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CAS WAVELETS STOCHASTIC OPERATIONAL MATRIX OF INTEGRATION AND ITS APPLICATION FOR SOLVING STOCHASTIC ITÔ-VOLTERRA INTEGRAL EQUATIONS

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Abstract. This article provides an effective technique for solving stochastic Itô-Volterra integral equations using Cosine and Sine (CAS) wavelets. A novel stochastic operational matrix of integration of CAS wavelets is developed in this article for solving stochastic Itô-Volterra integral equations. Stochastic Itô-Volterra integral equation can be reduced to a system of algebraic equations using the newly generated stochastic operational matrix of integration of CAS wavelets along with the operational matrix of integration of CAS wavelets. These system of algebraic equations can be solved using appropriate methods. Convergence and the error analysis of the proposed technique is studied in detail. Numerical examples are presented in order to show the efficiency and reliability of the proposed method.

1. Introduction

Wavelets have many interesting applications that are described below. Wavelets have also been used to analyze the coherent state of a specific quantum system [1]. In several science and engineering problems, integral equations arise. Various integral equations are studied by authors using various methods, of which the wavelet methods

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are one of them. CAS wavelets are used by many authors to solve integral and integro-differential equations. Some of them are found in [2–21]. Like integral equations, in studying many physical phenomena in life sciences and engineering stochastic or random integral equations are very important [22–24]. In recent years, the numerical simulation of stochastic integral equations has been an important area of research [25–29]. Many a times finding the analytic solution of stochastic integral equations is quite difficult. And hence providing computational methods to these equations are of great importance. Numerical methods to these equations using Block pulse functions and various wavelets have been studied by many authors, which are found in [30–39].

In this paper, a computational technique is given for solving stochastic Itô-Volterra integral equation,

$$y(x) = f(x) + \int_0^x k_1(x,t)\, y(t)\, dt + \int_0^x k_2(x,t)\, y(t)\, dW(t), \quad x \in [0,T), \tag{1.1}$$

using CAS wavelets. In equation (1.1), $f(x) \in L^2[0,1)$, $k_1(x,t), k_2(x,t) \in L^2[0,1) \times [0,1)$ for $x, t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P) and $y(x)$ is unknown. Also $W(x)$ is a Brownian motion process and $\int_0^x k_2(x,t)\, y(t)\, dW(t)$, is the Itô-integral.

We have derived a new stochastic operational matrix of integration of CAS wavelets. This technique uses the derived stochastic operational matrix of integration of CAS wavelets along with the operational matrix of integration of CAS wavelets to reduce the stochastic Itô-Volterra integral equations to a system of algebraic equations with unknown coefficients, which are solved by using efficient methods.

The paper is organized as follows: Definitions of Brownian motion and properties of CAS wavelets are studied and given in section 2. Method of solution is given in section 3. Convergence and error analysis of the proposed method is given in 4. Some numerical examples based on the proposed method are given in section 5. Finally. conclusion is drawn in section 6.

2. Brownian Motion and CAS Wavelets

2.1. **Brownian motion.** For definitions of Brownian motion see [40].

2.2. **CAS Wavelets.** CAS wavelets [13] $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments: $n = 0, 1, ..., 2^k - 1$, k is assumed to be any positive integer. They are defined on the interval $[0, 1)$ as follows:

$$\psi_{nm}(x) = \begin{cases} 2^{\frac{k}{2}} CAS_m \left(2^k x - n\right), & \frac{n}{2^k} \le x < \frac{n+1}{2^k}, \\ 0, & Otherwise, \end{cases} \tag{2.1}$$

with

$$CAS_m(x) = \cos(2m\pi x) + \sin(2m\pi x),$$

where $m = -M, -(M-1), ..., 0., ..., (M-1), M$. For instance, for $k = 1$ and $M = 1$, we get

$$\left.\begin{aligned} \psi_{0,(-1)}(x) &= \sqrt{2}\left(\cos\left(4\pi x\right) - \sin\left(4\pi x\right)\right) \\ \psi_{0,0}(x) &= \sqrt{2} \\ \psi_{0,1}(x) &= \sqrt{2}\left(\cos\left(4\pi x\right) + \sin\left(4\pi x\right)\right) \end{aligned}\right\} 0 \le x < \frac{1}{2},$$

$$\left.\begin{aligned} \psi_{1,(-1)}(x) &= \sqrt{2}\left(\cos\left(4\pi x\right) - \sin\left(4\pi x\right)\right) \\ \psi_{1,0}(x) &= \sqrt{2} \\ \psi_{1,1}(x) &= \sqrt{2}\left(\cos\left(4\pi x\right) + \sin\left(4\pi x\right)\right) \end{aligned}\right\} \frac{1}{2} \le x < 1.$$

2.3. **Function Approximation.** Suppose $q(x) \in L^2[0, 1)$ is expanded in terms of the CAS wavelets as

$$q(x) \simeq q^*(x) = \sum_{n=0}^{\infty} \sum_{m=-M}^{\infty} g_{n,m}\psi_{n,m}(x) = G^T \psi(x). \tag{2.2}$$

Truncating the above infinite series, we get

$$q(x) \simeq q^*(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} g_{nm}\psi_{nm}(x) = G^T \psi(x), \tag{2.3}$$

where, G and $\psi(x)$ are $\hat{m} \times 1$ $\left(\hat{m} = 2^k\,(2M+1)\right)$ matrices given by:

$$G = [g_{0,(-M)}, g_{0,(-(M-1))}, ..., g_{0,M}, g_{1,(-M)}, ..., g_{1,M}, ..., g_{2^k-1,(-M)}, ..., g_{2^k-1,M}\ ]^{\mathrm{T}}, \tag{2.4}$$

$$\begin{aligned}\psi(x) = [&\psi_{0,(-M)}(x), \psi_{0,(-(M-1))}(x), ..., \psi_{0,M}(x), \psi_{1,(-M)}(x), ..., \psi_{1,M}(x), ...,\\ &\psi_{2^k-1,(-M)}(x), ..., \psi_{2^k-1,M}(x)]^{\mathrm{T}}.\end{aligned} \tag{2.5}$$

And from equation (2.5) using the collocation point $x_i = \frac{(i-0.5)}{2^k(2M+1)}$, $i = 1, 2, ..., 2^k\,(2M+1)$, we can write for $k = 1$ and $M = 1$ $(\hat{m} = 6)$ as:

$$\psi(x) = \begin{bmatrix} \psi_{0,(-1)}(x) \\ \psi_{0,0}(x) \\ \psi_{0,1}(x) \\ \psi_{1,(-1)}(x) \\ \psi_{1,0}(x) \\ \psi_{1,1}(x) \end{bmatrix}$$

$$= \begin{bmatrix} -0.5176 & -1.4142 & 1.9319 & 0 & 0 & 0 \\ 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ 1.9319 & -1.4142 & -0.5176 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5176 & -1.4142 & 1.9319 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 1.9319 & -1.4142 & -0.5176 \end{bmatrix}.$$

2.4. **Operational Matrix of Integration of CAS Wavelets.** The operational matrix of integration of CAS wavelets P is given in detail in [13].

2.5. **Stochastic Operational Matrix of Integration of CAS Wavelets.** Here we derive a new stochastic operational matrix of CAS wavelets as follows:

The stochastic operational matrix of integration of CAS wavelets, P_S is defined as follows:

$$\int_0^x \psi(t)dW(t) = P_s\psi(x), \tag{2.6}$$

where P_s is a $\hat{m}\times\hat{m}$ matrix and is called the stochastic operational matrix of integration of CAS wavelets. In particular, for $M=1$ and $k=1$ ($\hat{m}=6$), we have

$$\begin{aligned}\int_0^x \psi_{0,(-1)}(x)dW(t) &= \begin{cases} \sqrt{2}((\cos(4\pi x)-\sin(4\pi x))\ W(x) & \\ -\int_0^x 4\pi\left(\sin(4\pi t)-\cos(4\pi t)\right)\ W(t)dt), & 0\le x<\dfrac{1}{2} \\ \sqrt{2}\left(W(x)+\displaystyle\int_0^{1/2} 4\pi W(t)dt\right), & \dfrac{1}{2}\le x<1 \end{cases} \\ &\simeq \left(-W\left(\frac{1}{4}\right)-\int_0^{\frac{1}{4}} 4\pi\left(\sin(4\pi t)-\cos(4\pi t)\right)\ W(t)dt\right)\psi_{0,0}(x) \\ &+\left(W\left(\frac{1}{2}\right)+\int_0^{1/2} 4\pi W(t)dt\right)\psi_{1,0}(x),\end{aligned} \tag{2.7}$$

$$\int_0^x \psi_{0,0}(t)dW(t) = \begin{cases} \sqrt{2}W\left(x\right), & 0\le x<\frac{1}{2} \\ \sqrt{2}W\left(\frac{1}{2}\right), & \frac{1}{2}\le x<1 \end{cases} \simeq W\left(\frac{1}{4}\right)\psi_{0,0}(x)+W\left(\frac{1}{2}\right)\psi_{1,0}(x), \tag{2.8}$$

$$\begin{aligned}\int_0^x \psi_{0,1}(x)dW(t) &= \begin{cases} \sqrt{2}((\cos(4\pi x)+\sin(4\pi x))\ W(x) & \\ +\int_0^x 4\pi\left(\sin(4\pi t)-\cos(4\pi t)\right)\ W(t)dt), & 0\le x<\dfrac{1}{2} \\ \sqrt{2}\left(W(x)-\int_0^{1/2} 4\pi W(t)dt\right), & \frac{1}{2}\le x<1 \end{cases} \\ &\simeq \left(-W\left(\frac{1}{4}\right)+\int_0^{\frac{1}{4}} 4\pi\left(\sin(4\pi t)-\cos(4\pi t)\right)\ W(t)dt\right)\psi_{0,0}(x) \\ &+\left(W\left(\frac{1}{2}\right)-\int_0^{1/2} 4\pi W(t)dt\right)\psi_{1,0}(x),\end{aligned} \tag{2.9}$$

$$\int_0^x \psi_{1,(-1)}(x)dW(t) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ \sqrt{2}\big((\cos(4\pi x) - \sin(4\pi x))\, W(x) - W\left(\frac{3}{4}\right) & \\ - \int_{1/2}^{x} 4\pi\left(\sin(4\pi\, t) - \cos(4\pi\, t)\right)\, W(t)dt\big), & 0 \leq x < 1 \end{cases}$$

$$\simeq \left(-\int_{\frac{1}{2}}^{\frac{3}{4}} 4\pi\left(\sin(4\pi t) - \cos(4\pi t)\right)\, W(t)dt\right)\psi_{1,0}(x), \tag{2.10}$$

$$\int_0^x \psi_{1,0}(t)dW(t) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ \sqrt{2}\left(W(x) - W\left(\frac{1}{2}\right)\right), & \frac{1}{2} \leq x < 1 \end{cases} \simeq \left(W\left(\frac{3}{4}\right) - W\left(\frac{1}{2}\right)\right)\psi_{1,0}(x), \tag{2.11}$$

$$\int_0^x \psi_{1,1}(x)dW(t) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ \sqrt{2}\big((\cos(4\pi x) + \sin(4\pi x))\, W(x) - W\left(\frac{3}{4}\right) & \\ + \int_{1/2}^{x} 4\pi\left(\sin(4\pi\, t) - \cos(4\pi\, t)\right)\, W(t)dt\big), & \dfrac{1}{2} \leq x < 1 \end{cases}$$

$$\simeq \left(\int_{\frac{1}{2}}^{\frac{3}{4}} 4\pi\left(\sin(4\pi t) - \cos(4\pi t)\right)\, W(t)dt\right)\psi_{1,0}(x). \tag{2.12}$$

Using equations (2.7) to (2.12), we get

$$\int_0^x \psi(t)dW(t) = \begin{bmatrix} \int_0^x \psi_{0,(-1)}(t)dW(t) \\ \int_0^x \psi_{0,0}(t)dW(t) \\ \int_0^x \psi_{0,1}(t)dW(t) \\ \int_0^x \psi_{1,(-1)}(t)dW(t) \\ \int_0^x \psi_{1,0}(t)dW(t) \\ \int_0^x \psi_{1,1}(t)dW(t) \end{bmatrix}.$$

Therefore,

$$\int_0^x \psi(t)dW(t)$$

$$= \underbrace{\begin{bmatrix} 0 & \left(-W\left(\frac{1}{4}\right) - \int_0^{\frac{1}{4}} 4\pi\left(\sin(4\pi t) - \cos(4\pi t)\right) W(t)dt\right. & 0 & 0 & W\left(\frac{1}{2}\right) + \int_0^{1/2} 4\pi W(t)dt & 0 \\ 0 & W\left(\frac{1}{4}\right) & 0 & 0 & W\left(\frac{1}{2}\right) & 0 \\ 0 & -W\left(\frac{1}{4}\right) + \int_0^{\frac{1}{4}} 4\pi\left(\sin(4\pi t) - \cos(4\pi t)\right) W(t)dt & 0 & 0 & W\left(\frac{1}{2}\right) - \int_0^{1/2} 4\pi W(t)dt & 0 \\ 0 & 0 & 0 & 0 & -\int_{\frac{1}{2}}^{\frac{3}{4}} 4\pi\left(\sin(4\pi t) - \cos(4\pi t)\right) W(t)dt & 0 \\ 0 & 0 & 0 & 0 & W\left(\frac{3}{4}\right) - W\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & \int_{\frac{1}{2}}^{\frac{3}{4}} 4\pi\left(\sin(4\pi t) - \cos(4\pi t)\right) W(t)dt & 0 \end{bmatrix}}_{P_S} \psi(x).$$

The stochastic operational matrix of integration of CAS wavelets is derived here in particular for $k = 1$ and $M = 1$ i.e., for $\hat{m} = 6$ and can extended for different values of k and M i.e., for different values of $\hat{m}$.

Remark 2.1. *For a $\hat{m}$-vector F, we have*

$$\psi(x)\psi^T(x)F = \tilde{F}\psi(x), \tag{2.13}$$

where, $\psi(x)$ is the CAS wavelet coefficient matrix and $\tilde{F}$ is an $\hat{m} \times \hat{m}$ matrix given by

$$\tilde{F} = \psi(x)\bar{F}\psi^{-1}(x), \tag{2.14}$$

where $\bar{F} = diag(\psi^{-1}(x)F)$. Also, for a $\hat{m} \times \hat{m}$ matrix X, we have

$$\psi^T(x)X\psi(x) = \hat{X}^T\psi(x), \tag{2.15}$$

where, $\hat{X}^T = V\psi^{-1}(x)$ and $V = diag(\psi^T(x)X\psi(x))$ is a $\hat{m}$-vector.

3. Method of Solution

Consider equation (1.1). In equation (1.1), let us approximate $f(x)$, $y(x)$, $k_1(x,t)$, and $k_2(x,t)$, with respect to CAS wavelets as follows:

$$y(x) \simeq G^T\psi(x) = G\psi^T(x), \tag{3.1}$$

where G is given in equation (2.4) and is the unknown vector to be determined.

$$f(x) \simeq F^T\psi(x) = F\psi^T(x), \tag{3.2}$$

$$k_1(x,t) \simeq \psi^T(x)K_1\psi(t) = \psi^T(t)K_1^T\psi(x), \tag{3.3}$$

and

$$k_2(x,t) \simeq \psi^T(x)K_2\psi(t) = \psi^T(t)K_2^T\psi(x), \tag{3.4}$$

where G and F are CAS wavelet coefficient vectors and K_1, K_2 are the CAS wavelet matrix. Substituting equations (3.1), (3.2), (3.3) and (3.4) in equation (1.1), we get

$$\begin{aligned} G^T\psi(x) =& F^T\psi(x) + \psi^T(x)K_1\left(\int_0^x \psi(t)\psi^T(t)Gdt\right) \\ &+ \psi^T(x)K_2\left(\int_0^x \psi(t)\psi^T(t)GdW(t)\right). \end{aligned}$$

Using the Remark 2.1, we get

$$G^T\psi(x) = F^T\psi(x) + \psi^T(x)K_1\left(\int_0^x \tilde{G}\psi(t)dt\right) + \psi^T(x)K_2\left(\int_0^x \tilde{G}\psi(t)dW(t)\right),$$

where $\tilde{G}$ is a $\hat{m}\times\hat{m}$ matrix. Using the operational matrix of integration and stochastic operational matrix of CAS wavelets, we get

$$G^T\psi(x) = F^T\psi(x) + \psi^T(x)K_1\tilde{G}P\psi(x) + \psi^T(x)K_2\tilde{G}P_s\psi(x).$$

Let $X_1 = K_1\tilde{G}P$ and $X_2 = K_2\tilde{G}P_s$. Again using the remark 2.1, we get,

$$G^T\psi(x) - \hat{X}_1^T\psi(x) - \hat{X}_2^T\psi(x) = F^T\psi(x),$$

where $\hat{X}_1$ and $\hat{X}_2$ are the $\hat{m}\times\hat{m}$ matrices and are linear functions of C and these equations are applicable for all $x \in [0,1)$, hence

$$G^T - \hat{X}_1^T - \hat{X}_2^T = F^T. \tag{3.5}$$

Solving this linear system of equations, we get the unknown vector C. Substituting this unknown vector in equation 3.1, we get the solution the stochastic volterra integral equation given in equation (1.1).

4. Convergence and Error Analysis

Lemma 4.1. *Let $q(x) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0,1)$ and $|q(x)| < \delta$, for every $x \in [0,1)$. Then, the CAS wavelet bases of $q(x)$ on equation (2.3) are bounded as:*

$$|g_{n,m}| \leq \frac{\delta}{2^{\frac{k}{2}}}, \tag{4.1}$$

where, δ is a constant.

Proof. Using CAS wavelets, any arbitrary function $q(x)$ can be approximated as:

$$q^*(x) \simeq \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} g_{n,m}\psi_{n,m}(x) = G^T\psi(x). \tag{4.2}$$

The coefficients $g_{n,m}$ in (4.2) are calculated as follows:

$$g_{n,m} = \int_0^1 q(x)\psi_{n,m}(x)dx. \tag{4.3}$$

Using the definition of $\psi_{n,m}(x)$ i.e., CAS wavelets, we have

$$\psi_{n,m}(x) = 2^{\frac{k}{2}} CAS_m(2^k x - n), \;\; \frac{n}{2^k} \leq x < \frac{n+1}{2^k}.$$

And therefore, equation (4.3) becomes,

$$g_{n,m} = \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} q(x) 2^{\frac{k}{2}} CAS_m(2^k x - n)dx. \tag{4.4}$$

Let $2^k x - n = v$, then equation (4.4) becomes:

$$g_{n,m} = \frac{1}{2^{\frac{k}{2}}} \int_0^1 y\left(\frac{v+n}{2^k}\right) CAS_m(v)dv. \tag{4.5}$$

Seeing the properties of CAS wavelets [10], we have

$$\int_0^1 |CAS_m(v)|dv \leq 1. \tag{4.6}$$

From equations (4.5) and using the assumption $|q(x)| \leq \delta$, we have

$$\begin{aligned} |g_{n,m}| &= \frac{1}{2^{\frac{k}{2}}} \left| \int_0^1 y\left(\frac{v+n}{2^k}\right) CAS_m(v) \right| dv \\ &\leq \frac{\delta}{2^{\frac{k}{2}}} \int_0^1 |CAS_m(v)|dv. \end{aligned} \tag{4.7}$$

Using equations (4.6) and (4.7), we get

$$|g_{n,m}| \leq \frac{\delta}{2^{\frac{k}{2}}}.$$

□

Theorem 4.1. *Let $q(x) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0,1)$ and $|q(x)| < \delta$ for every $x \in [0,1)$. By using the CAS wavelet expansion we approximate this function.*
Let $q^*(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} g_{nm}\psi_{nm}(x)$ be the CAS wavelet series. Then, the bound of the truncated error $E(x)$ is given as:

$$\|E(x)\|_2 = \|q(x) - q^*(x)\| \leq \left(\sum_{n=2^k}^{\infty} \sum_{m=-M}^{M} \alpha_m^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty} \sum_{m=-M}^{\infty} \alpha_m^2\right)^{\frac{1}{2}}, \tag{4.8}$$

where, $\alpha_m = \frac{\delta}{2^{\frac{k}{2}}}$.

Proof. Any function $q(x) \in L^2[0,1)$ can be expanded in terms of CAS wavelets as:

$$q(x) = \sum_{n=0}^{\infty} \sum_{m=-M}^{\infty} g_{n,m}\psi_{n,m}(x).$$

If $q^*(x)$ is the expansion truncated by using CAS wavelets, then the error obtained by truncating the above function can be computed as:

$$E(x) = q(x) - q^*(x) = \sum_{n=2^k}^{\infty} \sum_{m=-M}^{M} g_{n,m}\psi_{n,m}(x) + \sum_{n=0}^{\infty} \sum_{m=-M}^{\infty} g_{n,m}\psi_{n,m}(x). \tag{4.9}$$

From equation (4.9), we can write

$$\|E(x)\| \leq \|\sum_{n=2^k}^{\infty}\sum_{m=-M}^{M} g_{n,m}\psi_{n,m}(x)\| + \|\sum_{n=0}^{\infty}\sum_{m=-M}^{\infty} g_{n,m}\psi_{n,m}(x)\|$$

$$= \left(\int_0^1 |\sum_{n=2^k}^{\infty}\sum_{m=-M}^{M} g_{n,m}\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 |\sum_{n=0}^{\infty}\sum_{m=-M}^{\infty} g_{n,m}\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n=2^k}^{\infty}\sum_{m=-M}^{M} |g_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty}\sum_{m=-M}^{\infty} |g_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}}. \tag{4.10}$$

From Lemma 4.1, using the property, $|g_{n,m}| < \frac{\delta}{2^{\frac{k}{2}}}$, equation (4.10) reduces to,

$$\|E(x)\|_2 \leq \left(\sum_{n=2^k}^{\infty}\sum_{m=-M}^{M} |\frac{\delta}{2^{\frac{k}{2}}}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty}\sum_{m=-M}^{\infty} |\frac{\delta}{2^{\frac{k}{2}}}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}}. \tag{4.11}$$

Let us assume that

$$\alpha_m = \frac{\delta}{2^{\frac{k}{2}}}, \tag{4.12}$$

the definition of $\psi_{n,m}(x)$ i.e., CAS wavelets, we have

$$\psi_{n,m}(x) = 2^{\frac{k}{2}} CAS_m(2^k x - n), \;\; \frac{n}{2^k} \leq x < \frac{n+1}{2^k}.$$

Therefore,

$$\psi_{n,m}^2(x) = 2^k CAS_m^2(2^k x - n), \;\; \frac{n}{2^k} \leq x < \frac{n+1}{2^k}. \tag{4.13}$$

Integrating equation (4.13) with respect to x, we get

$$\int_0^1 \psi_{n,m}^2(x) dx = 2^k \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} CAS_m^2(2^k x - n) dx, \;\; \frac{n}{2^k} \leq x < \frac{n+1}{2^k}. \tag{4.14}$$

Let $2^k x - n = u$, equation (4.14) becomes,

$$\int_0^1 \psi_{n,m}^2(x) dx = \int_0^1 CAS_m^2(u) du. \tag{4.15}$$

Using the property of CAS wavelets [13], we have

$$\int_0^1 CAS_m^2(u)du = 1. \tag{4.16}$$

Substituting equation (4.16) in (4.15), we get

$$\int_0^1 \psi_{n,m}^2(x)dx = 1. \tag{4.17}$$

From (4.11), (4.12), and (4.17), we get

$$\|E(x)\|_2 \leq \left(\sum_{n=2^k}^{\infty} \sum_{m=-M}^{M} \alpha_m^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty} \sum_{m=-M}^{M} \alpha_m^2\right)^{\frac{1}{2}}.$$

□

Lemma 4.2. *Let $k(x,t) \in L^2(\mathbb{R} \times \mathbb{R})$ be a continuous function on $[0,1) \times [0,1)$ and $|k(x,t)| < \varsigma$, for each $[x,t) \in [0,1) \times [0,1)$. Then, the CAS wavelet bases of $k(x,t)$ are bounded as:*

$$|k_{n,m}| < \sum_{n_1=0}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=0}^{2^k-1} \sum_{m_2=-M}^{M} \frac{\varsigma}{2^k}, \tag{4.18}$$

where, ς is any constant.

Proof. Let us approximate $k(x,t)$ as $k^*(x,t) = \psi^T(t)K\psi(x)$. Here $K = [k_{n,m}]$ is a matrix of order $\hat{m} \times \hat{m}$ and

$$|k_{n,m}| \leq \sum_{n_1=0}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=0}^{2^k-1} \sum_{m_2=-M}^{M} \left|\left\langle\left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right)\right\rangle, \psi_{n_2,m_2}\left(t\right)\right\rangle\right|. \tag{4.19}$$

By the definition of inner product,

$$\left\langle\left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right)\right\rangle, \psi_{n_2,m_2}\left(t\right)\right\rangle = \int_0^1 \left[\int_0^1 k(t,x)\psi_{n_1,m_1}(x)\,dx\right]\psi_{n_2,m_2}(t)\,dt. \tag{4.20}$$

By the definition of CAS wavelets, equation (4.20) reduces to:

$$\begin{aligned} &\left\langle\left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right)\right\rangle, \psi_{n_2,m_2}\left(t\right)\right\rangle \\ &= 2^k \int_{\frac{n_2}{2^k}}^{\frac{n_2+1}{2^k}} \left[\int_{\frac{n_1}{2^k}}^{\frac{n_1+1}{2^k}} k(t,x) CAS_{m_1}\left(2^k x - n_1\right) dx\right] CAS_{m_2}\left(2^k t - n_2\right) dt. \end{aligned} \tag{4.21}$$

Let $2^k x - n_1 = v$ and $2^k t - n_2 = u$. Then equation (4.17) becomes:

$$\begin{aligned}&\left\langle\left\langle k\left(x,t\right),\psi_{n_1,m_1}\left(x\right)\right\rangle,\psi_{n_2,m_2}\left(t\right)\right\rangle\\&=\frac{1}{2^k}\int_0^1\left[\int_0^1\left[k\left(\frac{v+n_1}{2^k},\frac{u+n_2}{2^k}\right)\right]CAS_{m_1}\left(v\right)dv\right]CAS_{m_2}\left(u\right)du.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\langle\left\langle k\left(x,t\right),\psi_{n_1,m_1}\left(x\right)\right\rangle,\psi_{n_2,m_2}\left(t\right)\right\rangle\\&\leq\frac{1}{2^k}\int_0^1\int_0^1\left|k\left(\frac{v+n_1}{2^k},\frac{u+n_2}{2^k}\right)\right|\left|CAS_{m_1}\left(v\right)\right|\left|CAS_{m_2}\left(u\right)\right|dvdu.\end{aligned}\tag{4.22}$$

In the hypothesis it is assumed that $|k(x,t)| \leq \varsigma$ and hence, equation (4.22) becomes:

$$\left\langle\left\langle k\left(x,t\right),\psi_{n_1,m_1}\left(x\right)\right\rangle,\psi_{n_2,m_2}\left(t\right)\right\rangle<\frac{\varsigma}{2^k}\int_0^1\int_0^1\left|CAS_{m_1}\left(v\right)\right|\left|CAS_{m_2}\left(u\right)\right|dvdu.\tag{4.23}$$

From equation (4.6) and (4.23), we get

$$\left\langle\left\langle k\left(x,t\right),\psi_{n_1,m_1}\left(x\right)\right\rangle,\psi_{n_2,m_2}\left(t\right)\right\rangle<\frac{\varsigma}{2^k}.\tag{4.24}$$

And hence from (4.19), we get

$$|k_{n,m}|<\sum_{n_1=0}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=0}^{2^k-1}\sum_{m_2=-M}^{M}\frac{\varsigma}{2^k}.$$

□

Theorem 4.2. *Let $k(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ be a continuous function on $[0,1)\times[0,1)$ and $|k(x,t)| < \varsigma$ for all $[x,t) \in [0,1)\times[0,1)$. By using the CAS wavelet expansion we approximate this function.*
Let $k^*(x,t) = \sum_{n_1=0}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=0}^{2^k-1}\sum_{m_2=-M}^{M} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)$ be the CAS wavelet series. Then, the bound of the truncated error $E(x,t)$ can be given as:

$$\begin{aligned}\|E(x,t)\|_2&=\|k(x,t)-k^*(x,t)\|_2\\&<\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=2^k}^{2^k-1}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac12}+\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac12}\\&+\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac12}+\left(\sum_{n_1=2^k}^{\infty}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac12},\end{aligned}\tag{4.25}$$

where,

$$\rho_{n,m} = \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^k-1} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^k-1} \frac{\varsigma}{2^k}. \tag{4.26}$$

Proof. Any function $k(x,t) \in L^2(\mathbb{R} \times \mathbb{R})$ can be expanded in terms of CAS wavelets as:

$$k(x,t) = \sum_{m_1=0}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x) \psi_{n_2,m_2}(t).$$

If this expansion is truncated by using CAS wavelets, then then the error obtained by truncating the above function can be computed as:

$$\begin{aligned} E(x,t) &= k(x,t) - k^*(x,t) \\ &= \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{2^k-1} \sum_{m_2=-M}^{\infty} k_{n,m} \psi_{n_1,m_1}(x) \psi_{n_2,m_2}(t) \\ &+ \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m} \psi_{n_1,m_1}(x) \psi_{n_2,m_2}(t) \\ &+ \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{\infty} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m} \psi_{n_1,m_1}(x) \psi_{n_2,m_2}(t) \\ &+ \sum_{n_1=2^k}^{\infty} \sum_{m_1=-M}^{\infty} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m} \psi_{n_1,m_1}(x) \psi_{n_2,m_2}(t). \end{aligned}$$

Hence,

$$
\begin{aligned}
E(x,t) \leq & \left\| \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{2^k-1} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right\|_2 \\
& + \left\| \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right\|_2 \\
& + \left\| \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{\infty} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right\|_2 \\
& + \left\| \sum_{n_1=2^k}^{\infty} \sum_{m_1=-M}^{\infty} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right\|_2 \\
\leq & \left(\int_0^1 \int_0^1 \left| \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{2^k-1} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
& + \left(\int_0^1 \int_0^1 \left| \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
& + \left(\int_0^1 \int_0^1 \left| \sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{\infty} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
& + \left(\int_0^1 \int_0^1 \left| \sum_{n_1=2^k}^{\infty} \sum_{m_1=-M}^{\infty} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
\leq & \left(\sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{2^k-1} \sum_{m_2=-M}^{\infty} |k_{n,m}|^2 \int_0^1 \psi^2_{n_1,m_1}(x)\,dx \int_0^1 \psi^2_{n_2,m_2}(t)\,dt \right)^{\frac{1}{2}} \\
& + \left(\sum_{n_1=2^k}^{2^k-1} \sum_{m_1=-M}^{M} \sum_{n_2=2^k}^{\infty} \sum_{m_2=-M}^{\infty} |k_{n,m}|^2 \int_0^1 \psi^2_{n_1,m_1}(x)\,dx \int_0^1 \psi^2_{n_2,m_2}(t)\,dt \right)^{\frac{1}{2}}
\end{aligned}
$$

$$+\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}|k_{n,m}|^2\int_0^1\psi_{n_1,m_1}^2(x)\,dx\int_0^1\psi_{n_2,m_2}^2(t)\,dt\right)^{\frac{1}{2}}$$

$$+\left(\sum_{n_1=2^k}^{\infty}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}|k_{n,m}|^2\int_0^1\psi_{n_1,m_1}^2(x)\,dx\int_0^1\psi_{n_2,m_2}^2(t)\,dt\right)^{\frac{1}{2}}. \tag{4.27}$$

From equation (4.17), we have

$$\int_0^1\psi_{n_1,m_1}^2(x)dx=1,$$

and

$$\int_0^1\psi_{n_2,m_2}^2(x)dx=1.$$

And hence equation (4.27) reduces to,

$$\|E(x,t)\|_2\le\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=2^k}^{2^k-1}\sum_{m_2=-M}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}+\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}+\left(\sum_{n_1=2^k}^{\infty}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}. \tag{4.28}$$

Using Lemma 4.2, we have

$$|k_{n,m}|<\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^k-1}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^k-1}\frac{\varsigma}{2^k}. \tag{4.29}$$

Let

$$\rho_{n,m}=\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^k-1}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^k-1}\frac{\varsigma}{2^k}. \tag{4.30}$$

From (4.29) and (4.30), we have

$$|k_{n,m}|<\rho_{n,m}. \tag{4.31}$$

From equations (4.28) and (4.31), we get

$$\|E(x,t)\|_2 < \left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=2^k}^{2^k-1}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{M}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}}$$
$$+ \left(\sum_{n_1=2^k}^{2^k-1}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{n_1=2^k}^{\infty}\sum_{m_1=-M}^{\infty}\sum_{n_2=2^k}^{\infty}\sum_{m_2=-M}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}}.$$

□

Theorem 4.3. *Let $y(x)$ and $y^*(x)$ are the exact and approximate solution of equation (1.1), respectively. Also, let us assume that*

(1) $\|y(x)\|_2 \le \xi$,

(2) $\|k_i(x,t)\|_2 \le M_i, \ \ i = 1, 2$,

(3) $(M_1 + \gamma_1) + \|W(x)\|_\infty (M_2 + \gamma_2) < 1$,

then,

$$\|y(x) - y^*(x)\|_2 \le \frac{\lambda + \gamma_{1\xi} + \|W(x)\|_\infty \gamma_{2\xi}}{1 - (M_1 + \gamma_1) - \|W(x)\|_\infty (M_2 + \gamma_2)},$$

where

$$\lambda = \max \|q(x) - q^*(x)\|_2,$$

$$\gamma_i = \max \|k_i(x,t) - k_i^*(x,t)\|_2, \ \ i = 1, 2,$$

and λ and γ_i are given in Theorem 4.1 and Theorem 4.2, respectively.

Proof. Let us approximate all the known and unknown functions of equation (1.1) using CAS wavelets. Let us suppose that $f^*(x)$, $k_1^*(x,t)$, and $k_2^*(x,t)$ are approximations of $f(x)$, $k_1(x,t)$ and $k_2(x,t)$, respectively. Then,

$$y(x) - y^*(x) = q(x) - q^*(x) + \int_0^x (k_1(x,t)\,y(t) - k_1^*(x,t)\,y^*(t))\,dt$$
$$+ \int_0^x (k_2(x,t)\,y(t) - k_2^*(x,t)\,y^*(t))\,dW(t).$$

Thus,

$$\|y(x)-y^*(x)\|_2 \leq \|q(x)-q^*(x)\|_2 + \|(k_1(x,t)\,y(x)-k_1^*(x,t)\,y^*(x))\|_2 + \|W(x)\|_\infty \|(k_2(x,t)\,y(x)-k_2^*(x,t)\,y^*(x))\|_2 . \tag{4.32}$$

For $i=1,2$, we have

$$\|k_i(x,t)\,y(x)-k_i^*(x,t)\,y^*(x)\|_2 \leq \|k_i(x,t)\|_2 \|y(x)-y^*(x)\|_2 + \|k_i(x,t)-k_i^*(x,t)\|_2 \|y(x)-y^*(x)\|_2 + \|k_i(x,t)-k_i^*(x,t)\|_2 \|y(x)\|_2 .$$

Using Theorem 4.1 and assumptions 1 and 2, we have

$$\|k_i(x,t)\,y(x)-k_i^*(x,t)\,y^*(x)\|_2 \leq (M_i+\gamma_i)\|y(x)-y^*(x)\|_2 + \gamma_{i\xi}. \tag{4.33}$$

Using Theorem 4.2 and equations (4.32) and (4.33), we get

$$\|y(x)-y^*(x)\|_2 \leq \lambda + (M_1+\gamma_1)\|y(x)-y^*(x)\|_2 + \gamma_{1\xi} + \|W(x)\|_\infty ((M_2+\gamma_2)\|y(x)-y^*(x)\|_2 + \gamma_{2\xi}) .$$

By using the assumption (3), we finally conclude that

$$\|y(x)-y^*(x)\|_2 \leq \frac{\lambda+\gamma_{1\xi}+\|W(x)\|_\infty \gamma_{2\xi}}{1-(M_1+\gamma_1)-\|W(x)\|_\infty (M_2+\gamma_2)}.$$

□

5. Computational Experiments

In this section we present some numerical examples to show the efficiency and accuracy of the proposed method. The computations related to the test problems are performed in MATLAB 2016. Let $y_i(x)$ be the exact solution of the given test problem and $y_i^*(x)$ be its CAS wavelet solution obtained by the proposed method. Then the maximum absolute error is defined as:

$$\|E\|_\infty = \max_{1 \leq i < \hat{m}} |y_i(x) - y_i^*(x)|.$$

Test problem 5.1. *Let us consider [25],*

$$y(x) = \frac{1}{10} + \int_0^x ln(1+t)y(t)dt + \int_0^x ty(t)dW(t), \quad x, t \in [0,1). \tag{5.1}$$

Exact solution of (5.1) is $y(x) = \frac{1}{10} e^{(1+x)ln(1+x) - x - \frac{x^3}{6} + \int_0^x tdW(t)}$. *Table 1 compares the absolute values of test problem 5.1 obtained from method illustrated in section 3 at some selected points for* $\hat{m} = 6$ *and* $\hat{m} = 10$, *and table 2 shows the maximum absolute errors* ($\|E\|_\infty$) *of test problem 5.1. Figure 1 shows the absolute errors of test problem 5.1 for* $\hat{m} = 6, 10$.

TABLE 1. Absolute errors of test problem 5.1 for different values of $\hat{m}$.

x	$\hat{m}$=6	$\hat{m}$=10
0	1.6559e-02	5.5065e-04
0.1	1.5025e-03	1.0076e-04
0.2	3.0955e-03	1.0247e-04
0.3	7.7911e-03	5.8570e-04
0.4	1.5589e-02	1.0880e-03
0.5	1.9728e-02	2.0770e-03
0.6	1.9728e-02	2.0770e-03
0.7	3.4927e-02	2.7458e-03
0.8	5.2663e-02	5.3377e-03
0.9	7.7542e-02	4.3741e-03

TABLE 2. Comparison of maximum absolute errors of test problem 5.1.

$\hat{m}$	$\|E\|_\infty$
6	*7.7542e-02*
10	*5.3377e-03*

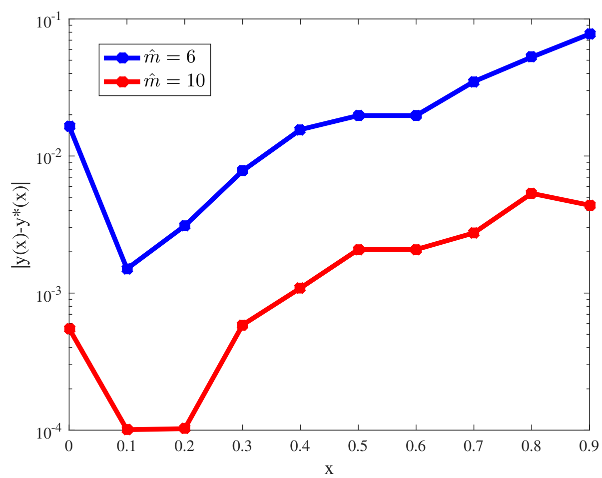

FIGURE 1. Comparison of maximum absolute errors of test problem 5.1.

Test problem 5.2. *Let us consider [25],*

$$y(x) = 1 + \int_0^x t^2 y(t)dt + \int_0^x ty(t)dW(t), \;\; x, t \in [0, 1). \tag{5.2}$$

Exact solution of (5.2) is $y(x) = e^{\frac{x^3}{6} + \int_0^x tdW(t)}$. *Table 3 compares the absolute values of test problem 5.2 obtained from method illustrated in section 3 at some selected points for* $\hat{m} = 6$ *and* $\hat{m} = 10$, *and table 4 shows the maximum absolute errors* ($\|E\|_\infty$) *of test problem 5.2. Figure 2 shows the absolute errors of test problem 5.2 for* $\hat{m} = 6, 10$.

TABLE 3. Absolute errors of test problem 5.2 for different values of $\hat{m}$.

x	$\hat{m}$=6	$\hat{m}$=10
0	7.3745e-03	4.3337e-04
0.1	9.0263e-03	1.0145e-04
0.2	9.0864e-03	1.0501e-04
0.3	9.2237e-03	1.1123e-04
0.4	9.4382e-03	6.8782e-04
0.5	5.2388e-03	1.2944e-03
0.6	5.2388e-03	1.2944e-03
0.7	8.0074e-03	1.9126e-03
0.8	1.1523e-02	2.0882e-03
0.9	1.2552e-02	2.4375e-03

TABLE 4. Comparison of maximum absolute errors of test problem 5.2.

$\hat{m}$	$\|E\|_\infty$
6	1.2552e-02
10	2.4375e-03

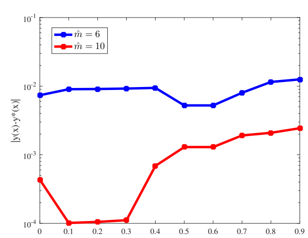

FIGURE 2. Comparison of maximum absolute errors of test problem 5.2.

6. Conclusion

In this article, we have provided an effective technique for solving stochastic Itô-Volterra integral equations using Cosine and Sine (CAS) wavelets. A novel stochastic operational matrix of integration of CAS wavelets is developed in this article for solving stochastic Itô-Volterra integral equations. Stochastic Itô-Volterra integral equation are reduced to a system of algebraic equations using the newly generated stochastic operational matrix of integration of CAS wavelets along with the operational matrix of integration of CAS wavelets. These system of algebraic equations are solved using appropriate methods. Convergence and the error analysis of the proposed technique is given in detail. Computational experiments show that the results obtained by using the proposed method are in good agreement with that of exact solution, and the maximum absolute error ($\|E\|_\infty$) decreases as the value of $\hat{m}$ increases, and hence we conclude that method proposed is efficient and reliable of the solving stochastic Itô-Volterra integral equations.

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