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ON A MEASURABLE SOLUTION OF A CLASS OF HIGHER–ORDER STOCHASTIC HEAT–TYPE EQUATION

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Abstract. We give a generalized measurable–predictable solution to a higher–order stochastic parabolic initial–value problem in terms of the further generalized Hermite polynomials. Condition and estimates on the existence and uniqueness of the solution are given. We prove the upper second moment growth bound estimate for the solution and consequently show that the second moment of the solution grows exponentially in time with respect to the parameter λ at the precise rate of $2 + c_3\lambda^2 \mathrm{Lip}_\sigma^2,\ c_3 > 0$ and ; $c_3\mathrm{Lip}_\sigma^2$ as the noise level increases.

1. Introduction

Consider the following higher-order stochastic heat-type equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^p u(x,t)}{\partial x^p} + \lambda\sigma(u(x,t))\dot{w}(x,t), \quad p \geq 1 \\ u(x,0) = \phi(x),\ x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The constant $\lambda > 0$ is a noise level, $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $\dot{w}(x,t)$ is a space–time white noise. Higher order parabolic equations have great modeling properties, for example, Cauchy problems associated to Kadomtsev–Petviashili (KP) equations having higher order dispersion in the main direction of propagation occur naturally in the modelling of certain long dispersive waves (see [26]), also, a

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third-order diffusion equation models the rapid solidification of undercooled melts, in which the crystallization front propagates with a high velocity, that is, a high-order diffusion equation describes impurity distribution in the solid and liquid phases as a function of the crystallization front velocity, (see [27]). Moreover, higher order diffusion equations can find their applications in seismology (in computing the traveltime (phase) and the amplitude in constant density acoustic media, (see [8]), in underwater acoustics and range-dependent acoustic calculations, etc, (see [9, 10, 11])). For more applications (see [7, 20, 30]) and therein references. The authors in [1] studied the following Cauchy problem in $\mathbb{R}^n$, $n \geq 1$ for higher order ($m \geq 2$) linear parabolic equations

$$\begin{cases} u_t(x,t) + (-1)^m \displaystyle\sum_{|\alpha|\leq m, |\beta|\leq m} D^\alpha \{a_{\alpha,\beta} D^\beta\} u(x,t) = 0, \; (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \; x \in \mathbb{R}^n, \end{cases}$$

and particularly the polyharmonic heat equation

$$\begin{cases} u_t(x,t) + (-\Delta)^m u(x,t) = 0, \; (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \; x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

They showed that for $u_0 \in C^0 \cap L^\infty(\mathbb{R}^n)$, equation (1.2) admits a unique global in time bounded solution given by

$$u(x,t) = \alpha t^{-\frac{n}{2m}} \int_{\mathbb{R}^n} u_0(x-y) f_{m,n}\left(\frac{|y|}{t^{\frac{1}{2m}}}\right) dy, \; (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$$

where $\alpha = \alpha_{m,n} > 0$ is a suitable normalization constant and

$$f_{m,n}(\eta) = \eta^{1-n} \int_0^\infty e^{-s^{2m}} (\eta s)^{\frac{n}{2}} J_{\frac{(n-2)}{2}}(\eta s) ds,$$

J_ν denotes the ν-Bessel function ([22]), (see also [24]). One of the challenges of the above solution is that the kernels are very complicated, and they do not exist in any simple form and also depend on n. Motivation for studying equation (1.1) arises from the connection between the generalized Hermite polynomials $\{\gamma_n^p(x)\}$ and the initial-value problem (see [3, 28]),

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^p u(x,t)}{\partial x^p}, \quad u(x,0) = \phi(x) \tag{1.3}$$

where $p \geq 1$ is an arbitrary positive number. It is known ([3], p. 348 & [28], p. 459), that if $\phi(x) = x^n$, $n \in \mathbb{N}$, the solution to equation (1.3) at time t is given by

$$u(x,t) = (-t)^{\frac{n}{p}} \gamma_n^p \left(\frac{x}{p(-t)^{\frac{1}{p}}} \right).$$

Extensive research works have been done on equation (1.1) for $p = 2$ (the classical stochastic heat equation) and for $p \in (1,2)$ (the stochastic fractional heat equation), (see [16, 17, 23, 29] and therein references), but to the best of our knowledge, little or no work exist for the higher–order stochastic heat–type equation. The difficulty in studying the above equation is that the solution is no longer given in terms of the known heat kernel or the fractional heat kernel but in terms of the Hermite generalized polynomials.

The solution to equation (1.1) can also be expressed in terms of the two parameter polynomials known as the further generalized polynomials $H_n^{(p)}(x,t)$ by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{(-t)^{\frac{n}{p}}}{n!} H_n^{(p)} \left(\frac{x}{(-t)^{\frac{1}{p}}}, -1 \right) = \exp\left(\left(\frac{x}{(-t)^{\frac{1}{p}}} + 1 \right) t - 1 \right).$$

In this paper, we study some properties of a generalized solution to equation (1.1) for $p = m, \ m \in \mathbb{N}$ and consider therefore

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^m u(x,t)}{\partial x^m} + \lambda \sigma(u(x,t)) \dot{w}(x,t) \\ u(x,0) = x^n = H_n^{(m)}(x,0), \ m \in \mathbb{N}. \end{cases} \tag{1.4}$$

Proposition 1.1 ([4], Proposition 4)**.** *The polynomials* $H_n^{(m)}(x,t), \ n, m \in \mathbb{N}$ *satisfy the following partial differential equation*

$$\frac{\partial H_n^{(m)}(x,t)}{\partial t} = \frac{\partial^m H_n^{(m)}(x,t)}{\partial x^m}.$$

In what follows, we aim to study some properties and behaviours of the solution to equation (1.4).

Remark 1. *The case of the generalized solution of equation* (1.1) *for* $p \geq 1$ *will be left for further research.*

Therefore, the fundamental solution of

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^m u(x,t)}{\partial x^m}$$

is given by

$$u(x,t) = H_n^{(m)}(x,t) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{t^k x^{n-mk}}{k!(n-mk)!},$$

which is known as further Hermite generalized polynomials with the generating function

$$\sum_{n=0}^{\infty} H_n^{(m)}(x,\alpha) = e^{xt+\alpha t^m}. \tag{1.5}$$

Some connections and relations between various generalized polynomials are given below, (see [3, 4, 5, 6, 15, 28]). If one considers a class of generalized Hermite polynomials $\{\gamma_n^m(x)\}$ defined by the generating function

$$e^{mxt-t^m} = \sum_{n=0}^{\infty} \gamma_n^m(x) t^n, \tag{1.6}$$

then equation (1.5) gives equation (1.6) when $x \sim mx$ and $\alpha = -1$, that is,

$$\gamma_n^m(x) = H_n^{(m)}(mx, -1).$$

Also, if one considers the polynomials $\{h_{n,m}(x)\}_{n=0}^{\infty}$ defined by $h_{n,m}(x) = \gamma_n^m\left(\frac{2x}{m}\right)$ with the generating function

$$F(x,t) = e^{2xt-t^m} = \sum_{n=0}^{\infty} h_{n,m}(x) t^n, \tag{1.7}$$

then equation (1.6) gives equation (1.7) when $x \sim \frac{2x}{m}$, and also have the following relations

$$h_{n,m}(x) = \frac{1}{n!} H_n^{(m)}(x)$$

and

$$H_n(x) = H_n^{(2)}(2x, -1).$$

For the classical series expansions

$$\exp(\alpha x - \frac{\alpha^2}{2} t) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(x, t),$$

where $H_n(x,t) = t^{\frac{n}{2}} h_n(\frac{2}{\sqrt{t}})$ with (h_n) the classical Hermite polynomials. Other representations of both the classical and the generalized Hermite polynomials can be found in ([12, 13, 18, 21]), (see also [19]) for a distributional properties of the generalized Hermite polynomials, ([25]) for an application of generalized Hermite polynomials to oscillator calculus, generalized Hermite polynomial as a family of orthogonal system in ([2]), and some characterizations of the generalized Hermite polynomials in ([14]).

Now we attempt to make sense of the solution to equation (1.4) by giving the following definitions, (see [16, 23, 29]).

Definition 1.1. We say that a process $\{u(x,t)\}_{x \in \mathbb{R}, t>0}$ is a mild solution of (1.4) if a.s, the following is satisfied

$$u(x,t) = \int_{\mathbb{R}} H_n^{(m)}(x-y,t) u_0(y) dy + \lambda \int_0^t \int_{\mathbb{R}} H_n^{(m)}(x-y, t-s) \sigma(u(s,y)) w(dyds).$$

If $\{u(x,t)\}_{x \in \mathbb{R},\, t>0}$ satisfies the following additional condition

$$\sup_{0 \le t \le T} \sup_{x \in \mathbb{R}} \mathbb{E}|u(x,t)|^2 < \infty,$$

for all $T > 0$, then we say that $\{u(x,t)\}_{x \in \mathbb{R},\, t>0}$ is a random field solution to (1.4).

Remark 2. *The above solution $u(x,t)$ is a function of n (depends on n), so rather than constructing a Picard iteration that converges to $u(x,t)$ uniformly over n; we* *assume a measurability condition on the solution by multiplying through by* $\frac{1}{n!}$ *and sum over* $n \in [0, \infty)$ *(expressing the solution in terms of a generating function), thus with* $c_0 := \sum_{n=0}^{\infty} \frac{1}{n!} < \infty$, *we have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} u(x,t) &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{y^n}{n!} H_n^{(m)}(x-y,t)dy \\ &+ \lambda \int_0^t \int_{\mathbb{R}} \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} H_n^{(m)}(x-y,t-s) \right\} \sigma(u(s,y)) w(dyds).\end{aligned}$$

Definition 1.2. We say that a process $\{u(x,t)\}_{x\in\mathbb{R},\, t>0}$ is a measurable predictable mild solution of (1.4) if a.s, the following is satisfied

$$u(x,t) = \frac{1}{c_0} \int_{\mathbb{R}} e^{(x-y).y+ty^m} dy + \frac{\lambda}{c_0} \int_0^t \int_{\mathbb{R}} e^{(x-y)+(t-s)} \sigma(u(s,y)) w(dyds).$$

If $\{u(x,t)\}_{x\in\mathbb{R},\, t>0}$ satisfies the following condition

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} \mathbb{E}|u(x,t)|^2 < \infty,$$

for all $T > 0$, then we say that $\{u(x,t)\}_{x\in\mathbb{R},t>0}$ is a measurable predictable random field solution to (1.4).

The present paper is organized in three sections as follows. The introduction which contains the problem formulation, some preliminary concepts and definitions are in Section 1. Section 2 gives statements and proofs of the main results. A very brief summary of results obtained were given in Section 3.

2. Main Results

Define the following norm on $L^2(\mathbb{P})$ by

$$\|u\|_{2,T} = \left\{ \sup_{0\leq t\leq T} \sup_{0\leq x<1} e^{-2t} \mathbb{E}|u(t,x)|^2 \right\}^{1/2}$$

and assume that $x \mapsto \mathbb{E}|u(t,x)|^2$ is supported on $[0,1)$ for all $t \geq 0$.

We state the existence and uniqueness result which follow by the following condition:

Condition 2.1. *The function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies $|\sigma(0)| = 0$ with*

$$|\sigma(x) - \sigma(y)| \leq \mathrm{Lip}_\sigma |x - y|.$$

Theorem 2.1. *Given that σ satisfies Condition 2.1 and $0 < c_3 T \lambda^2 \mathrm{Lip}_\sigma^2 < 1$, then there exists a unique measurable predictable random field solution $u \in L^2([0,1))$ to equation (1.4).*

Proof of Theorem 2.1. Define the operator

$$\mathcal{A}u(x,t) = \frac{1}{c_0}\int_{\mathbb{R}} e^{(x-y).y+ty^m}dy + \frac{\lambda}{c_0}\int_0^t\int_{\mathbb{R}} e^{(x-y)+(t-s)}\sigma(u(s,y))w(dyds),$$

then the fixed point of the operator $\mathcal{A}$ gives the solution of equation (1.4). Therefore, the proof of Theorem 2.1 follows by fixed point theorem, given the following propositions below, (see [23], Theorem 3.1.1, Proposition 3.1.4 and Proposition 3.1.5).

Proposition 2.1. *Let u be some measurable predictable random field solution with support on $L^2([0,1))$. Then there exist some positive constant*

$$\mathcal{C}_{2,m,T} := \sup_{0\leq x<1} \frac{1}{c_1\sqrt{2(1-x)}} 2^{-\frac{1}{2m}} T^{-\frac{1}{2m}} \sqrt{\Gamma\left(1 + \frac{1}{m}\right)}$$

such that

$$\|\mathcal{A}u\|_{2,T}^2 \leq \mathcal{C}_{2,m,T} + c_3 T \lambda^2 \mathrm{Lip}_\sigma^2 \|u\|_{2,T}^2.$$

Proof. Multiply through by $\frac{1}{n!}$, $n \in \mathbb{N}$ and take sum over n in $[0,\infty)$:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{A}u(x,t) &= \int_{\mathbb{R}}\sum_{n=0}^{\infty}\frac{y^n}{n!}H_n^{(m)}(x-y,t)dy \\ &+ \lambda\int_0^t\int_{\mathbb{R}}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}H_n^{(m)}(x-y,t-s)\right\}\sigma(u(s,y))w(dyds).\end{aligned}$$

By the generating function of the further Hermite generalized polynomials, and since $c_0 := \sum_{n=0}^{\infty} \frac{1}{n!} < \infty$,

$$c_0\mathcal{A}u(x,t) = \int_{\mathbb{R}} e^{(x-y).y+ty^m}dy + \lambda\int_0^t\int_{\mathbb{R}} e^{(x-y)+(t-s)}\sigma(u(s,y))w(dyds).$$

Take second moments of both sides and by Itô isometry,

$$c_1\mathbb{E}|\mathcal{A}u(x,t)|^2 \leq \left|\int_{\mathbb{R}} e^{(x-y).y+ty^m}dy\right|^2 + \lambda^2\int_0^t\int_{\mathbb{R}} e^{2\left\{(x-y)+(t-s)\right\}}\mathbb{E}|\sigma(u(s,y))|^2dyds.$$

Using the Lipschitz condition on σ and the fact that $-y^2 \leq -y$ we have

$$\begin{aligned}
c_1\mathbb{E}|\mathcal{A}u(x,t)|^2 &\leq \left|\int_{\mathbb{R}} e^{(x-1).y+ty^m}dy\right|^2 + \lambda^2\mathrm{Lip}_\sigma^2\int_0^t\int_{\mathbb{R}} e^{2\left\{(x-y)+(t-s)\right\}}\mathbb{E}|u(s,y)|^2dyds.\\
&\leq \left(\int_{\mathbb{R}} e^{2(x-1)y}dy\right)^{1/2}\left(\int_{\mathbb{R}} e^{2ty^m}dy\right)^{1/2}\\
&+ \lambda^2\mathrm{Lip}_\sigma^2\int_0^t\int_{\mathbb{R}} e^{2\left((x-y)+(t-s)\right)}\mathbb{E}|u(s,y)|^2dyds.\\
&\leq \sqrt{\frac{1}{2-2x}}\sqrt{2^{-\frac{1}{m}}\Gamma\big(1+\frac{1}{m}\big)}(-t)^{-\frac{1}{2m}}\\
&+ \lambda^2\mathrm{Lip}_\sigma^2\int_0^t dse^{2(t-s)}\sup_{0\leq y<1}\mathbb{E}|u(s,y)|^2\int_0^1 e^{2(x-y)}dy,
\end{aligned}$$

where the last inequality follows from the assumption that $\mathbb{E}|u(x,t)|^2$ is supported on $x \in [0,1)$. Thus

$$\mathbb{E}|\mathcal{A}u(x,t)|^2 \leq \mathcal{C}_{2,m}(x)(-t)^{-\frac{1}{2m}} + \lambda^2\mathrm{Lip}_\sigma^2\frac{(1-\frac{1}{e^2})}{2c_1}e^{2x}\int_0^t e^{2(t-s)}\sup_{y\in[0,1)}\mathbb{E}|u(s,y)|^2ds,$$

with $\mathcal{C}_{2,m}(x) := \frac{1}{c_1\sqrt{2(1-x)}}2^{-\frac{1}{2m}}\sqrt{\Gamma(1+\frac{1}{m})}$. Now take sup over $x \in [0,1)$ of both sides, then

$$\sup_{x\in[0,1)}\mathbb{E}|\mathcal{A}u(x,t)|^2 \leq \mathcal{C}_{2,m}\times(-t)^{-\frac{1}{2m}} + c_3\lambda^2\mathrm{Lip}_\sigma^2\int_0^t e^{2t}\sup_{y\in[0,1)}e^{-2s}\mathbb{E}|u(s,y)|^2ds,$$

where $\mathcal{C}_{2,m} := \sup_{x\in[0,1)} \mathcal{C}_{2,m}(x) < \infty$, and $c_3 := \frac{c_2(1-\frac{1}{e^2})}{2c_1}$ with $c_2 = \sup_{x\in[0,1)} e^{2x} = e^2$. Next, multiply both sides by e^{-2t} to obtain

$$\begin{aligned}\sup_{x\in[0,1)} e^{-2t}\mathbb{E}|\mathcal{A}u(x,t)|^2 &\leq \mathcal{C}_{2,m} \times (-t)^{-\frac{1}{2m}} e^{-2t} + c_3\lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t \sup_{y\in[0,1)} e^{-2s}\mathbb{E}|u(s,y)|^2 ds \\ &\leq \mathcal{C}_{2,m} \times (-t)^{-\frac{1}{2m}} e^{-2t} + c_3\lambda^2 \mathrm{Lip}_\sigma^2 \|u\|_{2,T}^2 t,\end{aligned}$$

and by taking sup over $t \in [0,T]$, $T < \infty$, we obtain

$$\|\mathcal{A}u\|_{2,T}^2 \leq \mathcal{C}_{2,m} T^{-\frac{1}{2m}} + c_3 T\lambda^2 \mathrm{Lip}_\sigma^2 \|u\|_{2,T}^2.$$

□

Proposition 2.2. *Let u and v be some measurable predictable random field solutions such that $\|u\|_{2,T} + \|v\|_{2,T} < \infty$. Then there exist some positive constant c_3 such that*

$$\|\mathcal{A}u - \mathcal{A}v\|_{2,T}^2 \leq c_3 T\lambda^2 \mathrm{Lip}_\sigma^2 \|u-v\|_{2,T}^2.$$

Proof. The proof follows the argument above. □

Now, we use the estimates of Proposition 2.1 and Proposition 2.2 to complete the proof as follows: By fixed point theorem, we have $u(x,t) = \mathcal{A}u(x,t)$ and

$$\|u\|_{2,T}^2 = \|\mathcal{A}u\|_{2,T}^2 \leq \mathcal{C}_{2,m,T} + c_3 T\lambda^2 \mathrm{Lip}_\sigma^2 \|u\|_{2,T}^2,$$

which follows that

$$\|u\|_{2,T}^2 \leq \frac{\mathcal{C}_{2,m,T}}{1 - c_3 T\lambda^2 \mathrm{Lip}_\sigma^2},$$

and $\|u\|_{2,T} < \infty$ whenever $1 - c_3 T\lambda^2 \mathrm{Lip}_\sigma^2 > 0$.

Similary,

$$\|u-v\|_{2,T}^2 = \|\mathcal{A}u - \mathcal{A}v\|_{2,T}^2 \leq c_3 T\lambda^2 \mathrm{Lip}_\sigma^2 \|u-v\|_{2,T}^2,$$

which shows that $\|u-v\|_{2,T}^2[1 - c_3 T\lambda^2 \mathrm{Lip}_\sigma^2] \leq 0$ and $\|u-v\|_{2,T}^2 \leq 0$ if and only if $1 - c_3 T\lambda^2 \mathrm{Lip}_\sigma^2 > 0$. Thus the existence–uniqueness result follows by Banach's contraction principle. □

Now, we give the energy growth bound result of the generalized solution. We are able to prove the upper bound for the moment estimate and the lower bound is open for further research. The theorem is given as follows:

Theorem 2.2. *Suppose Condition 2.1 on $\sigma : \mathbb{R} \to \mathbb{R}$ holds, then*

$$\mathbb{E}|u(t,x)|^2 \leq \mathcal{C}_{2,m,T} e^{\left(2+c_3(\lambda \mathrm{Lip}_\sigma)^2\right)t}, \ 0 \leq x < 1$$

for some positive constants c_3 and $\mathcal{C}_{2,m,T}$ as previously defined.

Proof. Following similar steps as above, we have

$$\sup_{x\in[0,1)} e^{-2t}\mathbb{E}|u(x,t)|^2 \leq \mathcal{C}_{2,m}(-t)^{-\frac{1}{2m}}e^{-2t} + c_3\lambda^2\mathrm{Lip}_\sigma^2 \int_0^t \sup_{y\in[0,1)} e^{-2s}\mathbb{E}|u(s,y)|^2 ds.$$

Let $F_t := \sup_{x\in[0,1)} e^{-2t}\mathbb{E}|u(x,t)|^2$, then

$$\begin{aligned} F_t &\leq \mathcal{C}_{2,m}(-t)^{-\frac{1}{2m}}e^{-2t} + c_3\lambda^2\mathrm{Lip}_\sigma^2 \int_0^t F_s ds \\ &\leq \mathcal{C}_{2,m}T^{-\frac{1}{2m}} + c_3\lambda^2\mathrm{Lip}_\sigma^2 \int_0^t F_s ds \end{aligned}$$

for all $0 \leq t \leq T$ and $T < \infty$. Thus for all $t \geq 0$,

$$F_t \leq \mathcal{C}_{2,m,T} + c_3\lambda^2\mathrm{Lip}_\sigma^2 \int_0^t F_s ds.$$

Applying Gronwall's inequality we have

$$F_t \leq \mathcal{C}_{2,m,T} \ e^{c_3(\lambda\mathrm{Lip}_\sigma)^2 t}$$

and therefore

$$\sup_{x\in[0,1)} \mathbb{E}|u(x,t)|^2 \leq \mathcal{C}_{2,m,T} \ e^{\left(2+c_3(\lambda\mathrm{Lip}_\sigma)^2\right)t}$$

and the result follows □

Now we give an immediate consequence result of Theorem 2.2

Corollary 2.1. *There exists a positive constant c_3 such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u(x,t)|^2 \leq 2 + c_3\lambda^2 \mathrm{Lip}_\sigma^2 \tag{2.1}$$

and

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda^2} \log \mathbb{E}|u(x,t)|^2 \leq c_3 \mathrm{Lip}_\sigma^2 t. \tag{2.2}$$

Equation (2.1) states that the second moment of the measurable predictable solution grows at most exponentially at a rate with respect to the parameter λ (known as the noise level) given by $2 + c_3\lambda^2\mathrm{Lip}_\sigma^2$. We also estimate the upper bound on the noise level. That is, in equation (2.2), as the noise level increases, the second moment grows at most at a rate of $c_3\mathrm{Lip}_\sigma^2 t$ for all $0 \leq t \leq T$ and $T < \infty$.

3. Conclusion

We attempt to make sense of a solution to a higher order stochastic heat equation. The existence and uniqueness result are given and we find an upper energy growth bound for the integral solution.

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