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ON CONDUCTOR IDEALS OF QUADRATIC ORDERS

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Abstract. In this paper we study the orders in quadratic number fields. We give a complete description of conductor ideals of certain orders in quadratic number fields and investigate some of their important arithmetic properties.

1. Introduction

Let K be an algebraic number field. The ring of integers of K is denoted by $\mathcal{O}_K$. A subring of $\mathcal{O}_K$ with quotient field K is called an order in K and is denoted by $\mathcal{O}$. The orders can also be defined equivalently in many other ways, cf. [15, page: 73-81]. Clearly $\mathcal{O}_K$ is an order in K and hence it is the maximal order in the sense of set inclusion. We say that an order $\mathcal{O}$ is proper if $\mathcal{O} \neq \mathcal{O}_K$. In algebraic number theory, the rings $\mathcal{O}$ and $\mathcal{O}_K$ are important objects to study because of their remarkable properties. Also in algebraic geometry, the affine space $Spec(\mathcal{O})$ is non-singular if and only if $\mathcal{O} = \mathcal{O}_K$, where $Spec(\mathcal{O})$ denotes the set of all prime ideals of the ring $\mathcal{O}$, cf. [9, Theorem 5.1]. Therefore, we have a correspondence between the study of $\mathcal{O}$ and $\mathcal{O}_K$ with the study of singular and non-singular curves, respectively. For detailed study about the facts related to $\mathcal{O}$ and $\mathcal{O}_K$, see [1] and [15]. The conductor of $\mathcal{O}$, denoted by $\mathcal{O} : \mathcal{O}_K$, is defined by the set $\{x \in K |\ x\mathcal{O}_K \subseteq \mathcal{O}\}$ which is always non-zero and is a proper ideal of $\mathcal{O}$ if and only if $\mathcal{O}$ is a proper subring of

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$\mathcal{O}_K$. Moreover, $\mathcal{O} : \mathcal{O}_K$ is a maximal common ideal of both $\mathcal{O}$ and $\mathcal{O}_K$. It plays a significant role in the factorization based properties of orders, cf. [8]. The orders in algebraic number fields and their conductor ideals have been investigated by many authors (see [3, 6, 11, 12, 13, 14, 16, 18]).

Let $d \neq 0, 1$ be an integer. The ring $\mathbb{Z}[\sqrt{d}]$ is an order in the quadratic number field $\mathbb{Q}(\sqrt{d}\ )$ and is often named as quadratic order. Moreover, every quadratic order can be written as $\mathbb{Z}[f\sqrt{d}]$ or $\mathbb{Z}[f(1+\sqrt{d})/2]$ for some integer $f \geq 1$, cf. [14, Proposition 4] or [4, page: 133-134]. The quadratic order $\mathbb{Z}[\sqrt{d}]$ is proper order if either d is non-square-free or d is square-free with $d \equiv 1\ (mod\ 4)$. In this paper we completely describe the conductor ideals of the proper quadratic orders of the form $\mathbb{Z}[\sqrt{d}]$ and investigate some of their important arithmetic properties. We obtain the following results:

If $d = m^2 d' \in \mathbb{Z} - \{0, 1\}$, where d' is square-free integer with $d' \equiv\ 1\ (mod\ 4)$ and $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(2m, m + \sqrt{d}\big)$ which is a non-principal ideal in $\mathcal{O}$ but principal in $\mathcal{O}_K$ generated by $2m$. Moreover, if $m > 1$, then $\mathcal{O} : \mathcal{O}_K$ is always non-maximal in both $\mathcal{O}$ and $\mathcal{O}_K$ (Theorems 3.1, 3.2, Remark 3.1). If $d \in \mathbb{Z} - \{0, 1\}$ is square-free with $d \equiv 1\ (mod\ 4)$ and $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(2, 1 + \sqrt{d}\big)$ which is a non-principal ideal in $\mathcal{O}$ but principal in $\mathcal{O}_K$ generated by 2. Moreover, in this case $\mathcal{O} : \mathcal{O}_K$ is always maximal in $\mathcal{O}$ but maximal in $\mathcal{O}_K$ if and only if $d \equiv 5\ (mod\ 8)$ (Corollaries 3.1, 3.2, Remark 3.1, Theorem 3.3). If $d = m^2 d' \in \mathbb{Z} - \{0, 1\}$, where d' is square-free with $d' \equiv 2, 3\ (mod\ 4)$ and $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(m, \sqrt{d}\big)$ which is a non-principal ideal in $\mathcal{O}$ for $m \neq \pm 1$ but always principal in $\mathcal{O}_K$. Moreover, in this case $\mathcal{O} : \mathcal{O}_K$ is maximal ideal in $\mathcal{O}$ if and only if m is prime and it is maximal ideal in $\mathcal{O}_K$ if and only if m is prime and $x^2 - d'$ is irreducible in $\mathbb{Z}_m[x]$ (Theorems 3.4, 3.5, 3.6, 3.7, Remark 3.2). If $d \in \mathbb{Z} - \{0, 1\}$, then $\big(n, 1 + \sqrt{d}\big)$ is a maximal ideal in $\mathbb{Z}[\sqrt{d}]$ if and only if $gcd(n, 1 - d)$ is prime (Theorem 3.8).

2. Preliminaries

For the reader's convenience we give a working introduction here for the notions involved. Let $A \subseteq B$ be integral domains. An element $b \in B$ is said to be *integral* over A it is root of some monic polynomial with coefficients in A and is said to be *algebraic* over A if it is root of some polynomial with coefficients in A. Clearly, an element is integral over a field K if and only if it is algebraic over K. The integral closure of A in B is denoted by A'_B and is defined as $A'_B = \{x \in B \mid x \textit{ is integral over } A\}$. Clearly, A'_B is a subdomain of B containing A. The integral closure D'_K in its quotient field K is simply denoted by D' and the domain D is said to be integrally closed if $D = D'$. A complex number which is integral over $\mathbb{Z}$ is called *algebraic integer* and a complex number which is algebraic over $\mathbb{Q}$ is called *algebraic number.* An *algebraic number field* is a subfield of $\mathbb{C}$ of the form $\mathbb{Q}(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_1, \alpha_2, ..., \alpha_n$ are algebraic numbers. Every algebraic number field can be written as $\mathbb{Q}(\alpha)$ for some algebraic integer α. If α is an algebraic number of degree 2 (root of some irreducible quadratic polynomial $x^2 + ax + b \in \mathbb{Q}[x]$), then $\mathbb{Q}(\alpha) = \{a + b\alpha \mid a, b \in \mathbb{Q}\}$ is called the *quadratic field.* Every quadratic field can be be written as $\mathbb{Q}(\sqrt{d}\,)$ for some (unique) square-free integer d. The set of all algebraic integers that lie in an algebraic number field K is denoted by $\mathcal{O}_K$; that is $\mathcal{O}_K = \mathbb{Z}'_{\mathbb{C}} \cap K$. The set $\mathcal{O}_K$ is an integral domain with quotient field K and is called the *ring of integers of the algebraic number field* K. If $K = \mathbb{Q}(\sqrt{d})$, where d is square-free integer, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$ if $d \equiv 2, 3$ $(mod\ 4)$ and $\mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{d})/2]$ if $d \equiv 1$ $(mod\ 4)$. For each algebraic number field K, $\mathcal{O}_K$ is a Dedekind domain; that is $\mathcal{O}_K$ is Noetherian, integrally closed and one-dimesnional. A reader in need of a quick review on these topics may consult [1]. A subring of $\mathcal{O}_K$ with quotient field K is called an order in K and is denoted by $\mathcal{O}$. It can also be defined as the subring of $\mathcal{O}_K$ which is also a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$. The order $\mathcal{O}$ in K is Noetherian and one-dimesnional integral domain. However, it is not necessarily integrally closed. For detailed introduction about the orders, see [14].

Throughout this paper all rings are (commutative unitary) integral domains. Any unexplained material is standard as in [1], [2] [7], and [10].

3. Main Results

Let $d \in \mathbb{Z} - \{0, 1\}$. Then we can write $d = d^{'}m^2$ for some integer m and a square-free integer $d^{'}$. Note that $\mathbb{Z}[\sqrt{d}] = \mathbb{Z}[m\sqrt{d'}] \leq \mathbb{Z}[\sqrt{d'}]$ and $K = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{d'})$. Hence $\mathbb{Z}[\sqrt{d}]$ will be an order in $K = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{d'})$. In the next results we are going to study the behavior of conductor ideal $\mathcal{O} : \mathcal{O}_K$ in both $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ and $\mathcal{O}_K = \mathbb{Z}[\sqrt{d'}]$ or $\mathbb{Z}[(1 + \sqrt{d'})/2]$ depending on the specified congruence classes of $d^{'}$.

Theorem 3.1. *Let $d = m^2d^{'} \in \mathbb{Z} - \{0, 1\}$, where $d^{'}$ is square-free integer with $d^{'} \equiv 1 \ (mod\ 4)$. If $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(2m, m + \sqrt{d}\big)$.*

Proof. We have $\mathcal{O}_K = \mathbb{Z}[\sqrt{d'}]$ and $\mathcal{O} : \mathcal{O}_K = \{x \in K |\ x\mathcal{O}_K \subseteq \mathcal{O}\}$. Since $2m\big(a + b(\frac{1+\sqrt{d'}}{2})\big) = (2a + b)m + bm\sqrt{d'} \in \mathcal{O}$ for all $a + b(\frac{1+\sqrt{d'}}{2}) \in \mathcal{O}_K$, so $2m \in \mathcal{O} : \mathcal{O}_K$. Also note that $(m + m\sqrt{d'})\big(a + b(\frac{1+\sqrt{d'}}{2})\big) = ma + mb(\frac{1+d^{'}}{2}) + (a + b)m\sqrt{d'}$. As $d^{'} \equiv 1(mod\ 4)$, so $\frac{1+d^{'}}{2} \in \mathbb{Z}$. Thus $(m+m\sqrt{d'})\big(a+b(\frac{1+\sqrt{d'}}{2})\big) \in \mathcal{O}$ for all $a+b(\frac{1+d^{'}}{2}) \in \mathcal{O}_K$. Hence $m + m\sqrt{d'} \in \mathcal{O} : \mathcal{O}_K$. This implies $\big(2m, m + m\sqrt{d'}\big) \subseteq \mathcal{O} : \mathcal{O}_K$. Let $\alpha = a+bm\sqrt{d'} \in \mathcal{O} : \mathcal{O}_K$. As $\alpha \cdot \left(\frac{1+\sqrt{d'}}{2}\right) \in \mathcal{O}$ then $(a+bm\sqrt{d'}) \left(\frac{1+\sqrt{d'}}{2}\right) \in \mathcal{O}$. This implies $\frac{a+bmd^{'}}{2} + \frac{(a+bm)\sqrt{d'}}{2} \in \mathcal{O}$; i.e. $\frac{a+bmd^{'}}{2} = s$ and $\frac{a+bm}{2} = mk$ for some $k \in \mathbb{Z}$. This implies $a = 2s-bmd^{'}$ and also $a = 2mk-bm$. Thus we have $\alpha = 2mk-bm+bm\sqrt{d'} = 2m(k-b)+(m+m\sqrt{d'})b \in \big(2m, m+m\sqrt{d'}\big)$. This implies $\mathcal{O} : \mathcal{O}_K \subseteq \big(2m, m+m\sqrt{d'}\big)$ and hence $\mathcal{O} : \mathcal{O}_K = \big(2m, m + m\sqrt{d'}\big) = \big(2m, m + \sqrt{d}\big)$. □

Corollary 3.1. *Let $d \in \mathbb{Z} - \{0, 1\}$ be a square-free with $d \equiv 1 \ (mod\ 4)$ and $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$. Then $\mathcal{O} : \mathcal{O}_K = \big(2, 1 + \sqrt{d}\big)$.*

Theorem 3.2. *Let $d = m^2d' \in \mathbb{Z} - \{0, 1\}$, where d' is square-free integer with $d' \equiv 1$ (mod 4). If $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(2m, m+\sqrt{d}\big)$ is a non-principal ideal in $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$.*

Proof. Let $N = \big(2m, m + m\sqrt{d'}\big)$. As $N^2 = \big(4m^2, 2m^2 + 2m^2\sqrt{d'}, m^2 + m^2d' + 2m^2\sqrt{d'}\big)$. Since $d' \equiv 1$ (mod 4), so $N = \big(4m^2, 2m^2 + 2m^2\sqrt{d'}, m^2 + m^2(4k + 1) + 2m^2\sqrt{d'}\big) = \big(4m^2, 2m^2 + 2m^2\sqrt{d'}, 4km^2 + 2m^2 + 2m^2\sqrt{d'}\big) = \big(4m^2, 2m^2 + 2m^2\sqrt{d'}, 4km^2\big)$. Now $N = 2m\big(2m, 2m+2m\sqrt{d'}\big)$. This implies $N^2 = 2mN$. Suppose N is principal; i.e. $N = \alpha\mathbb{Z}[\sqrt{d}]$ for some $0 \neq \alpha \in \mathbb{Z}[\sqrt{d}]$. So $\alpha^2\mathbb{Z}[\sqrt{d}] = 2m\alpha\mathbb{Z}[\sqrt{d}]$. This implies $\alpha\mathbb{Z}[\sqrt{d}] = 2m\mathbb{Z}[\sqrt{d}]$. Therefore $N = \big(2m\big)$ and so $m + m\sqrt{d'} \in \big(2m\big)$; i.e. $2m \mid m + m\sqrt{d'}$, a contradiction. □

Corollary 3.2. *Let $d \in \mathbb{Z} - \{0, 1\}$ be square-free with $d \equiv 1$ (mod 4). If $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(2, 1 + \sqrt{d}\big)$ is a non-principal ideal in $\mathbb{Z}[\sqrt{d}]$.*

Remark 3.1. *If $d = m^2d' \in \mathbb{Z} - \{0, 1\}$, where d' is square-free with $d' \equiv 1$ (mod 4), and $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K$ is a non-principal in $\mathcal{O}$ but principal in $\mathcal{O}_K$ generated by $2m$. If $m = 1$, then $\mathcal{O} : \mathcal{O}_K$ is maximal ideal in $\mathcal{O}$. Indeed, because by successively applying the isomorphisms $\sqrt{d} \mapsto x : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}[x]/\big(x^2-d\big)$ and $x \mapsto -1 : \mathbb{Z}[x]/\big(x+1\big) \to \mathbb{Z}$, we get that $\mathbb{Z}[\sqrt{d}]/\big(2, 1 + \sqrt{d}\big) \cong \mathbb{Z}/\big(2, 1 - d\big) \cong \mathbb{Z}_2$. But in this case; i.e. $m = 1$, $\mathcal{O} : \mathcal{O}_K$ need not to be a maximal ideal in $\mathcal{O}_K$. For example if $d = -3$, then $\mathcal{O}_K/\mathcal{O} : \mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{-3})/2]/\big(2\big) \cong \mathbb{Z}_2[x]/\big(1 + x + x^2\big)$ is a field and so $\mathcal{O} : \mathcal{O}_K$ is maximal in $\mathcal{O}_K$. But if $d = -7$, then $\mathcal{O}_K/\mathcal{O} : \mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{-7})/2]/\big(2\big) \cong \mathbb{Z}_2[x]/\big(x + x^2\big)$ which is not a field. Hence $\mathcal{O} : \mathcal{O}_K$ is not a maximal ideal in $\mathcal{O}_K$. However, if $m > 1$ then $\mathcal{O} : \mathcal{O}_K$ is always non-maximal in both $\mathcal{O}$ and $\mathcal{O}_K$.*

From the above observations a natural question arise: For which values of square-free integer d the ideal $\mathcal{O} : \mathcal{O}_K$ is maximal in $\mathcal{O}_K$? The answer to this question is given in the next Theorem.

Theorem 3.3. *Let $d \in \mathbb{Z} - \{0,1\}$ be a square-free with $d \equiv 1\ (mod\ 4)$ and $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$. Then $\mathcal{O} : \mathcal{O}_K = \big(2, 1+\sqrt{d}\big) \in Max(\mathcal{O}_K)$ if and only if $d \equiv 5\ (mod\ 8)$.*

Proof. Since $d \equiv 1\ (mod\ 4)$, we have $\mathcal{O}_K = \mathbb{Z}\big[\frac{1+\sqrt{d}}{2}\big]$, cf. [1, Theorem 5.4.2] and by Remark 3.1 above, we have $\mathcal{O} : \mathcal{O}_K = \big(2\big)$ in $\mathcal{O}_K$. Now $\frac{\mathcal{O}_K}{\mathcal{O}:\mathcal{O}_K} = \frac{\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]}{\left(2\right)} \simeq \frac{\mathbb{Z}_2[x]}{\left(x^2+x+\frac{1-d}{4}\right)}$. Note that $x^2+x+\frac{1-d}{4}$ is irreducible in $\mathbb{Z}_2[x]$ if and only if $\frac{1-d}{4} \equiv 1(mod\ 2)$. Therefore, $x^2+x+\frac{1-d}{4}$ is irreducible if and only if $d \equiv 5\ (mod\ 8)$. This implies that $\frac{\mathbb{Z}_2[x]}{\left(x^2+x+\frac{1-d}{4}\right)}$, and hence $\frac{\mathcal{O}_K}{\mathcal{O}:\mathcal{O}_K}$, is a field if and only if $d \equiv 5\ (mod\ 8)$. Therefore $\mathcal{O} : \mathcal{O}_K$ is a maximal ideal in $\mathcal{O}_K$ if and only if $d \equiv 5\ (mod\ 8)$. □

Theorem 3.4. *Let $d = m^2 d' \in \mathbb{Z} - \{0,1\}$, where d' is square-free with $d' \equiv 2,3\ (mod\ 4)$. If $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(m, \sqrt{d}\big)$.*

Proof. We have $\mathcal{O}_K = \mathbb{Z}[\sqrt{d'}]$. As $m(a+b\sqrt{d'}) = ma + mb\sqrt{d'} \in \mathcal{O}$ and $m\sqrt{d'}(a+b\sqrt{d'}) = ma\sqrt{d'} + mbd' \in \mathcal{O}$ for all $a+b\sqrt{d'} \in \mathcal{O}_K$. This implies $\big(m, m\sqrt{d'}\big) \subseteq \mathcal{O} : \mathcal{O}_K$. Let $\beta = x + ym\sqrt{d'} \in \mathcal{O} : \mathcal{O}_K$, where $x, y \in \mathbb{Z}$. Then $\beta \cdot \sqrt{d'} = x\sqrt{d'} + ymd' \in \mathcal{O}$. This implies x is a multiple of m and so $\beta = mk + ym\sqrt{d'}$ for some $k \in \mathbb{Z}$. Hence $\beta \in \big(m, m\sqrt{d'}\big)$. Thus $\mathcal{O} : \mathcal{O}_K = \big(m, m\sqrt{d'}\big)$. □

Theorem 3.5. *Let $d = m^2 d' \in \mathbb{Z} - \{0,1\}$, where d' is square-free integer with $d' \equiv 2,3\ (mod\ 4)$ and $m \in \mathbb{Z} - \{\pm 1\}$. If $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, then $\mathcal{O} : \mathcal{O}_K = \big(m, \sqrt{d}\big)$ is a non-principal ideal in $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$.*

Proof. Let $M = \big(m, m\sqrt{d'}\big)$. As $M^2 = \big(m^2, m^2\sqrt{d'}, m^2 d'\big) = \big(m^2, m^2\sqrt{d'}\big) = m\big(m, m\sqrt{d'}\big)$ $\Rightarrow M^2 = mM$. Suppose M is principal that is $M = \big(\alpha\big)$ for some $0 \neq \alpha \in \mathcal{O}$. Then we have $\alpha^2\mathbb{Z}[\sqrt{d}] = m\alpha\mathbb{Z}[\sqrt{d}]$. This implies $\alpha\mathbb{Z}[\sqrt{d}] = m\mathbb{Z}[\sqrt{d}]$. Therefore $M = \big(m\big)$ and so $m \mid \sqrt{d}$ in $\mathcal{O}$, a contradiction. □

Remark 3.2. *If $d \in \mathbb{Z} - \{0,1\}$ is a square-free with $d \equiv 2,3\ (mod\ 4)$, then $\mathcal{O} = \mathcal{O}_K$. So there is nothing to study about conductor ideals in this case. If $d = m^2 d' \in$*

$\mathbb{Z}-\{0,1\}$, where d' is square-free with $d' \equiv 2,3 \pmod 4$, $m \in \mathbb{Z}-\{\pm 1\}$ and $\mathcal{O}=\mathbb{Z}[\sqrt{d}]$, then $\mathcal{O}:\mathcal{O}_K$ is a non-principal ideal in $\mathcal{O}$ but principal ideal in $\mathcal{O}_K$ generated by m. Moreover, $\mathcal{O}:\mathcal{O}_K$ is not necessarily a maximal ideal in $\mathcal{O}$ and in $\mathcal{O}_K$. Further, we discuss that under which conditions $\mathcal{O}:\mathcal{O}_K$ is maximal ideal in $\mathcal{O}$ and in $\mathcal{O}_K$, respectively.

Theorem 3.6. *Let $d=m^2d' \in \mathbb{Z}-\{0,1\}$, where d' is square-free with $d' \equiv 2,3 \pmod 4$. Then $\mathcal{O}:\mathcal{O}_K=\big(m,\sqrt{d}\big)$ is maximal ideal in $\mathcal{O}=\mathbb{Z}[\sqrt{d}]$ if and only if m is prime.*

Proof. The map $x \mapsto \sqrt{d}:\mathbb{Z}[x]\to\mathbb{Z}[\sqrt{d}]$ is an epimorphism which induce an isomorphism between $\mathbb{Z}[x]/\big(x^2-d\big)$ and $\mathbb{Z}[\sqrt{d}]$. Applying this isomorphism, we get that

$$\frac{\mathbb{Z}[\sqrt{d}]}{\big(m,\sqrt{d}\big)} \cong \frac{\mathbb{Z}[x]/\big(x^2-d\big)}{\big(m,x,x^2-d\big)/(x^2-d)} \cong \frac{\mathbb{Z}[x]}{\big(m,x,x^2-d\big)}$$

Moreover, the map $x\mapsto 0 \ (mod\ m):\mathbb{Z}[x]\to\mathbb{Z}_m$ is an epimorphism which induce an isomorphism between $\dfrac{\mathbb{Z}[x]}{\big(m,x,x^2-d\big)}$ and $\dfrac{\mathbb{Z}}{\big(m,-d\big)}$. Hence $\dfrac{\mathbb{Z}[\sqrt{d}]}{\big(m,\sqrt{d}\big)} \cong \dfrac{\mathbb{Z}}{\big(m,-m^2d'\big)} \cong \mathbb{Z}_m$. Therefore $\mathcal{O}:\mathcal{O}_K$ is maximal in $\mathcal{O}$ if and only if m is prime. □

Theorem 3.7. *Let $d \in \mathbb{Z}-\{0,1\}$ and $d=m^2d'$, where d' is square-free with $d' \equiv 2,3 \pmod 4$. The ideal $\mathcal{O}:\mathcal{O}_K=\big(m,\sqrt{d}\big)$ is maximal ideal in $\mathcal{O}_K=\mathbb{Z}[\sqrt{d'}]$ if and only if m is prime and x^2-d' is irreducible in $\mathbb{Z}_m[x]$.*

Proof. We have $\mathcal{O}:\mathcal{O}_K=\big(m,\sqrt{d}\big)=\big(m\big)$ in $\mathcal{O}_K$. Therefore,

$$\frac{\mathcal{O}_K}{\mathcal{O}:\mathcal{O}_K} \cong \frac{\mathbb{Z}[\sqrt{d'}]}{\big(m\big)} \cong \mathbb{Z}_m[\sqrt{d'}]$$

Moreover, the map $x\mapsto\sqrt{d'}:\mathbb{Z}[x]\to\mathbb{Z}[\sqrt{d'}]$ is an epimorphism which induce an isomorphism between $\mathbb{Z}_m[x]/\big(x^2-d'\big)$ and $\mathbb{Z}_m[\sqrt{d'}]$. Applying this isomorphism, we get that $\dfrac{\mathcal{O}_K}{\mathcal{O}:\mathcal{O}_K} \cong \dfrac{\mathbb{Z}_m[x]}{\big(x^2-d'\big)}$. Now apply well known result [2, Proposition 15 of Section 9.5]. □

We have noticed that the ideal $\left(2, 1+\sqrt{d}\right)$ is a conductor ideal in $\mathbb{Z}[\sqrt{d}]$, where $d \in \mathbb{Z} - \{0, 1\}$ be a square-free with $d \equiv 1$ $(mod\ 4)$. Note that $\left(2, 1+\sqrt{d}\right)$ is maximal ideal in $\mathbb{Z}[\sqrt{d}]$ for every odd integer d. Now a question arise that what happened with the properties of the ideal $\left(2, 1+\sqrt{d}\right)$ if we replace 2 by any arbitrary positive integer? By using our results, it can be easily conclude that $\left(n, 1+\sqrt{d}\right)$ is a conductor ideal in $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ if and only if $n = 2$ and $d \equiv 1$ $(mod\ 4)$. Our next results is about maximality of the ideal $\left(n, 1+\sqrt{d}\right)$ in the ring $\mathbb{Z}[\sqrt{d}]$.

Theorem 3.8. *Let $d \in \mathbb{Z} - \{0, 1\}$. Then $\left(n, 1+\sqrt{d}\right)$ is maximal ideal in $\mathbb{Z}[\sqrt{d}]$ if and only if $gcd(n, 1-d)$ is prime.*

Proof. By successively applying the isomorphisms $\sqrt{d} \mapsto x : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}[x]/\left(x^2 - d\right)$ and $x \mapsto -1 : \mathbb{Z}[x]/\left(x+1\right) \to \mathbb{Z}$, we get that $\mathbb{Z}[\sqrt{d}]/\left(n, 1+\sqrt{d}\right) \cong \mathbb{Z}[x]/\left(n, 1+x, x^2 - d\right) \cong \mathbb{Z}/\left(n, 1-d\right) \cong \mathbb{Z}_\delta$, where $\delta = gcd(n, 1-d)$. Hence $\left(n, 1+\sqrt{d}\right) \in Max(\mathbb{Z}[\sqrt{d}])$ if and only if $\delta = gcd(n, 1-d)$ is prime. □

We conclude the paper with some specific remarks about quadratic orders.

Remark 3.3. *Let $d \in \mathbb{Z} - \{0, 1\}$ be square-free with $d \equiv 1$ $(mod\ 4)$. Then the order $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ is a one-dimensional Noetherian domain which is not integrally closed and hence it cannot be a PID. In this case, the conductor ideal is given by $\mathcal{O} : \mathcal{O}_K = \left(2, 1+\sqrt{d}\right)$ which is non-principal. However, if $d \equiv 5$ $(mod\ 8)$, then $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ can be described by a certain factorization property of non-principal ideals; i.e. each non-principal ideal in $\mathcal{O}$ can be written as $\alpha \cdot (\mathcal{O} : \mathcal{O}_K)$ for some $0 \neq \alpha \in \mathcal{O}$, cf.* [5, Section 2, Example 2.13]. *This shows that for $d \equiv 5$ $(mod\ 8)$, each non-principal ideal in $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ is contained in the conductor ideal $\mathcal{O} : \mathcal{O}_K$.*

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