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SUM 2-IRREDUCIBLE SUBMODULES OF A MODULE

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Abstract. Let R be a commutative ring with identity and M be an R-module. M is said to be *sum-irreducible* precisely when it is non-zero and cannot be expressed as the sum of two proper submodules of itself. In this paper, we will introduce the concept of sum 2-irreducible submodules of M as a generalization of sum irreducible submodules of M and investigate some basic properties of this class of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

An ideal I of R is said to be *irreducible* if $I = J_1 \cap J_2$ for ideals J_1 and J_2 of R implies that either $I = J_1$ or $I = J_2$. A proper ideal I of R is said to be *strongly irreducible* if for ideals J_1, J_2 of R, $J_1 \cap J_2 \subseteq I$ implies that $J_1 \subseteq I$ or $J_2 \subseteq I$.

Strongly irreducible ideals were first studied by Fuchs, [13], under the name primitive ideals. Apparently the name "strongly irreducible" was first used by Blair in [8]. In [9, p. 177, Exercise 34] the strongly irreducible ideals are called quasi prime. More information about these classes of ideals can be found in [7], [15], and [17].

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An ideal I of R is said to be *2-irreducible* if whenever $I = J_1 \cap J_2 \cap J_3$ for ideals J_1, J_2 and J_3 of R, then either $I = J_1 \cap J_2$ or $I = J_1 \cap J_3$ or $I = J_2 \cap J_3$. Clearly, any irreducible ideal is a 2-irreducible ideal [11].

An R-module M is said to be *sum-irreducible* precisely when it is non-zero and cannot be expressed as the sum of two proper submodules of itself [16].

The main purpose of this paper is to introduce the concept of sum 2-irreducible submodules of an R-module M as a generalization of sum-irreducible submodules of M and obtain some related results.

We say that a non-zero submodule N of an R-module M is a *sum 2-irreducible submodule* if whenever $N = H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 + H_2$ or $N = H_2 + H_3$ or $N = H_1 + H_3$ (Definition 2.1).

For an R-module M, among other results, we prove that if N is a sum 2-irreducible submodule of an Artinian R-module M, then either N is sum-irreducible or N is a sum of exactly two sum-irreducible submodules of M (Theorem 2.6). Example 2.5 shows that the concepts of sum-irreducible submodules and of sum 2-irreducible submodules are different in general. Also, it is shown that if M is a strong comultiplication R-module, then every non-zero proper ideal of R is a 2-irreducible ideal if and only if every non-zero proper submodule of M is a sum 2-irreducible submodule of M (Theorem 2.11). Finally, let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposition of the ring R and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. We proved that if N is a sum 2-irreducible submodule of M, then either $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a sum 2-irreducible submodule of M_k, and $N_i = 0$ for every $i \in \{1, 2, ..., n\} \backslash \{k\}$ or $N = \times_{i=1}^{n} N_i$ such that for some $k, m \in \{1, 2, ..., n\}$, N_k is a sum-irreducible submodule of M_k, N_m is a sum-irreducible submodule of M_m, and $N_i = 0$ for every $i \in \{1, 2, ..., n\} \setminus \{k, m\}$ (Theorem 2.13).

2. Main results

Definition 2.1. We say that a non-zero submodule N of an R-module M is a *sum 2-irreducible submodule* if whenever $N = H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 + H_2$ or $N = H_2 + H_3$ or $N = H_1 + H_3$. Also, we say that M is a *sum 2-irreducible module* if M is a sum 2-irreducible submodule of itself.

Proposition 2.2. Let N be a sum 2-irreducible submodule of an R-module M. Then N is also a sum 2-irreducible submodule of T and N/K is a sum 2-irreducible submodule of M/K for any submodules T and K of M with $K \subseteq N \subseteq T$.

Proof. The first assertion is clear. Now let $N/K = H_1/K + H_2/K + H_3/K$ for submodules H_1, H_2 and H_3 of M. Then $N = H_1 + H_2 + H_3$ and so as N is a sum 2-irreducible submodule of M, either $N = H_1 + H_2$ or $N = H_2 + H_3$ or $N = H_1 + H_3$. Hence either $N/K = H_1/K + H_2/K$ or $N/K = H_2/K + H_3/K$ or $N/K = H_1/K + H_3/K$, as needed. □

Remark 2.3. It is easy to see that any strongly irreducible ideal is an irreducible ideal but the converse need not be true in general (see [17, 1.2]).

Let M be an R-module. M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I) = \{m \in M : Im = 0\}$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$, where $Ann_R(N) = \{r \in R : rN = 0\}$ [2]. M satisfies the *double annihilator condition* (DAC for short) if for each ideal I of R we have $I = Ann_R(0 :_M I)$ [12]. M is said to be a *strong comultiplication module* if M is a comultiplication R-module and satisfies the DAC condition [6].

Theorem 2.4. *Let M be an R-module. Then we have the following.*

(a) *If M is a strong comultiplication R-module and N is a sum 2-irreducible submodule of M, then $Ann_R(N)$ is a 2-irreducible ideal of R.*

(b) *If M is a comultiplication R-module and N is a non-zero submodule of M such that $Ann_R(N)$ is a 2-irreducible ideal of R, then N is a sum 2-irreducible submodule of M.*

Proof. (a) Let M be a strong comultiplication R-module, N be a sum 2-irreducible submodule of M, and $Ann_R(N) = J_1 \cap J_2 \cap J_3$ for some ideals J_1, J_2, and J_3 of R. Then $(0 :_M Ann_R(N)) = (0 :_M J_1 \cap J_2 \cap J_3)$. Now as M is a strong comultiplication R-module, $N = (0 :_M J_1) + (0 :_M J_2) + (0 :_M J_3)$. Thus by assumption, either $N = (0 :_M J_1) + (0 :_M J_2)$ or $N = (0 :_M J_1) + (0 :_M J_3)$ or $N = (0 :_M J_2) + (0 :_M J_3)$. Therefore, either $Ann_R(N) = J_1 \cap J_2$ or $Ann_R(N) = J_1 \cap J_3$ or $Ann_R(N) = J_2 \cap J_3$ since M is a strong comultiplication R-module.

(b) Let M be a comultiplication R-module and N be a non-zero submodule of M such that $Ann_R(N)$ be a 2-irreducible ideal of R. Let $N = H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M. Then $Ann_R(N) = Ann_R(H_1) \cap Ann_R(H_2) \cap Ann_R(H_3)$. Thus either $Ann_R(N) = Ann_R(H_1) \cap Ann_R(H_2)$ or $Ann_R(N) = Ann_R(H_1) \cap Ann_R(H_3)$ or $Ann_R(N) = Ann_R(H_2) \cap Ann_R(H_3)$ since $Ann_R(N)$ is a 2-irreducible ideal of R. Therefore, since M is a comultiplication R-module, either $(0 :_M Ann_R(N)) = (0 :_M Ann_R(H_1)) + (0 :_M Ann_R(H_2))$ or $(0 :_M Ann_R(N)) = (0 :_M Ann_R(H_1)) + (0 :_M Ann_R(H_3))$ or $(0 :_M Ann_R(N)) = (0 :_M Ann_R(H_2)) + (0 :_M Ann_R(H_3))$ by [4, 3.3]. Thus again since M is a comultiplication R-module, $N = H_1 + H_2$ or $N = H_1 + H_3$ or $N = H_2 + H_3$, as needed. □

The following example shows that the concepts of sum-irreducible submodules and sum 2-irreducible submodules are different in general.

Example 2.5. Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$. Then $\mathbb{Z}_6 = \bar{3}\mathbb{Z}_6 + \bar{2}\mathbb{Z}_6$ implies that $\mathbb{Z}_6$ is not a sum-irreducible $\mathbb{Z}$-module. But $Ann_{\mathbb{Z}}(\mathbb{Z}_6) = 6\mathbb{Z}$ is a 2-irreducible ideal of $\mathbb{Z}$ by [11, Example 1]. Since by [3, 3.8], the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a comultiplication $\mathbb{Z}$-module, $\mathbb{Z}_6$ is a sum 2-irreducible $\mathbb{Z}$-module by Theorem 2.4 (b).

Theorem 2.6. *Let M be an Artinian R-module. If N is a sum 2-irreducible submodule of M, then either N is sum-irreducible or N is a sum of exactly two sum-irreducible submodules of M.*

Proof. Let N be a sum 2-irreducible submodule of M. By [16, 5.3], N has a representation of a finite sum of sum-irreducible submodules $N = N_1 + N_2 + \cdots + N_k$. We show that either $k = 1$ or $k = 2$. If $k > 3$, then since N is sum 2-irreducible, $N = N_i + N_j$ for some $1 \leq i, j \leq k$, say $i = 1$ and $j = 2$. Therefore $N_3 \subseteq N_1 + N_2$, which is a contradiction. □

A non-zero submodule S of an R-module M is said to be *second* if for each $a \in R$, the endomorphism of M given by multiplication by a is either surjective or zero [18].

Let M be an R-module. For a submodule N of M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0). $N \neq 0$ is said to be a *second radical submodule of* M if $sec(N) = N$ (see [10] and [5]).

A non-zero submodule N of an R-module M is said to be a *strongly 2-absorbing secondary submodule* of M if whenever I, J are ideals of R, K is a submodule of M and $IJN \subseteq K$, then $I(sec(N)) \subseteq K$ or $J(sec(N)) \subseteq K$ or $IJ \subseteq Ann_R(N)$ [1].

Corollary 2.7. Let M be an Artinian comultiplication R-module. If N is a sum 2-irreducible submodule of M, then N is a strongly 2-absorbing secondary submodule of M.

Proof. Let N be a sum 2-irreducible submodule of M. By the fact that every sum-irreducible submodule of an Artinian R-module is secondary and regarding Theorem 2.6, we have either N is a secondary submodule or is a sum of two secondary submodules. It is clear that every secondary submodule is strongly 2-absorbing secondary, also the sum of two secondary submodules is a strongly 2-absorbing secondary submodule, by [1, 2.17 (c)]. □

Proposition 2.8. Let M be a comultiplication R-module and let N_1, N_2, and N_3 be second submodules of M such that $N_1 \cap N_2 = N_1 \cap N_3 = N_2 \cap N_3 = 0$. Then $N_1 + N_2 + N_3$ is not a sum 2-irreducible submodule of M.

Proof. This is clear. □

Corollary 2.9. Let M be a comultiplication R-module such that every non-zero submodule of M is sum 2-irreducible. Then M has at most two minimal submodules.

Proof. This follows from Proposition 2.8. □

Theorem 2.10. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a sum 2-irreducible submodule of M, then $f(N)$ is a sum 2-irreducible submodule of $f(M)$.*
(b) *If $\acute{N}$ is a sum 2-irreducible submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a sum 2-irreducible submodule of M.*

Proof. (a) Let N be a sum 2-irreducible submodule of M. Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Suppose that $f(N) = \acute{H}_1 + \acute{H}_2 + \acute{H}_3$ for submodules $\acute{H}_1$, $\acute{H}_2$ and $\acute{H}_3$ of $f(M)$. Then

$$N = f^{-1}(f(N)) = f^{-1}(\acute{H}_1) + f^{-1}(\acute{H}_2) + f^{-1}(\acute{H}_3)$$

since f is monomorphism. Thus by assumption, either $N = f^{-1}(H_1) + f^{-1}(\acute{H}_2)$ or $N = f^{-1}(\acute{H}_1) + f^{-1}(\acute{H}_3)$ or $N = f^{-1}(\acute{H}_2) + f^{-1}(\acute{H}_3)$. Now as $\acute{H}_1$, $\acute{H}_2$ and $\acute{H}_3$ are submodules of $f(\acute{M})$, we have either $f(N) = \acute{H}_1 + \acute{H}_2$ or $f(N) = \acute{H}_1 + \acute{H}_3$ or $f(N) = \acute{H}_2 + \acute{H}_3$ as needed.

(b) Let $\acute{N}$ be a sum 2-irreducible submodule of $f(M)$. If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $f^{-1}(\acute{N}) = H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M. Then $\acute{N} = f(f^{-1}(\acute{N})) = f(H_1)+f(H_2)+f(H_3)$. Thus by assumption, either $\acute{N} = f(H_1)+f(H_2)$ or $\acute{N} = f(H_1) + f(H_3)$ or $\acute{N} = f(H_2) + f(H_3)$. Now as f is monomorphism, either $f^{-1}(\acute{N}) = H_1 + H_2$ or $f^{-1}(\acute{N}) = H_1 + H_3$ or $f^{-1}(\acute{N}) = H_2 + H_3$, as desired. □

Theorem 2.11. *Let M be a strong comultiplication R-module. Then every non-zero proper ideal of R is a 2-irreducible ideal if and only if every non-zero proper submodule of M is a sum 2-irreducible submodule of M.*

Proof. " $\Rightarrow$ " Let $N = H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M. Then $Ann_R(N) = Ann_R(H_1) \cap Ann_R(H_2) \cap Ann_R(H_3)$. This implies that either $Ann_R(N) = Ann_R(H_1) \cap Ann_R(H_2)$ or $Ann_R(N) = Ann_R(H_1) \cap Ann_R(H_3)$ or $Ann_R(N) = Ann_R(H_2) \cap Ann_R(H_3)$ since by assumption, $Ann_R(N)$ is a 2-irreducible ideal. Therefore, either $N = H_1 + H_2$ or $N = H_1 + H_3$ or $N = H_2 + H_3$ because M is a comultiplication R-module.

" $\Leftarrow$ " Let I be a non-zero proper ideal of R and let $I = I_1 \cap I_2 \cap I_3$ for ideals I_1, I_2 and I_3 of R. Then by using [14, 2.6],

$$(0 :_M I) = (0 :_M I_1) + (0 :_M I_2) + (0 :_M I_3).$$

Thus by assumption, either $(0 :_M I) = (0 :_M I_1) + (0 :_M I_2)$ or $(0 :_M I) = (0 :_M I_1) + (0 :_M I_3)$ or $(0 :_M I) = (0 :_M I_2) + (0 :_M I_3)$. This implies that either $(0 :_M I) = (0 :_M I_1 \cap I_2)$ or $(0 :_M I) = (0 :_M I_1 \cap I_3)$ or $(0 :_M I) = (0 :_M I_2 \cap I_3)$. Thus either $I = I_1 \cap I_2$ or $I = I_1 \cap I_3$ or $I = I_2 \cap I_3$ since M is a strong comultiplication R-module. □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.12. *Let $R = R_1 \times R_2$ be a decomposition of the ring R and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M. If N is a sum 2-irreducible submodule of M, then either $N_1 = 0$ and N_2 sum 2-irreducible submodule of M_2 or $N_2 = 0$ and N_1 is a sum 2-irreducible submodule of M_1 or N_1, N_2 are sum-irreducible submodules of M_1, M_2, respectively.*

Proof. Let $N = N_1 \times N_2$ be a sum 2-irreducible submodule of M such that $N_2 = 0$. From our hypothesis, N is non-zero, so $N_1 \neq 0$. Set $\acute{M} = M_1 \times 0$. One can see that $\acute{N} = N_1 \times 0$ is a sum 2-irreducible submodule of $\acute{M}$. Also, observe that $\acute{M} \cong M_1$ and $\acute{N} \cong N_1$. Thus N_1 is a sum 2-irreducible submodule of M_1. Suppose that $N_1 \neq 0$ and $N_2 \neq 0$. We show that N_1 is a sum-irreducible submodule of M_1. Suppose that $N_1 = H_1 + K_1$ for some submodules H_1, K_1 of M_1. Then

$$N_1 \times N_2 = (H_1 + K_1) \times N_2 = (H_1 \times 0) + (K_1 \times 0) + (0 \times N_2).$$

Thus by assumption, either $N_1 \times N_2 = (H_1 \times 0) + (K_1 \times 0)$ or $N_1 \times N_2 = (H_1 \times 0) + (0 \times N_2)$ or $N_1 \times N_2 = (K_1 \times 0) + (0 \times N_2)$. Therefore, $N_1 = H_1$ or $N_1 = K_1$ since $N_2 \neq 0$. Hence, N_1 is a sum-irreducible submodule of M_1. Similarly, one can see that N_2 is sum-irreducible submodule of M_2. □

Theorem 2.13. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposition of the ring R and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. If N is a sum 2-irreducible submodule of M, then either $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a sum 2-irreducible submodule of M_k, and $N_i = 0$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$ or $N = \times_{i=1}^{n} N_i$ such* *that for some* $k, m \in \{1, 2, ..., n\}$, N_k *is a sum-irreducible submodule of* M_k, N_m *is a sum-irreducible submodule of* M_m, *and* $N_i = 0$ *for every* $i \in \{1, 2, ..., n\} \setminus \{k, m\}$.

Proof. We use induction on n. For $n = 2$ the result holds by Theorem 2.12. Now let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \cdots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 2.12, N is a sum 2-irreducible submodule of M if and only if either $N = L \times 0$ for some sum 2-irreducible submodule L of K or $N = 0 \times L_n$ for some sum 2-irreducible submodule L_n of M_n or $N = L \times L_n$ for some sum-irreducible submodule L of K and some sum-irreducible submodule L_n of M_n. Note that a non-zero submodule L of K is a sum-irreducible submodule of K if and only if $L = \times_{i=1}^{n-1} N_i$ such that for some $k \in \{1, 2, ..., n-1\}$, N_k is a sum-irreducible submodule of M_k, and $N_i = 0$ for every $i \in \{1, 2, ..., n-1\} \setminus \{k\}$. Consequently we reach the claim. □

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