Since G does not contain θ_7, then $N(v_5) \cap N(v_i) \cap W = \phi$ and $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_5) \cap W$, $i = 1, 2, 3, 4$.

(3) If $N(v_a)\cap N(v_b)\cap W \neq \phi$ for some $1 \leq a < b \leq 4$, then $N(v_i)\cap N(v_j)\cap W = \phi$, $1 \leq i < j \leq 4$ and $\{i, j\} \neq \{a, b\}$. As otherwise θ_7 is produced.

(4) Since G does not contain θ_7, then $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $x \neq y$ and $1 \leq i < j \leq 4$.

(5) By observations 3 and 4, $\alpha((N(v_a) \cup N(v_b)) \cap W) \geq 2$.

(6) By Lemmas 2.1 and 2.6, $\alpha(N(v_5) \cap W) \geq 2$ and $\alpha(N[N(v_i) \cap W] \cap W) \geq 2$, $1 \leq i \leq 4$ and $i \neq a, b$.

Therefore, since G does not contain θ_7, then $\alpha(G) \geq 2 \times 4 = 8$, a contradiction. The proof is complete. □

Lemma 2.8. *Let G be a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$. If $v_1v, v_2v \in E(G)$ and G does not contain θ_7, then $N(\{v_1, v_2\}) - \{v\}$ does not contain C_4.*

Proof. Suppose that $N(\{v_1, v_2\}) - \{v\}$ contains $C_4 = c_1c_2c_3c_4$. By Lemma 2.7, the vertices of C_4 are not adjacent to one vertex. Thus, $c_iv_1, c_{i+1}v_2 \in E(G)$ for some $1 \leq i \leq 4$, say $i = 1$. Therefore, $c_1c_4c_3c_2v_2vv_1c_1$ is a θ_7, a contradiction. The proof is complete. □

Lemma 2.9. *If H is a sub-graph of G of order 10 that contains neither θ_7 nor K_5, then $\alpha(H) \geq 3$.*

Proof. Suppose that $|H| = 10$ and $\alpha(H) \leq 2$. Since H does not contain K_5, then $\alpha(H) = 2$, say v_1, v_2 are independent vertices in H. Thus, there is a vertex v_i such that $|N(v_i)| \geq 4$ and hence $\alpha(N(v_i)) = 2$, $i = 1, 2$, say $i = 1$. Now, let u_1 and u_2 be an independent vertices in $N(v_1)$. Then, $|N(\{u_1, u_2\}) - \{v_1\}| = 7 = R(C_4, K_3)$. Therefore, by Lemma 2.8, $N(\{u_1, u_2\}) - \{v_1\}$ contains three independent vertices, a contradiction. The proof is complete. □

Lemma 2.10. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain $K_1 + P_4$.*

Proof. Suppose that G contains $K_1 + P_4$. Let $U = \{v_1, v_2, v_3, v_4\}$ and v be the vertices of P_4 and K_1, respectively. Also, let $W = G - (U \cup \{v\})$. Since $\delta(G) \geq 6$, then $|N(v) \cap W| \geq 2$. Note that, $v_i x \notin E(G)$ for any $x \in N(v) \cap W$, $i = 1, 4$. Now, We consider two cases.

Case 2.10.1. $\alpha(N(v) \cap W) \geq 2$.

Proof. Let $u_1, u_2 \in N(v) \cap W$ be an independent vertices. Then, v_1, v_4, u_1, u_2 are independent vertices. By Lemma 2.9, if $|G - N[v_1, v_4, u_1, u_2]| \geq 10$, then $G - N[v_1, v_4, u_1, u_2]$ contains three independent vertices. Those vertices with $\{v_1, v_4, u_1, u_2\}$ are a 7 independent vertices, a contradiction. Thus, $|G - N[v_1, v_4, u_1, u_2]| \leq 9$ and $|N(\{v_1, v_4, u_1, u_2\}) - \{v\}| \geq 23 \geq R(C_4, K_7)$. Therefore, by Lemma 2.8, G contains a 7 independent vertices, a contradiction. □

Case 2.10.2. $\alpha(N(v) \cap W) = 1$.

Proof. We have the the following observations:

(1) By Lemma 2.5, $v_i y \notin E(G)$ for any $y \in N(v) \cap W$, $i = 1, \cdots, 4$.

(2) Since G does not contain θ_7, then $N(N(v) \cap W) \cap N(v_i) \cap W = \phi$ and $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(N(v) \cap W) \cap W$, $i = 1, 4$.

(3) By Lemma 2.2, $|N(\{v_1, v_4\}) - \{v, v_2, v_3\}| \geq 8 \geq R(C_4, K_3)$. Therefore, by Lemma 2.8, $N(\{v_1, v_4\}) - \{v, v_2, v_3\}$ contains a three independent vertices, w_1, w_2, w_3, such that $v_1 w_1, v_1 w_2, v_4 w_3 \in E(G)$.

(4) By Lemma 2.2, $|N(\{w_1, w_2\}) - \{v_1, v_2, v_3, v_4\}| \geq 7 = R(C_4, K_3)$. Therefore, by Lemma 2.8, $N(\{w_1, w_2\}) - \{v_1, v_2, v_3, v_4\}$ contains a three independent vertices, s_1, s_2, s_3.

(5) $N(N(v) \cap W) \cap W$ contains a two independent vertices, r_1, r_2.

□

Therefore, since G does not contain θ_7, then $\{s_1, s_2, s_3, r_1, r_2, w_3, v\}$ is a 7 independent set of vertices, a contradiction. The proof is complete. □

Lemma 2.11. *If G is a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$, then G does not contain K_4.*

Proof. Suppose that G contains K_4. Let $U = \{v_1, v_2, v_3, v_4\}$ be the vertices of K_4 and let $W = G - U$. By Lemma 2.10, $N(v_i) \cap N(v_j) \cap W = \phi$, $1 \leq i < j \leq 4$. Now, we consider two cases.

Case 2.11.1. *$xy \in E(G)$ for some $x \in N(v_1) \cap W$ and $y \in N(v_2) \cap W$.*

Proof. Since G does not contain θ_7, then $xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $1 \leq i < j \leq 4$ and $\{i, j\} \neq \{1, 2\}$. By Lemma 2.5, $\alpha(N(\{v_1, v_2\}) \cap W) \geq 3$. Moreover, by Lemmas 2.1 and 2.6, $\alpha(N[N(v_i) \cap W] \cap W) \geq 2$, $i = 3, 4$. Therefore, since G does not contain θ_7, then $\alpha(G) \geq 3 + 2 \times 2 = 7$, a contradiction. □

Case 2.11.2. *$xy \notin E(G)$ for any $x \in N(v_i) \cap W$ and $y \in N(v_j) \cap W$, $1 \leq i < j \leq 4$.*

Proof. Since G does not contain θ_7, then $N[N(v_i) \cap W] \cap N[N(v_j) \cap W] \cap W = \phi$ and $xy \notin E(G)$ for any $x \in N[N(v_i) \cap W] \cap W$ and $y \in N[N(v_j) \cap W] \cap W$, $1 \leq i < j \leq 4$. Note that, by Lemmas 2.1 and 2.6, $\alpha(N[N(v_i) \cap W] \cap W) \geq 2$, $i = 1, 2, 3, 4$. Therefore, since G does not contain θ_7, then $\alpha(G) \geq 2 \times 4 = 8$, a contradiction. □

The proof is complete. □

Theorem 2.1. *$R(\theta_s, K_7) = 6(s - 1) + 1$, $s = 7$ and $s \geq 14$.*

Proof. The graph $(s - 1)K_6$ contains neither θ_7 nor 7 independent vertices. Thus, $R(\theta_s, K_7) \geq 6(s - 1) + 1$. For $s \geq 14$, let G be a graph of order $6(s - 1) + 1 =$

$R(C_s, K_7)$. If G contains C_s, then G contains θ_s or a 7 independent vertices. Therefore, $R(\theta_s, K_7) = 6(s-1)+1$, $s \geq 14$. For $s = 7$, let G be a graph of order 37 that contains neither θ_7 nor $\overline{K_7}$. By Lemmas 2.1, 2.10 and 2.11, $\alpha(N(u)) \geq 3$ for any $u \in V(G)$. Now, we consider two cases.

Case 2.11.3. $\alpha(N(u)) \geq 4$ *for some* $u \in V(G)$.

Proof. Without loss of generality, assume that $\alpha(N(u)) = 4$ for some $u \in V(G)$. Let $\{v_1, v_2, v_3, v_4\}$ be a set of independent vertices in $N(u)$. By Lemma 2.9, if $|G - N[v_1, v_4, u_1, u_2]| \geq 10$, then $G - N[v_1, v_2, v_3, v_4]$ contains three independent vertices. Those vertices with $\{v_1, v_2, v_3, v_4\}$ are a 7 independent vertices, a contradiction. Thus, $|G - N[v_1, v_4, u_1, u_2]| \leq 9$ and $|N(\{v_1, v_2, v_3, v_4\}) - \{u\}| \geq 23 \geq R(C_4, K_7)$. Therefore, by Lemma 2.8 G contains a 7 independent vertices, a contradiction. □

Case 2.11.4. $\alpha(N(u)) = 3$ *for all* $u \in V(G)$.

Proof. Suppose that G contains neither θ_7 nor a 7 independent vertices. If $N(u)$ contains P_3 for some $u \in V(G)$, then by Lemmas 2.1, 2.10 and 2.11, $\alpha(N(u)) \geq 4$, a contradiction. Thus, $N(u)$ does not contain P_3 for any $u \in V(G)$. Moreover, by lemma 2.1, $N(u)$ does not contain any isolated vertex for any $u \in V(G)$. Since $\alpha(N(u)) = 3$ for any $u \in V(G)$, then $|N(u)| = 6$ and each vertex in $N(u)$ is adjacent to only one vertex in $N(u)$. Now, let u be a vertex in $V(G)$ and let $v_1, \cdots, v_6 \in N(u)$ such that $v_1v_2, v_3v_4, v_5v_6 \in E(G)$. Since $N(u)$ does not contain P_3 for any $u \in V(G)$ and G does not contain θ_7, then $N(v_i) \cap N(v_j) \cap (G - N[u]) = \phi$, $1 \leq i < j \leq 6$. Therefore, since G does not contain θ_7, then $\alpha(G) \geq 2 \times 6 = 12$, a contradiction. □

The proof of the Theorem is complete. □

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Department of Mathematics, Yarmouk University, Irbid, Jordan

Email address: ahmad_a@yu.edu.jo