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POMPEIU TYPE INEQUALITIES USING CONFORMABLE FRACTIONAL CALCULUS AND ITS APPLICATIONS

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Abstract. We establish Pompeiu's mean value theorem for α-fractional differentiable mappings. Then, some Pompeiu type inequalities including conformable fractional integrals are obtained, and the weighted versions of this Pompeiu type inequalities are presented. Finally, some applications for quadrature rules and special means are given.

1. Introduction

In 1938, a famous integral inequality, which was named Ostrowski inequality, introduced by Ostrowski [17] as follows:

Theorem 1.1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\rightarrow\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

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Inequality (1.1) has wide applications in numerical analysis and the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence, the inequality (1.1) has attracted considerable attention and interest from mathematicians and researchers.

In 1946, Pompeiu [19] derived a variant of Lagrange's mean value theorem, which is known as *Pompeiu's mean value theorem.*

Theorem 1.2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0, there exist a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi),$$

for all pairs $x_1 \neq x_2$ in $[a,b]$

In [6], Dragomir proved the following Pompeiu type inequality by using Pompeiu's mean value theorem.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then, for any $x \in [a,b]$, we have the inequality*

$$\left| \frac{a+b}{2} \frac{f(x)}{x} + \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \| f - lf' \|_\infty , \tag{1.2}$$

where $l(t) = t$ for all $t \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In recent years, many authors have worked on the inequalities obtained by using Cauchy's mean value theorem and the above variant of the Lagrange's mean value theorem given by Pompeiu in [19]. For example, the authors presented some Ostrowski type inequalities by using mean value theorem in [4] and [20]. In addition to the inequality (1.2), Pečarić and Ungar provided a new Ostrowski type inequality for p-norm by using Pompeiu's mean value theorem in [18]. What's more, Dragomir obtained some power Pompeiu's type and exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions in [8] and [9]. Also, Dragomir gave some generalizations of Pompeiu inequality, and these results are used to obtain some new Ostrowski type inequalities in [7]. In [25], Sarikaya obtained some new Pompeiu type inequalities for twice differentiable mappings. Afterwards, some researchers examined some new Ostrowski and Grüss type inequalities via a variant of Lagrange's mean value theorem for two-variable functions in [22]-[24]. On the other side, Erden and Sarikaya established generalized Pompeiu mean value theorem and Pompeiu type inequalities for local fractional calculus in [10]. For recent other results obtained realated to similar inequalities, we refer the reader to [11], [15], [16], [12] and the references therein.

2. Definitions and properties of conformable fractional derivative and integral

Recently, the authors introduced a new simple well-behaved definition of the fractional derivative called the "conformable fractional derivative" depending just on the basic limit definition of the derivative in [14]. Namely, for given a function $f:[0,\infty)\to\mathbb{R}$, the conformable fractional derivative of order $0<\alpha\leq 1$ of f at $t>0$ was defined by

$$D_{\alpha}(f)(t)=\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon}.$$

If f is $\alpha-$differentiable in some $(0,a)$ with $\alpha>0$, $\lim\limits_{t\to 0^{+}}f^{(\alpha)}(t)$ exists such that

$$f^{(\alpha)}(0)=\lim_{t\to 0^{+}}f^{(\alpha)}(t).$$

Also, we note that if f is differentiable, then one has

$$D_{\alpha}(f)(t)=t^{1-\alpha}f'(t), \tag{2.1}$$

where

$$f'(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon)-f(t)}{\epsilon}.$$

We can write $f^{(\alpha)}(t)$ for $D_{\alpha}(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

In order to prove the main results, we use the mean value theorem for conformable fractional derivatives. This theorem is established by Iyiola and Nwaeze [13] as follows.

Theorem 2.1 (Mean value theorem for conformable fractional differentiable functions)**.** *Let $\alpha\in(0,1]$ and $f:[a,b]\to\mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b) with $0\le a<b$. Then, there exists $c\in(a,b)$, such that*

$$D_{\alpha}(f)(c)=\frac{f(b)-f(a)}{\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}}.$$

The following definitions and theorems related to conformable fractional derivative and integral were referred in [1]-[3], [5], [13] and [14].

Theorem 2.2. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then, we possess*

i. $D_{\alpha}(af+bg)=aD_{\alpha}(f)+bD_{\alpha}(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_{\alpha}(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_{\alpha}(fg)=fD_{\alpha}(g)+gD_{\alpha}(f)$, *iv.* $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$.

Definition 2.1 (Conformable fractional integral)**.** Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$, if the integral

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx,$$

exists and is finite.

Remark 1. We have

$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 2.3. *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$, we possess*

$$I_\alpha^a D_\alpha^a f(t) = f(t) - f(a).$$

Theorem 2.4. ***(Integration by parts)*** *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\, D_\alpha^a(g)(x)\, d_\alpha x = fg|_a^b - \int_a^b g(x)\, D_\alpha^a(f)(x)\, d_\alpha x.$$

Theorem 2.5. *Assume that $f : [a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$D_\alpha^a I_\alpha^a f(t) = f(t).$$

Theorem 2.6. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ with $0 \le a < b$. Then,*

$$\left|I_\alpha^a(f)(x)\right| \le I_\alpha^a |f|(x).$$

In [5], Anderson provided Ostrowski's α-fractional inequality using a Motgomery identity as follows:

Theorem 2.7. *Let $a, b, s, t \in \mathbb{R}$ with $0 \leq a < b$, and $f : [a, b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0, 1]$. Then, one has*

$$\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{M}{2\alpha (b^\alpha - a^\alpha)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha) \right],$$

where

$$M = \sup_{t \in (a,b)} |D_\alpha f(t)| < \infty.$$

In this study, Pompeiu's mean value theorem for conformable fractional derivatives is obtained. Later, we present Pompeiu type inequalities involving conformable fractional integrals with applications Ostrowski's inequalities. Finally, by means of the inequalities given in this work, some applications in numerical integration and for conformable special means are given.

3. Main Results

We prove Pompeiu's mean value theorem for conformable fractional differentiable functions.

Theorem 3.1. *Let $\alpha \in (0, 1]$ and $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be an α-fractional differentiable mapping on (a, b) with $0 < a < b$, for all pairs $x_1 \neq x_2$ in $[a, b]$, there exist a point ξ in (x_1, x_2) such that the following equality holds:*

$$\frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{\frac{x_1^\alpha}{\alpha} - \frac{x_2^\alpha}{\alpha}} = \alpha f(\xi) - \xi^{2-\alpha} D_\alpha (f)(\xi). \tag{3.1}$$

Proof. We first define the function F on $\left[\frac{1}{b}, \frac{1}{a}\right]$ by

$$F(t) = t^\alpha f(\frac{1}{t}). \tag{3.2}$$

By using the third item of Theorem 2.2, we find that

$$D_\alpha(F)(t) = \alpha f(\frac{1}{t}) - \frac{1}{t^{2-\alpha}} D_\alpha(f)(\frac{1}{t}). \tag{3.3}$$

In addition, by applying the mean value theorem given for conformable fractional differentiable functions to F on the interval $[x, y] \subset \left[\frac{1}{b}, \frac{1}{a}\right]$, it follows that

$$\frac{F(x) - F(y)}{\frac{x^\alpha}{\alpha} - \frac{y^\alpha}{\alpha}} = D_\alpha(F)(c), \tag{3.4}$$

for all $c \in (x, y)$.

Now, if we use the identities (3.2)-(3.4), we obtain

$$\frac{x^\alpha f(\frac{1}{x}) - y^\alpha f(\frac{1}{y})}{\frac{x^\alpha}{\alpha} - \frac{y^\alpha}{\alpha}} = \alpha f(\frac{1}{c}) - \frac{1}{c^{2-\alpha}} D_\alpha(f)(\frac{1}{c}).$$

Let $x_2 = \frac{1}{x}$, $x_1 = \frac{1}{y}$ and $\xi = \frac{1}{c}$. Then, since $c \in (x, y)$, we have

$$x_1 < \xi < x_2,$$

and we can write

$$\frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{\frac{x_1^\alpha}{\alpha} - \frac{x_2^\alpha}{\alpha}} = \alpha f(\xi) - \xi^{2-\alpha} D_\alpha(f)(\xi),$$

which completes the proof. □

Now, we give an Ostrowski type inequality by using Pompeiu's mean value theorem which is given for conformable fractional differentiable functions in the following theorem.

Theorem 3.2. *Let $\alpha \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ be a continuous on $[a, b]$ and an α-fractional differentiable mapping on (a, b) with $0 < a < b$. Then, for any $x \in [a, b]$,* *we have the inequality*

$$\left|\frac{a^{\alpha}+b^{\alpha}}{2\alpha}\frac{f(x)}{x^{\alpha}}-\frac{1}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f(t)d_{\alpha}t\right| \tag{3.5}$$

$$\leq \frac{(b^{\alpha}-a^{\alpha})}{\alpha x^{\alpha}}\left[\frac{1}{4}+\left(\frac{x^{\alpha}-\frac{a^{\alpha}+b^{\alpha}}{2}}{b^{\alpha}-a^{\alpha}}\right)^{2}\right]\left\|f-uD_{\alpha}(f)\right\|_{\infty},$$

where $u(t)=\frac{t^{2-\alpha}}{\alpha}$, $t\in[a,b]$, *and* $\left\|f-uD_{\alpha}(f)\right\|_{\infty}=\sup_{\xi\in(a,b)}\left|f(\xi)-uD_{\alpha}(f)(\xi)\right|<\infty.$

Proof. Using Pompeiu's mean value theorem for conformable fractional differentiable functions for any $x,t\in[a,b]$, there is a point ξ between x and t such that

$$t^{\alpha}f(x)-x^{\alpha}f(t)=\left[f(\xi)-\frac{\xi^{2-\alpha}}{\alpha}D_{\alpha}(f)(\xi)\right](t^{\alpha}-x^{\alpha}). \tag{3.6}$$

Because of the equality (3.6) and the inequality

$$\begin{aligned}\left|f(\xi)-\frac{\xi^{2-\alpha}}{\alpha}D_{\alpha}(f)(\xi)\right| &\leq \sup_{\xi\in(a,b)}\left|f(\xi)-\frac{\xi^{2-\alpha}}{\alpha}D_{\alpha}(f)(\xi)\right| \\ &= \left\|f-uD_{\alpha}(f)\right\|_{\infty},\end{aligned}$$

it follows that

$$\left|t^{\alpha}f(x)-x^{\alpha}f(t)\right|\leq\left\|f-uD_{\alpha}(f)\right\|_{\infty}\left|t^{\alpha}-x^{\alpha}\right|. \tag{3.7}$$

Integrating both sides of (3.7) with respect to t from a to b for conformable fractional integrals, it is found that

$$\left| f(x) \int_a^b t^\alpha d_\alpha t - x^\alpha \int_a^b f(t) d_\alpha t \right| \tag{3.8}$$

$$\leq \|f - uD_\alpha(f)\|_\infty \int_a^b |t^\alpha - x^\alpha| \, d_\alpha t \tag{3.9}$$

$$= \|f - uD_\alpha(f)\|_\infty \left(\int_a^x (x^\alpha - t^\alpha) \, d_\alpha t + \int_x^b (t^\alpha - x^\alpha) \, d_\alpha t \right).$$

Using the definition 2.1 and the inequality (3.8), we obtain

$$\left| \frac{b^{2\alpha} - a^{2\alpha}}{2\alpha} f(x) - x^\alpha \int_a^b f(t) d_\alpha t \right| \tag{3.10}$$

$$\leq \|f - uD_\alpha(f)\|_\infty \left[\frac{(x^\alpha - a^\alpha)^2 + (b^\alpha - x^\alpha)^2}{2\alpha} \right].$$

If we divide the inequality (3.10) by $x^\alpha (b^\alpha - a^\alpha)$, we easily deduce the required result (3.5). □

Corollary 3.1. *Under the same assumptions of Theorem 3.2 with* $x^\alpha = \frac{a^\alpha + b^\alpha}{2}$. *Then, we have*

$$\left| \frac{1}{\alpha} f\left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) - \frac{1}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|$$

$$\leq \frac{(b^\alpha - a^\alpha)}{2\alpha (a^\alpha + b^\alpha)} \|f - uD_\alpha(f)\|_\infty .$$

We consider the weighted version of the inequality (3.5) in the following theorem.

Theorem 3.3. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b) with $0 < a < b$. If $w : [a,b] \to \mathbb{R}$ is nonnegative and α-fractional integrable on $[a,b]$, then one has*

$$\left| \frac{f(x)}{x^\alpha} \int_a^b t^\alpha w(t) d_\alpha t - \int_a^b f(t) w(t) d_\alpha t \right| \tag{3.11}$$
$$\leq \|f - uD_\alpha(f)\|_\infty \left[\int_a^x w(t) d_\alpha t - \int_x^b w(t) d_\alpha t + \frac{1}{x^\alpha} \left(\int_x^b t^\alpha w(t) d_\alpha t - \int_a^x t^\alpha w(t) d_\alpha t \right) \right],$$

for each $x \in [a,b]$ and where $u(t) = \frac{t^{2-\alpha}}{\alpha}$, $t \in [a,b]$, and
$\|f - uD_\alpha(f)\|_\infty = \sup_{\xi \in (a,b)} |f(\xi) - uD_\alpha(f)(\xi)| < \infty.$

Proof. Multiplying both sides of the inequality (3.7) by $w(t)$, and later integrating both sides of the resulting inequality with respect to t from a to b for conformable fractional integrals, we have

$$\left| f(x) \int_a^b t^\alpha w(t) d_\alpha t - x^\alpha \int_a^b f(t) w(t) d_\alpha t \right|$$
$$\leq \|f - uD_\alpha(f)\|_\infty \int_a^b w(t) |t^\alpha - x^\alpha| d_\alpha t$$
$$= \|f - uD_\alpha(f)\|_\infty x^\alpha \left(\int_a^x w(t) d_\alpha t - \int_x^b w(t) d_\alpha t \right)$$
$$+ \|f - uD_\alpha(f)\|_\infty \left(\int_x^b t^\alpha w(t) d_\alpha t - \int_a^x t^\alpha w(t) d_\alpha t \right),$$

by this way, we obtain the inequality (3.11). □

Theorem 3.4. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable mapping on (a,b) with $0 < a < b$. Then, for any $x \in [a,b]$, we have the inequality*

$$\left| \frac{f(x)}{\alpha x^{\alpha}} - \frac{1}{b^{\alpha} - a^{\alpha}} \int_a^b \frac{f(t)}{t^{\alpha}} d_{\alpha} t \right| \leq \frac{2}{\alpha \left(b^{\alpha} - a^{\alpha} \right)} \left(\ln \frac{x^{\alpha}}{\sqrt{a^{\alpha} b^{\alpha}}} + \frac{\frac{a^{\alpha}+b^{\alpha}}{2} - x^{\alpha}}{x^{\alpha}} \right) \left\| f - uD_{\alpha}(f) \right\|_{\infty}, \tag{3.12}$$

where $u(t) = \frac{t^{2-\alpha}}{\alpha}$, $t \in [a,b]$, and
$\left\| f - uD_{\alpha}(f) \right\|_{\infty} = \sup_{\xi \in (a,b)} \left| f(\xi) - uD_{\alpha}(f)(\xi) \right| < \infty.$

Proof. If we divide both sides of (3.6) by $t^{\alpha} x^{\alpha}$, we obtain the inequality

$$\left| \frac{f(x)}{x^{\alpha}} - \frac{f(t)}{t^{\alpha}} \right| \leq \left\| f - uD_{\alpha}(f) \right\|_{\infty} \left| \frac{1}{x^{\alpha}} - \frac{1}{t^{\alpha}} \right|, \tag{3.13}$$

for any $t, x \in [a,b]$.

Integrating both sides of the above result over $t \in [a,b]$ by considering conformable fractional integrals, we find that

$$\begin{aligned} \left| \frac{f(x)}{x^{\alpha}} \frac{b^{\alpha} - a^{\alpha}}{\alpha} - \int_a^b \frac{f(t)}{t^{\alpha}} d_{\alpha} t \right| &\leq \int_a^b \left| \frac{f(x)}{x^{\alpha}} - \frac{f(t)}{t^{\alpha}} \right| d_{\alpha} t \\ &\leq \left\| f - uD_{\alpha}(f) \right\|_{\infty} \int_a^b \left| \frac{1}{x^{\alpha}} - \frac{1}{t^{\alpha}} \right| d_{\alpha} t. \end{aligned} \tag{3.14}$$

We observe that

$$\int_a^b \left|\frac{1}{x^\alpha}-\frac{1}{t^\alpha}\right| d_\alpha t = \int_a^b \left(\frac{1}{t^\alpha}-\frac{1}{x^\alpha}\right) d_\alpha t + \int_a^b \left(\frac{1}{x^\alpha}-\frac{1}{t^\alpha}\right) d_\alpha t \tag{3.15}$$

$$= \ln\frac{x}{a}-\frac{x^\alpha-a^\alpha}{\alpha x^\alpha}+\frac{b^\alpha-x^\alpha}{\alpha x^\alpha}-\ln\frac{b}{x}$$

$$= \frac{2}{\alpha}\left(\ln\frac{x^\alpha}{\sqrt{a^\alpha b^\alpha}}+\frac{\frac{a^\alpha+b^\alpha}{2}-x^\alpha}{x^\alpha}\right),$$

for any $x \in [a,b]$. If we sabstitute (3.15) in (3.14), then we deduce the desired inequality (3.12). □

Corollary 3.2. *Under the same assumptions of Theorem 3.4 with* $x^\alpha = \frac{a^\alpha+b^\alpha}{2}$*. Then, we have*

$$\left|\frac{f\left(\left(\frac{a^\alpha+b^\alpha}{2}\right)^{\frac{1}{\alpha}}\right)}{\alpha\frac{a^\alpha+b^\alpha}{2}}-\frac{1}{b^\alpha-a^\alpha}\int_a^b \frac{f(t)}{t^\alpha}d_\alpha t\right|$$

$$\leq \frac{2}{\alpha\left(b^\alpha-a^\alpha\right)}\left(\ln\frac{a^\alpha+b^\alpha}{2}-\ln\sqrt{a^\alpha b^\alpha}\right)\left\|f-uD_\alpha(f)\right\|_\infty .$$

We consider now the weighted version of the inequality (3.12).

Theorem 3.5. *Let* $\alpha \in (0,1]$ *and* $f:[a,b]\to\mathbb{R}$ *be a continuous on* $[a,b]$ *and an* α*-fractional differentiable mapping on* (a,b) *with* $0<a<b$*. If* $w:[a,b]\to\mathbb{R}$ *is nonnegative and* α*-fractional integrable on* $[a,b]$*, then one possesses*

$$\left|\frac{f(x)}{x^\alpha}\int_a^b w(t)d_\alpha t-\int_a^b \frac{f(t)}{t^\alpha}w(t)d_\alpha t\right|$$

$$\leq \left\|f-uD_\alpha(f)\right\|_\infty\left[\int_a^x \frac{w(t)}{t^\alpha}d_\alpha t-\int_x^b \frac{w(t)}{t^\alpha}d_\alpha t\right.$$

$$\left.+\frac{1}{x^\alpha}\left(\int_x^b w(t)d_\alpha t-\int_a^x w(t)d_\alpha t\right)\right],$$

for each $x \in [a, b]$ *and where* $u(t) = \frac{t^{2-\alpha}}{\alpha}$, $t \in [a, b]$ *and*
$\|f - uD_\alpha(f)\|_\infty = \sup_{\xi \in (a,b)} |f(\xi) - uD_\alpha(f)(\xi)| < \infty.$

Proof. If we use the inequality (3.13), we attain

$$\begin{aligned} & \left| \frac{f(x)}{x^\alpha} \int_a^b w(t) d_\alpha t - \int_a^b \frac{f(t)}{t^\alpha} w(t) d_\alpha t \right| \\ \leq & \int_a^b \left| \frac{f(x)}{x^\alpha} - \frac{f(t)}{t^\alpha} \right| w(t) d_\alpha t \\ \leq & \|f - uD_\alpha(f)\|_\infty \int_a^b \left| \frac{1}{x^\alpha} - \frac{1}{t^\alpha} \right| w(t) d_\alpha t. \end{aligned}$$

By simple calculations, the required inequality can be easily deduced, and thus the theorem is proved. □

4. Applications to Numerical Integration

In this section, we obtain some estimates of composite quadrature rules by taking into account the results given in the previous section.

We consider the partition of the interval $[a, b]$, $0 < a < b$, given by

$$I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b,$$

and $\xi_i \in [x_i, x_{i+1}]$, $i = 0, ..., n-1$ a sequence of intermediate points. We also define the quadrature

$$S(f, I_n, \xi) := \frac{1}{2\alpha} \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i^\alpha} \left(x_{i+1}^\alpha + x_i^\alpha \right) h_i, \tag{4.1}$$

where $h_i = (x_{i+1}^\alpha - x_i^\alpha)$, $i = 0, ..., n-1$.

Theorem 4.1. *Let* $\alpha \in (0, 1]$ *and* $f : [a, b] \to \mathbb{R}$ *be a continuous on* $[a, b]$ *and an* α*-fractional differentiable mapping on* (a, b) *with* $0 < a < b$*. Then we have the* *representation*

$$\int_a^b f(t)d_\alpha t = S(f, I_n, \xi) + R(f, I_n, \xi),$$

where $S(f, I_n, \xi)$ *is as defined in (4.1) and the remainder satisfies the estimation:*

$$|R(f, I_n, \xi)| \leq \frac{1}{\alpha}\|f - uD_\alpha(f)\|_\infty \sum_{i=0}^{n-1} \frac{h_i^2}{\xi_i^\alpha}\left[\frac{1}{4} + \left(\frac{\xi_i^\alpha - \frac{x_i^\alpha + x_{i+1}^\alpha}{2}}{h_i}\right)^2\right]. \tag{4.2}$$

Proof. Applying Theorem 3.2 on the interval $[x_i, x_{i+1}]$ for the intermediate points ξ_i, we obtain

$$\left|\frac{x_{i+1}^\alpha + x_i^\alpha}{2\alpha}\frac{f(\xi_i)}{\xi_i^\alpha}h_i - \int_{x_i}^{x_{i+1}} f(t)d_\alpha t\right|$$

$$\leq \frac{1}{\alpha}\frac{h_i^2}{\xi_i^\alpha}\left[\frac{1}{4} + \left(\frac{\xi_i^\alpha - \frac{x_i^\alpha + x_{i+1}^\alpha}{2}}{h_i}\right)^2\right]\|f - uD_\alpha(f)\|_\infty,$$

for all $i = 0, ..., n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality, we obtain the estimation (4.2). □

Now, we define the mid-point rule as follows:

$$M(f, I_n) := \frac{1}{\alpha}\sum_{i=0}^{n-1} f\left(\left(\frac{x_i^\alpha + x_{i+1}^\alpha}{2}\right)^{\frac{1}{\alpha}}\right)h_i,$$

where $h_i = (x_{i+1}^\alpha - x_i^\alpha)$, $i = 0, ..., n-1$.

Corollary 4.1. *Under the same assumptions of Theorem 4.1 with* $\xi_i^\alpha = \frac{x_i^\alpha + x_{i+1}^\alpha}{2}$. *Then, we have*

$$\int_a^b f(t)d_\alpha t = M(f, I_n) + R(f, I_n),$$

where the remainder satisfies the estimation:

$$|R(f, I_n)| \leq \frac{1}{2\alpha}.$$

5. Applications to Some Special Means

We define conformable arithmetic, geometric, and p-logarithmic means, respectively:

$$A_\alpha(a,b) = \frac{a^\alpha + b^\alpha}{2},$$

$$G_\alpha(a,b) = \sqrt{a^\alpha b^\alpha},$$

$$L_p^\alpha(a,b) = \left[\frac{b^{\alpha(p+1)} - a^{\alpha(p+1)}}{\alpha\,(p+1)\,(b^\alpha - a^\alpha)}\right]^{\frac{1}{p}},\ p \in \mathbb{R}\backslash\{-1,0\}.$$

In order to attain the results in this section, we will use inequalities obtained in Corollary 3.1 and Corollary 3.2.

Consider the mapping $f : (0,\infty) \to \mathbb{R}$, $f(t) = t^{\alpha p}$, $p \in \mathbb{R}\backslash\{-1,0\}$. Then, $0 < a < b$, we have

$$f\left(\left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}\right) = [A_\alpha(a,b)]^p,$$

$$\frac{1}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t = \left[L_p^\alpha(a,b)\right]^p,$$

and

$$\frac{1}{b^\alpha - a^\alpha}\int_a^b \frac{f(t)}{t^\alpha}d_\alpha t = \left[L_{p-1}^\alpha(a,b)\right]^{p-1}.$$

Also, if we use the identity (2.1), then we obtain

$$\begin{aligned}\|f - uD_\alpha(f)\|_\infty &= \delta(a,b)\\ &= \begin{cases}\left(1 - pa^{2-2\alpha}\right)a^{\alpha p}, & \text{if } \ p \in (-\infty,0)\setminus\{-1\},\\ \\ \left|1 - pb^{2-2\alpha}\right|b^{\alpha p}, & \text{if } \ p \in (0,1)\cup(1,\infty).\end{cases}\end{aligned}$$

Finally, if we use the corollary 3.1 and corollary 3.2, then we derive the inequalities

$$\begin{aligned}&\left|\frac{1}{\alpha}\left[A_\alpha(a,b)\right]^p - \left[L_p^\alpha(a,b)\right]^p\right|\\ \leq\ &\frac{(b^\alpha - a^\alpha)}{4\alpha A_\alpha(a,b)}\delta(a,b),\end{aligned}$$

and

$$\left|\frac{1}{\alpha}\left[A_{\alpha}(a,b)\right]^{p-1}-\left[L_{p-1}^{\alpha}(a,b)\right]^{p-1}\right|$$

$$\leq \frac{\delta(a,b)}{\alpha\left(b^{\alpha}-a^{\alpha}\right)}\ln\left[\frac{A_{\alpha}(a,b)}{G_{\alpha}(a,b)}\right]^{2},$$

respectively.

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