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LOCAL LINEAR PRESERVERS OF MATRIX MAJORIZATIONS

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Abstract. In this paper, we characterize the local linear preservers of right (resp. left) matrix majorization on $\mathbb{R}_n$ (resp. $\mathbb{R}^n$).

1. Introduction

Vector majorization is a much studied concept in linear algebra and its applications. The reader can find that majorization has been connected with combinatorics, analytic inequalities, numerical analysis, matrix theory, probability and statistics in a book written by Marshall, Olkin, and Arnold [6].

Let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices, and is abbreviated $\mathbf{M}_{n,n}$ to $\mathbf{M}_n$. We denote the set of 1-by-n (resp. n-by-1) real vectors by $\mathbb{R}_n$ (resp. $\mathbb{R}^n$).

A matrix $R = [r_{ij}] \in \mathbf{M}_{n,m}$ with nonnegative entries is called a row stochastic matrix if $\sum_{j=1}^{n} r_{ij} \leq 1$ for all i. For vectors $x, y \in \mathbb{R}_n$ (resp. $\mathbb{R}^n$), it is said that x is right (resp. left) matrix majorized by y (denoted by $x \prec_r y$ (resp. $x \prec_l y$)) if $x = yR$ (resp. $x = Ry$) for some n-by-n row stochastic matrix R.

Let V be a linear space of matrices, T be a linear function on V, and R be a relation on V. The linear function T is said to preserve R, if $R(TX, TY)$ whenever $R(X, Y)$.

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The following conventions will be fixed throughout the paper.

We will denote the notation $\mathbb{P}(n)$ for the collection of all n-by-n permutation matrices. e denotes an all ones vector. Also, let A^t be the transpose of a given matrix A. Let $[A_1/\ldots/A_n]$ be the n-by-n matrix with rows $A_1,\ldots,A_n\in\mathbb{R}_n$. Let $\mathbb{N}_k$ be the set $\{1,\ldots,k\}\subset\mathbb{N}$. For every $x=(x_1,\ldots,x_n)\in\mathbb{R}_n$ $(\mathbb{R}^n)$ define $\mathrm{tr(x)}:=\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{x_i}$. Let $[T]$ be the matrix representation of a linear function $T:\mathbb{R}_n\to\mathbb{R}_n$ $(T:\mathbb{R}^n\to\mathbb{R}^n)$ with respect to the standard basis. In this case, $Tx=xA$ $(Tx=Ax)$, where $A=[T]$.

In [4, 5], the authors obtained all linear preservers of $\prec_r$ and $\prec_l$ on $\mathbb{R}_n$ and $\mathbb{R}^n$, respectively. The following theorems characterize the linear preservers of right (left) matrix majorization. The case $n=1$ is evident.

Theorem 1.1. Let $T:\mathbb{R}_n\to\mathbb{R}_n$ be a linear function. Then T preserves $\prec_r$ if and only if one of the following conditions holds.

(i) If $n=2$;

- There exists some $\boldsymbol{a}\in\mathbb{R}_2$ such that

$$T(x)=\mathrm{tr(x)}\boldsymbol{a},\quad\forall\mathrm{x}\in\mathbb{R}_2.$$

- There exist $a,b\in\mathbb{R}$ $(ab\geq 0)$ such that

$$T(x)=x\begin{pmatrix}a&b\\b&a\end{pmatrix},\quad\forall x\in\mathbb{R}_2.$$

(ii) If $n\geq 3$;

- There exists some $\boldsymbol{a}\in\mathbb{R}_n$ such that

$$T(x)=\mathrm{tr(x)}\boldsymbol{a},\quad\forall\mathrm{x}\in\mathbb{R}_\mathrm{n}.$$

- There exist $\alpha\in\mathbb{R}$ and $P\in\mathbb{P}(n)$ such that

$$T(x)=\alpha xP,\quad\forall x\in\mathbb{R}_n.$$

Theorem 1.2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T preserves $\prec_l$ if and only if one of the following conditions holds.
(i) If $n = 2$; there exist $a, b \in \mathbb{R}$ $(ab \leq 0)$ such that

$$T(x) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} x, \quad \forall x \in \mathbb{R}^2.$$

(ii) If $n \geq 3$; there exist $\alpha \in \mathbb{R}$ and $P \in \mathbb{P}(n)$ such that

$$T(x) = \alpha P x, \quad \forall x \in \mathbb{R}^n.$$

A nonnegative real matrix D is called doubly stochastic if sum of entries of each row and each column of D is one.

Let X and Y be two matrices in $\mathbf{M}_{n,m}$. We say that X is multivariate majorized by Y, denoted by $X \prec Y$, if there exists a doubly stochastic matrix D in $\mathbf{M}_n$ such that $X = DY$.

Let $\phi : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear function. ϕ is said to be of local preserving multivariate majorization if for each $X \in \mathbf{M}_{n,m}$ there exists a linear function T_X of preserving multivariate majorization, which is depended on X, such that $\phi(X) = T_X(X)$.

In [7], the authors characterized all of linear functions that locally preserve the multivariate majorization. For more information about majorization see [1, 2, 3, 6].

Definition 1.3. A linear function $\phi : \mathbb{R}_n \to \mathbb{R}_n$ (resp. $\phi : \mathbb{R}^n \to \mathbb{R}^n$) is said to be a local preserving right (resp. left) matrix majorization if for each $x \in \mathbb{R}_n$ (resp. $\mathbb{R}^n$) there exists a linear function T_x of preserving right (resp. left) matrix majorization, which is depended on x, such that $\phi(x) = T_x(x)$.

In Section 2, we characterize all local linear preservers of right matrix majorization on $\mathbb{R}_n$. In section 3, we obtain the local linear preservers of left matrix majorization on $\mathbb{R}^n$.

The case $n = 1$ is evident. So we assume that $n \geq 2$.

2. Local right matrix majorization preserving linear functions on $\mathbb{R}_n$

In this section, we study local linear preservers of right matrix majorization on $\mathbb{R}_n$, and characterize all of the linear functions $\phi : \mathbb{R}_n \to \mathbb{R}_n$ local preserving right matrix majorization.

First, we obtain the local linear preservers of right matrix majorization on $\mathbb{R}_2$.

Theorem 2.1. Let $\phi : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear function. Then ϕ locally preserves $\prec_r$ if and only if $\mathrm{tr}(\mathrm{x}) = 0$ implies that $\mathrm{tr}\phi(\mathrm{x}) = 0$, $\forall x \in \mathbb{R}_2$.

Proof. First, assume that ϕ locally preserves $\prec_r$. Theorem 1.1 ensures that for each linear preserver T on $\mathbb{R}_2$ and for all $x \in \mathbb{R}_2$ that $\mathrm{tr}(\mathrm{x}) = 0$. So, we have $\mathrm{tr}\phi(\mathrm{x}) = 0$.

Next, suppose ϕ is a linear function which $\mathrm{tr}(\mathrm{x}) = 0$ implies that $\mathrm{tr}\phi(\mathrm{x}) = 0$, $\forall x \in \mathbb{R}_2$. Set

$$[\phi] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By the hypothesis, since $\mathrm{tr}(1, -1) = 0$, we see

$$\begin{aligned} 0 &= \mathrm{tr}\phi(1, -1) \\ &= a - c + b - d, \end{aligned}$$

and so

$$d = (a + b) - c.$$

Define the linear function T on $\mathbb{R}_2$ as

$$T(x) = (a - c)x, \quad \forall x \in \mathbb{R}_2.$$

Clearly, T preserves $\prec_r$.

Let $x_0 \in \mathbb{R}_2$. If $\mathrm{tr(x_0)} = 0$, then $\phi(x_0) = T(x_0)$. If $\mathrm{tr(x_0)} \neq 0$; then we set the linear function T_{x_0} on $\mathbb{R}_2$ as

$$T_{x_0}(x) = \mathrm{tr(x)}[\frac{1}{\mathrm{tr(x_0)}}\phi(\mathrm{x_0})], \quad \forall \mathrm{x} \in \mathbb{R}_2.$$

We observe that T_{x_0} preserves $\prec_r$ and $\phi(x_0) = T(x_0)$. Therefore, ϕ locally preserves $\prec_r$. □

In the following theorem, we characterize the linear functions $\phi : \mathbb{R}_n \to \mathbb{R}_n$ local preserving right matrix majorization whenever $(n \geq 3)$.

Theorem 2.2. Let $\phi : \mathbb{R}_n \to \mathbb{R}_n$ $(n \geq 3)$ be a linear function. Then ϕ locally preserves $\prec_r$ if and only if there exist $\boldsymbol{a} \in \mathbb{R}_n$, $\alpha \in \mathbb{R}$, and $P \in \mathbb{P}(n)$ such that

$$\phi(x) = \mathrm{tr(x)}\boldsymbol{a} + \alpha \mathrm{xP}, \quad \forall \mathrm{x} \in \mathbb{R}_\mathrm{n}.$$

Proof. We only prove the necessity of the condition. Assume that ϕ locally preserves $\prec_r$. Set

$$W = \{x \in \mathbb{R}_n \mid \mathrm{tr(x)} = 0\},$$

and

$$f_{ij} = e_i - e_j, \quad \forall i, j \in \mathbb{N}_n.$$

We see $\mathrm{dim(W) = n - 1}$ and the set $\{f_{1j} \mid j = 2, \ldots, n\}$ is a basic for W. If $x \in W$, then $\phi(x) \in W$. So $\phi\mid_W : W \longrightarrow W$ is a linear function.

We claim that there exist $\alpha \in \mathbb{R}$ and $P \in \mathbb{P}(n)$ such that

$$\phi\mid_W (x) = \alpha x P, \quad \forall x \in W.$$

If ϕ at the point $x \in W$ has the form (i) of Theorem 1.1, then $\phi(x) = 0$. By choosing $\alpha_x = 0$ for each $P \in \mathbb{P}(n)$, we have $\phi(x) = \alpha_x x P$. So ϕ in each point $x \in W$

has the form (ii) of Theorem 1.1, and then there exist $\alpha_x \in \mathbb{R}$ and $P_x \in \mathbb{P}(n)$ such that

$$\phi(x) = \alpha_x x P_x.$$

If $\phi\mid_W = 0$, then the proof is complete. So, suppose $\phi\mid_W \neq 0$, and we prove $\phi\mid_W$ is invertible. As $\phi\mid_W \neq 0$, there exist $\alpha \in \mathbb{R}\setminus\{0\}$, $i_0, j_0 \in \mathbb{N}_n$ $(i_0 \neq j_0)$, and $P \in \mathbb{P}(n)$ such that

$$\phi(f_{i_0j_0}) = \alpha f_{i_0j_0} P.$$

If $\phi\mid_W$ is not invertible; then there exists some $x = (x_1, \ldots, x_n) \in W\setminus\{0\}$ where $\mid x_i \mid < 1$ $(i \in \mathbb{N}_n)$ such that $\phi(x) = 0$. We have

$$\phi(x + f_{i_0j_0}) = \alpha f_{i_0j_0} P.$$

Since $\phi(x) = 0$, we observe that $x \notin \mathrm{span}\{\mathrm{f_{i_0j_0}}\}$. So $x + f_{i_0j_0} \notin \mathrm{span}\{\mathrm{f_{i_0j_0}}\}$.

On the other hand, since ϕ locally preserves $\prec_r$ and $x + f_{i_0j_0} \in W$, there exist $\beta \in \mathbb{R}$, and $Q \in \mathbb{P}(n)$ such that

$$\phi(x + f_{i_0j_0}) = \beta(x + f_{i_0j_0})Q.$$

Hence

$$\alpha f_{i_0j_0} P = \beta(x + f_{i_0j_0})Q,$$

which is a contradiction, because the set of all nonzero components of $\alpha f_{i_0j_0} P$ is two numbers with different sign, but, as $x + f_{i_0j_0} \notin \mathrm{span}\{\mathrm{f_{i_0j_0}}\}$ and $\mid x \mid < 1$, the set of all nonzero components of $\beta(x + f_{i_0j_0})Q$ is not two numbers with different sign. Thus, $\phi\mid_W$ is invertible.

Assume that $i, j, k \in \mathbb{N}_n$ are mutually distinct. The hypothesis ensures that there exist $r, s, p, q, u, v \in \mathbb{N}_n$, and $a, b, c \in \mathbb{R}$ such that

$$\phi(f_{ij}) = a f_{rs}, \quad \phi(f_{ki}) = b f_{pq}, \quad \text{and} \quad \phi(\mathrm{f_{jk}}) = \mathrm{cf_{uv}}.$$

Since ϕ is invertible, this implies that

$$a, b, c \neq 0, \quad r \neq s, \quad p \neq q, \quad \text{and} \quad \mathrm{u} \neq \mathrm{v}.$$

Without loss of generality, we assume that a, b, and c have the same sign. As ϕ is invertible,

$$\{f_{ij}, f_{ki}\}, \quad \{f_{ij}, f_{jk}\}, \text{and} \quad \{\mathrm{f_{ki}}, \mathrm{f_{jk}}\}$$

are linear independ, we deduce

$$\{\phi(f_{ij}), \phi(f_{ki})\}, \quad \{\phi(f_{ij}), \phi(f_{jk})\}, \text{and} \quad \{\phi(\mathrm{f_{ki}}), \phi(\mathrm{f_{jk}})\}$$

are linear independ, too.

The relation $f_{ij} + f_{ki} + f_{jk} = 0$ shows that

$$\phi(f_{ij}) + \phi(f_{ki}) + \phi(f_{jk}) = 0,$$

and so

$$a(e_r - e_s) + b(e_p - e_q) + c(e_u - e_v) = 0.$$

Now, we conclude $a = b = c$, and we have one of the following cases.

- $q = r$, $u = s$, $v = p$.
- $p = s$, $u = q$, $v = r$.

Thus,

- $\phi(f_{ij}) = af_{rs}$, $\phi(f_{jk}) = af_{pr}$, $\phi(f_{ki}) = af_{sp}$.
- $\phi(f_{ij}) = af_{rs}$, $\phi(f_{jk}) = af_{sq}$, $\phi(f_{ki}) = af_{qr}$.

So there is a unique permutation δ such that one of the following cases holds.

- $\phi(f_{ij}) = af_{\delta_j \delta_i} = a(e_{\delta_j} - e_{\delta_i}) = -a(e_{\delta_i} - e_{\delta_j})$, $\forall i, j \in \mathbb{N}_n$.
- $\phi(f_{ij}) = af_{\delta_i \delta_j} = a(e_{\delta_i} - e_{\delta_j})$, $\forall i, j \in \mathbb{N}_n$.

By putting $\alpha = a$ or $\alpha = -a$, in both cases, we have

$$A_i - A_j = \alpha(e_{\delta_i} - e_{\delta_j}), \quad \forall i, j \in \mathbb{N}_n.$$

We define the vector $\boldsymbol{a} \in \mathbb{R}_n$ as $\boldsymbol{a} = A_1 - \alpha e_{\delta_1}$. Since for all $i, j \in \mathbb{N}_n$

$$A_i - \alpha e_{\delta_i} = A_j - \alpha e_{\delta_j},$$

we see

$$A_i = \boldsymbol{a} + \alpha e_{\delta_i}, \forall i \in \mathbb{N}_n.$$

Choose $P = [e_{\delta_1} / \ldots / e_{\delta_n}] \in \mathbb{P}(n)$. This follows that $e_{\delta_i} = e_i P$, $\forall i \in \mathbb{N}_n$.

Now, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$ we observe that

$$\begin{aligned} T(x) &= xA \\ &= \textstyle\sum_{i=1}^n x_i A_i \\ &= \textstyle\sum_{i=1}^n x_i (\boldsymbol{a} + \alpha e_{\delta_i}) \\ &= (\textstyle\sum_{i=1}^n x_i) \boldsymbol{a} + \alpha \textstyle\sum_{i=1}^n x_i e_{\delta_i} \\ &= \mathrm{tr(x)} \boldsymbol{a} + \alpha \textstyle\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_i e_i P} \\ &= \mathrm{tr(x)} \boldsymbol{a} + \alpha \mathrm{xP}, \end{aligned}$$

as desired. □

3. Local left matrix majorization preserving linear functions on $\mathbb{R}^n$

Here, we characterize the linear functions $\phi : \mathbb{R}_n \longrightarrow \mathbb{R}_n$ local preserving left matrix majorization. For $n = 2$, we can give the following Theorem.

Theorem 3.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function. Then the following conditions are equivalent.

(i) ϕ preserves $\prec_l$.

(ii) ϕ locally preserves $\prec_l$.

(iii) There exist $a, b \in \mathbb{R}$ $(ab \leq 0)$ such that

$$Tx = \begin{pmatrix} a & b \\ b & a \end{pmatrix} x, \quad \forall x \in \mathbb{R}^2.$$

Proof. By the use of Theorem 1.2, we only show that *(ii)* implies *(iii)*. Theorem 1.2 ensures that for each linear preserver $\prec_l$, namely T, on $\mathbb{R}^2$ we have

- $Te \in \text{span(e)}$.
- If $\text{tr(x)} = 0$, then $\text{trT(x)} = 0$, $\forall x \in \mathbb{R}^2$.
- $(Te_1)_1(Te_1)_2 \leq 0$.

Suppose that ϕ locally preserves $\prec_l$ and $[\phi] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. We see $\phi(e) = (a+c, b+d)^t$, and hence

$$a + c = b + d \tag{3.1}$$

Since $\text{tr}(\mathrm{e}_1 - \mathrm{e}_2) = 0$, we have

$$\begin{aligned} 0 &= \text{tr}\phi(\mathrm{e}_1 - \mathrm{e}_2) \\ &= \text{tr}(\mathrm{a} - \mathrm{c}, \mathrm{b} - \mathrm{d})^{\mathrm{t}} \\ &= (a - c) + (b - d). \end{aligned}$$

This follows that

$$a - c = -b + d. \tag{3.2}$$

The relations (3.1) and (3.2) ensure that $a = d$ and $b = c$, and then

$$[\phi] = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

On the other hand, since $(\phi e_1)_1(\phi e_1)_2 \leq 0$, we observe that $ab \leq 0$, and the proof is complete. □

In the following theorem, we obtain the local linear preservers of left matrix majorization on $\mathbb{R}^n$ $(n \geq 3)$.

Theorem 3.2. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 3)$ be a linear function. Then the following conditions are equivalent.
(i) ϕ preserves $\prec_l$.
(ii) ϕ locally preserves $\prec_l$.
(iii)There exist $\alpha \in \mathbb{R}$ and $P \in \mathbb{P}(n)$ such that

$$\phi(x) = \alpha P x, \quad \forall x \in \mathbb{R}^n.$$

Proof. The equivalence of the relations (i) and (iii) results from Theorem 1.2. It is easily seen that the condition (iii) is a result the condition (ii). . So, it suffices to prove if ϕ locally preserves $\prec_l$, then there exist $\alpha \in \mathbb{R}$ and $P \in \mathbb{P}(n)$ such that

$$\phi(x) = \alpha P x, \quad \forall x \in \mathbb{R}^n.$$

Suppose that ϕ locally preserves $\prec_l$. In this case, for each $i \in \mathbb{N}_n$ there exist $\alpha_i \in \mathbb{R}$ and $P_i \in \mathbb{P}(n)$ such that

$$\phi(e_i) = \alpha_i P_i e_i.$$

Let $f : \mathbb{N}_n \longrightarrow \mathbb{N}_n$ be a function that is defined as

$$e_{f(i)} = P_i e_i, \quad \forall i \in \mathbb{N}_n.$$

So, we have

$$\phi(e_i) = \alpha_i e_{f(i)}, \quad \forall i \in \mathbb{N}_n.$$

Assume that $i \neq j$ $(i, j \in \mathbb{N}_n)$. The hypothesis ensures that there exist $\alpha \in \mathbb{R}$ and $Q \in \mathbb{P}(n)$ such that

$$\begin{aligned} \phi(e_i + e_j) &= \alpha Q(e_i + e_j) \\ &= \alpha e_k + \alpha e_l, \end{aligned}$$

where $e_k = Q e_i$, and $e_l = Q e_j$. This shows that $k \neq l$.

On the other hand,

$$\begin{aligned}\phi(e_i+e_j) &= \phi(e_i)+\phi(e_j)\\ &= \alpha_i e_{f(i)}+\alpha_j e_{f(j)}.\end{aligned}$$

So,

$$\alpha e_k+\alpha e_l=\alpha_i e_{f(i)}+\alpha_j e_{f(j)}.$$

As $k \neq l$, this follows that

$$\alpha_i=\alpha_j=\alpha,$$

and

$$\{f(i), f(j)\}=\{k, l\},$$

especially

$$f(i)\neq f(j).$$

Then

$$\alpha_1=\cdots=\alpha_n=\alpha,$$

and f is a permutation function on $\mathbb{N}_n$.

Suppose $P \in \mathbb{P}(n)$ is a permutation which its i^{th} column is $e_{f(i)}$. Thus,

$$\phi(e_i)=\alpha P e_i, \quad \forall i \in \mathbb{N}_n.$$

Then

$$\phi(x)=\alpha P x, \quad \forall x \in \mathbb{R}^n.$$

□

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