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GROUP S_3 CORDIAL REMAINDER LABELING FOR WHEEL AND SNAKE RELATED GRAPHS

A. LOURDUSAMY [(1)], S. JENIFER WENCY [(2)] AND F. PATRICK [(3)]

ABSTRACT. The concept of group S_3 cordial remainder labeling was recently introduced by Lourdusamy, Jenifer Wency and Patrick in [5]. In this paper, we prove that helm, flower, closed helm, gear, sunflower, triangular snake and quadrilateral snake are a group S_3 cordial remainder graphs.

1. INTRODUCTION

All graphs considered here are simple, finite, connected and undirected. Graph labeling was first introduced in 1960's. Most of the graph labeling trace their origins in the paper presented by Alex Rosa in 1967 [7]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices then the labeling is called vertex labeling. If the domain is the set of edges then the labeling is called edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by I. Cahit in [1]. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. The function f is called a cordial labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ denotes the number of vertices labeled with i ($i = 0, 1$) and $e_f(i)$ denotes the number of edges

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labeled with i $(i = 0, 1)$. An extensive survey of various graph labeling problems is available in Gallian [2].

Definition 1.1. Let A be a group. The order of $a \in A$ is the least positive integer n such that $a^n = e$. We denote the order of a by $o(a)$.

Definition 1.2. Consider the symmetric group S_3. Let the elements of S_3 be e, a, b, c, d, f where

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$
$$c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

We have $o(e) = 1$, $o(a) = o(b) = o(c) = 2$, $o(d) = o(f) = 3$.

Lourdusamy *et al.* introduced the concept of group S_3 cordial remainder labeling in [5]. Also they proved that path, cycle, star, bistar, complete bipartite $K_{2,n}$, wheel, fan, comb and crown graphs admit a group S_3 cordial remainder labeling. In [6], Lourdusamy *et al.* discussed the behaviour of group S_3 cordial remainder labeling of subdivision of star, subdivision of bistar, subdivision of wheel, subdivision of comb, subdivision of crown, subdivision of fan and subdivision of ladder. In [4], Jenifer *et al.* proved that shadow graph of cycle and path, splitting graph of cycle, armed crown, umbrella graph and dumbbell graph admit a group S_3 cordial remainder labeling. Also they proved that snake related graphs are a group S_3 cordial remainder graphs. For undefined terms the reader is referred to Harary [3].

Definition 1.3. Let $G = (V(G), E(G))$ be a graph and let $g : V(G) \to S_3$ be a function. For each edge xy assign the label r where r is the remainder when $o(g(x))$ is divided by $o(g(y))$ or $o(g(y))$ is divided by $o(g(x))$ according as $o(g(x)) \geq o(g(y))$

or $o(g(y)) \geq o(g(x))$. The function g is called a group S_3 cordial remainder labeling of G if $|v_g(x) - v_g(y)| \leq 1$ and $|e_g(1) - e_g(0)| \leq 1$, where $v_g(x)$ denotes the number of vertices labeled with x and $e_g(i)$ denotes the number of edges labeled with i $(i = 0, 1)$. A graph G which admits a group S_3 cordial remainder labeling is called a group S_3 cordial remainder graph.

In this paper, we prove that helm, flower, closed helm, gear, sunflower, triangular snake and quadrilateral snake are a group S_3 cordial remainder graphs. First, we introduce these graph.

Definition 1.4. The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set is $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 1.5. The wheel W_n is defined as the join $C_n + K_1$. The vertex K_1 is the apex vertex and the vertices on the underlying cycle are called rim vertices. The edges of the underlying cycle are called the rim edges and the edges joining the apex and the rim vertices are called spoke edges.

Definition 1.6. The helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle C_n.

Definition 1.7. The flower graph Fl_n is the graph obtained from a Helm by joining each pendent vertex to the central vertex of the Helm.

Definition 1.8. The closed helm CH_n is a graph obtained from a Helm H_n by joining each pendent vertex to form a cycle.

Definition 1.9. The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.

Definition 1.10. Let W_n be the wheel with central vertex u and cycle $C_n : u_1u_2 \dots u_nu_1$. Then the sunflower graph SF_n is obtained from W_n by adding the vertices $v_1v_2 \dots v_n$ where v_i is adjacent to $u_i,\ u_{i+1},\ 1 \le i \le n-1$ and v_n is adjacent to $u_n,\ u_1$.

Definition 1.11. A K_n-snake is defined as a connected graph in which all blocks are isomorphic to K_n and the block-cut point graph is a path. A K_3-snake is called triangular snake.

Definition 1.12. The quadrilateral snake is obtained from a path $u_1u_2 \cdots u_{n+1}$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and w_i.

2. Main Results

Theorem 2.1. *The Helm graph H_n is a group S_3 cordial remainder graph for $n \ge 3$.*

Proof. Let H_n be the Helm with $V(H_n) = \{u, u_i, v_i : 1 \le i \le n\}$ and $E(H_n) = \{uu_i, u_iv_i : 1 \le i \le n\} \bigcup \{u_iu_{i+1} : 1 \le i \le n-1\} \bigcup \{u_nu_1\}$. Therefore, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. Define $g : V(H_n) \to S_3$ as follows:

Case 1. $n = 3$.

Assign the labels d, a, d, f, b, e, c to the vertices $u, u_1, u_2, u_3, v_1, v_2, v_3$ respectively. The values of $v_g(x),\ x \in \{e, a, b, c, d, f\},\ e_g(0)$ and $e_g(1)$ are given in Table 1. According to these values from Table 1, g is a group S_3 cordial reminder labeling.

Case 2. $n = 4$.

Assign the labels $d, a, f, b, e, d, b, e, c$ to the vertices $u, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ respectively. The values of $v_g(x),\ x \in \{e, a, b, c, d, f\},\ e_g(0)$ and $e_g(1)$ are given in Table 1. According to these values from Table 1, g is a group S_3 cordial reminder labeling.

Case 3. $n = 5$.

Assign the labels $d, a, b, f, c, e, e, f, a, b, c$ to the vertices $u, u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5$ respectively. The values of $v_g(x),\ x \in \{e, a, b, c, d, f\},\ e_g(0)$ and $e_g(1)$ are given in Table 1. According to these values from Table 1, g is a group S_3 cordial reminder labeling.

Case 4. $n \geq 6$.

Subcase (i). $n \equiv 0\ (mod\ 6)$. Let $n = 6k$ and $k \geq 1$. $g(u) = d$; for $1 \leq i \leq n$,

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \\ d & \text{if } i \equiv 2\ (mod\ 6) \\ b & \text{if } i \equiv 3\ (mod\ 6) \\ c & \text{if } i \equiv 4\ (mod\ 6) \\ f & \text{if } i \equiv 5\ (mod\ 6) \\ e & \text{if } i \equiv 0\ (mod\ 6); \end{cases}$$

$$g(v_i) = \begin{cases} d & \text{if } i \equiv 1\ (mod\ 6) \\ a & \text{if } i \equiv 2\ (mod\ 6) \\ f & \text{if } i \equiv 3\ (mod\ 6) \\ b & \text{if } i \equiv 4\ (mod\ 6) \\ e & \text{if } i \equiv 5\ (mod\ 6) \\ c & \text{if } i \equiv 0\ (mod\ 6). \end{cases}$$

Subcase (ii). $n \equiv 5\ (mod\ 6)$.

Let $n = 6k + 5$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last five vertices u_{6k+1}, u_{6k+2}, u_{6k+3}, u_{6k+4} and u_{6k+5} are labeled by a, b, d, c, e respectively and the last five vertices v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4} and v_{6k+5} are labeled by e, f, a, b, c respectively.

Subcase (iii). $n \equiv 4\ (mod\ 6)$.

Let $n = 6k + 4$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last four vertices u_{6k+1}, u_{6k+2}, u_{6k+3} and u_{6k+4} are labeled by a, f, b, e respectively and the last four vertices v_{6k+1}, v_{6k+2}, v_{6k+3} and v_{6k+4}

are labeled by d, b, e, c respectively.

Subcase (iv). $n \equiv 3$ $(mod\ 6)$.

Let $n = 6k + 3$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last three vertices u_{6k+1}, u_{6k+2} and u_{6k+3} are labeled by a, d, f respectively and the last three vertices v_{6k+1}, v_{6k+2} and v_{6k+3} are labeled by b, e, c respectively.

Subcase (v). $n \equiv 2$ $(mod\ 6)$.

Let $n = 6k + 2$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last two vertices u_{6k+1}, u_{6k+2} are labeled by f, a respectively and the last two vertices v_{6k+1}, v_{6k+2} are labeled by b, e respectively.

Subcase (vi). $n \equiv 1$ $(mod\ 6)$.

Let $n = 6k + 1$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the two vertices u_{6k+1}, v_{6k+1} are labeled by a, b respectively.

From Table 1, it is clear that g is a group S_3 cordial remainder labeling. □

TABLE 1

n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
3	1	1	1	2	1	1	5	4
4	1	2	1	2	2	1	6	6
5	2	2	2	1	2	2	8	7
$6k$	$2k$	$2k$	$2k$	$2k+1$	$2k$	$2k$	$9k$	$9k$
$6k+1$	$2k+1$	$2k+1$	$2k$	$2k+1$	$2k$	$2k$	$9k+2$	$9k+1$
$6k+2$	$2k+1$	$2k+1$	$2k$	$2k+1$	$2k+1$	$2k+1$	$9k+3$	$9k+3$
$6k+3$	$2k+1$	$2k+1$	$2k+1$	$2k+2$	$2k+1$	$2k+1$	$9k+5$	$9k+4$
$6k+4$	$2k+1$	$2k+2$	$2k+1$	$2k+2$	$2k+2$	$2k+1$	$9k+6$	$9k+6$
$6k+5$	$2k+2$	$2k+2$	$2k+2$	$2k+2$	$2k+2$	$2k+1$	$9k+8$	$9k+7$

Example 2.1. *A group S_3 cordial remainder labeling of helm graph H_8 is shown in Figure 1.*

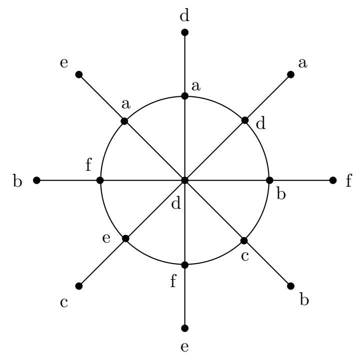

FIGURE 1

Theorem 2.2. *The flower graph Fl_n is a group S_3 cordial remainder graph for $n \geq 3$.*

Proof. The same labeling pattern is followed as in Theorem 2.1. □

Theorem 2.3. *The closed helm graph CH_n is a group S_3 cordial remainder graph for $n \geq 3$.*

Proof. Let H_n be the helm with $V(CH_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(CH_n) = \{uu_i, u_iv_i : 1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1\}$. Therefore, $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$. Define $g : V(CH_n) \to S_3$ as follows:

Case 1. $n = 3$.

Assign the labels d, b, d, c, f, e, a to the vertices $u, u_1, u_2, u_3, v_1, v_2, v_3$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table 2. According to these values from Table 2, g is a group S_3 cordial reminder labeling.

Case 2. $n = 4$.

Assign the labels $d, c, f, b, e, d, a, e, c$ to the vertices $u, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table

2. According to these values from Table 2, g is a group S_3 cordial reminder labeling.

Case 3. $n = 5$.

Assign the labels $d, a, b, f, c, e, d, f, a, e, c$ to the vertices $u, u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table 2. According to these values from Table 2, g is a group S_3 cordial reminder labeling.

Case 4. $n \geq 6$.

Subcase (i). $n \equiv 0$ $(mod\ 6)$. Let $n = 6k$ and $k \geq 1$. $g(u) = d$; for $1 \leq i \leq n$,

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \\ d & \text{if } i \equiv 2\ (mod\ 6) \\ b & \text{if } i \equiv 3\ (mod\ 6) \\ c & \text{if } i \equiv 4\ (mod\ 6) \\ f & \text{if } i \equiv 5\ (mod\ 6) \\ e & \text{if } i \equiv 0\ (mod\ 6); \end{cases}$$

$$g(v_i) = \begin{cases} d & \text{if } i \equiv 1\ (mod\ 6) \\ a & \text{if } i \equiv 2\ (mod\ 6) \\ b & \text{if } i \equiv 3\ (mod\ 6) \\ f & \text{if } i \equiv 4\ (mod\ 6) \\ e & \text{if } i \equiv 5\ (mod\ 6) \\ c & \text{if } i \equiv 0\ (mod\ 6). \end{cases}$$

Subcase (ii). $n \equiv 5$ $(mod\ 6)$.

Let $n = 6k + 5$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last five vertices u_{6k+1}, u_{6k+2}, u_{6k+3}, u_{6k+4} and u_{6k+5} are labeled by a, b, f, c, e respectively and the last five vertices v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4}

and v_{6k+5} are labeled by d, f, a, e, c respectively.

Subcase (iii). $n \equiv 4$ $(mod\ 6)$.

Let $n = 6k + 4$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last four vertices u_{6k+1}, u_{6k+2}, u_{6k+3} and u_{6k+4} are labeled by c, f, b, e respectively and the last four vertices v_{6k+1}, v_{6k+2}, v_{6k+3} and v_{6k+4} are labeled by d, a, e, c respectively.

Subcase (iv). $n \equiv 3$ $(mod\ 6)$.

Let $n = 6k + 3$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last three vertices u_{6k+1}, u_{6k+2} and u_{6k+3} are labeled by b, d, c respectively and the last three vertices v_{6k+1}, v_{6k+2} and v_{6k+3} are labeled by f, e, a respectively.

Subcase (v). $n \equiv 2$ $(mod\ 6)$.

Let $n = 6k + 2$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last two vertices u_{6k+1}, u_{6k+2} are labeled by b, f respectively and the last two vertices v_{6k+1}, v_{6k+2} are labeled by a, c respectively.

Subcase (vi). $n \equiv 1$ $(mod\ 6)$.

Let $n = 6k + 1$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the two vertices u_{6k+1}, v_{6k+1} are labeled by a, f respectively.

From Table 2, it is clear that g is a group S_3 cordial remainder labeling. □

Example 2.2. *A group S_3 cordial remainder labeling of closed helm graph CH_8 is shown in Figure 2.*

Theorem 2.4. *The gear graph G_n is a group S_3 cordial remainder graph for $n \geq 3$.*

Proof. Let $V(G_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(G_n) = \{uu_i, u_iv_i : 1 \leq i \leq n\} \bigcup \{v_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_nv_1\}$. Therefore, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. Define $g : V(G_n) \to S_3$ as follows:

Case 1. $n = 3$.

TABLE 2

n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
3	1	1	1	2	1	1	6	6
4	1	1	2	2	2	1	8	8
5	2	1	2	2	2	2	10	10
$6k$	$2k$	$2k$	$2k$	$2k+1$	$2k$	$2k$	$12k$	$12k$
$6k+1$	$2k+1$	$2k$	$2k$	$2k+1$	$2k$	$2k+1$	$12k+2$	$12k+2$
$6k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k$	$2k+1$	$12k+4$	$12k+4$
$6k+3$	$2k+1$	$2k+1$	$2k+1$	$2k+2$	$2k+1$	$2k+1$	$12k+6$	$12k+6$
$6k+4$	$2k+1$	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$2k+1$	$12k+8$	$12k+8$
$6k+5$	$2k+2$	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$2k+2$	$12k+10$	$12k+10$

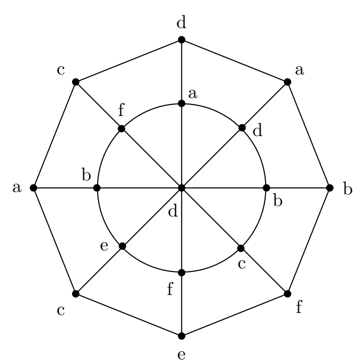

FIGURE 2

Assign the labels d, a, d, e, b, c, f to the vertices $u, u_1, u_2, u_3, v_1, v_2, v_3$ respectively. The values of $v_g(x),\ x \in \{e, a, b, c, d, f\},\ e_g(0)$ and $e_g(1)$ are given in Table 3. According to these values from Table 3, g is a group S_3 cordial reminder labeling.

Case 2. $n = 4$.

Assign the labels $d, a, c, f, d, f, b, e, c$ to the vertices $u, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ respectively. The values of $v_g(x),\ x \in \{e, a, b, c, d, f\},\ e_g(0)$ and $e_g(1)$ are given in Table

3. According to these values from Table 3, g is a group S_3 cordial reminder labeling.

Case 3. $n = 5$.

Assign the labels $d, a, d, b, c, f, a, b, f, e, c$ to the vertices $u, u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table 3. According to these values from Table 3, g is a group S_3 cordial reminder labeling.

Case 4. $n \geq 6$.

Subcase (i). $n \equiv 0\ (mod\ 6)$.

Let $n = 6k$ and $k \geq 1$.

$$g(u) = d;$$

for $1 \leq i \leq n$,

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \\ d & \text{if } i \equiv 2\ (mod\ 6) \\ b & \text{if } i \equiv 3\ (mod\ 6) \\ c & \text{if } i \equiv 4\ (mod\ 6) \\ f & \text{if } i \equiv 5\ (mod\ 6) \\ e & \text{if } i \equiv 0\ (mod\ 6); \end{cases}$$

$$g(v_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \\ b & \text{if } i \equiv 2\ (mod\ 6) \\ d & \text{if } i \equiv 3\ (mod\ 6) \\ e & \text{if } i \equiv 4\ (mod\ 6) \\ c & \text{if } i \equiv 5\ (mod\ 6) \\ f & \text{if } i \equiv 0\ (mod\ 6). \end{cases}$$

Subcase (ii). $n \equiv 5\ (mod\ 6)$.

Let $n = 6k+5$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last five vertices u_{6k+1}, u_{6k+2}, u_{6k+3}, u_{6k+4} and u_{6k+5} are labeled by a, d, b, c, f respectively and the last five vertices v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4} and v_{6k+5} are labeled by a, b, f, e, c respectively.

Subcase (iii). $n \equiv 4\ (mod\ 6)$.

Let $n = 6k+4$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last four vertices u_{6k+1}, u_{6k+2}, u_{6k+3} and u_{6k+4} are labeled by a, c, f, d respectively and the last four vertices v_{6k+1}, v_{6k+2}, v_{6k+3} and v_{6k+4} are labeled by f, b, e, c respectively.

Subcase (iv). $n \equiv 3\ (mod\ 6)$.

Let $n = 6k+3$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last three vertices u_{6k+1}, u_{6k+2} and u_{6k+3} are labeled by a, d, e respectively and the last three vertices v_{6k+1}, v_{6k+2} and v_{6k+3} are labeled by b, c, f respectively.

Subcase (v). $n \equiv 2\ (mod\ 6)$.

Let $n = 6k+2$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last two vertices u_{6k+1}, u_{6k+2} are labeled by a, f respectively and the last two vertices v_{6k+1}, v_{6k+2} are labeled by b, c respectively.

Subcase (vi). $n \equiv 1\ (mod\ 6)$.

Let $n = 6k+1$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the two vertices u_{6k+1}, v_{6k+1} are labeled by b, c respectively.

From Table 3, it is clear that g is a group S_3 cordial remainder labeling. □

Example 2.3. *A group S_3 cordial remainder labeling of gear graph G_7 is shown in Figure 3.*

Theorem 2.5. *The sunflower graph SF_n is a group S_3 cordial remainder graph for $n \geq 3$.*

TABLE 3

n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
3	1	1	1	2	1	1	5	4
4	1	1	2	2	1	2	6	6
5	2	2	2	2	1	2	7	8
$6k$	$2k$	$2k$	$2k$	$2k+1$	$2k$	$2k$	$9k$	$9k$
$6k+1$	$2k$	$2k+1$	$2k+1$	$2k+1$	$2k$	$2k$	$9k+2$	$9k+1$
$6k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k$	$2k+1$	$9k+3$	$9k+3$
$6k+3$	$2k+1$	$2k+1$	$2k+1$	$2k+2$	$2k+1$	$2k+1$	$9k+5$	$9k+4$
$6k+4$	$2k+1$	$2k+1$	$2k+2$	$2k+2$	$2k+1$	$2k+2$	$9k+6$	$9k+6$
$6k+5$	$2k+2$	$2k+2$	$2k+2$	$2k+2$	$2k+1$	$2k+2$	$9k+7$	$9k+8$

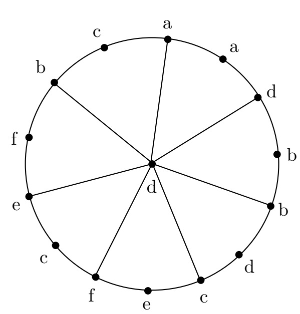

FIGURE 3

Proof. Let u be the center of the wheel and $u_1, u_2, \cdots, u_n$ be the vertices on the cycle of the wheel. Let $v_1, v_2, \cdots, v_n$ be the additional vertices so that v_i is adjacent to $u_i,\ u_{i+1},\ 1 \leq i \leq n-1$ and v_n is adjacent to $u_n,\ u_1$. Therefore, $|V(SF_n)| = 2n+1$ and $|E(SF_n)| = 4n$. Define $g : V(SF_n) \to S_3$ as follows:

Case 1. $n = 3$.

Assign the labels d, a, d, f, e, b, c to the vertices $u, u_1, u_2, u_3, v_1, v_2, v_3$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table 4. According to these values from Table 4, g is a group S_3 cordial reminder labeling.

Case 2. $n = 4$.

Assign the labels $d, a, f, b, e, b, c, d, f$ to the vertices $u, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table 4. According to these values from Table 4, g is a group S_3 cordial reminder labeling.

Case 3. $n = 5$.

Assign the labels $d, a, b, f, c, e, d, a, b, f, e$ to the vertices $u, u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5$ respectively. The values of $v_g(x)$, $x \in \{e, a, b, c, d, f\}$, $e_g(0)$ and $e_g(1)$ are given in Table 4. According to these values from Table 4, g is a group S_3 cordial reminder labeling.

Case 4. $n \geq 6$.

Subcase (i). $n \equiv 0\ (mod\ 6)$. Let $n = 6k$ and $k \geq 1$. $g(u) = d$; for $1 \leq i \leq n$,

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \\ d & \text{if } i \equiv 2\ (mod\ 6) \\ b & \text{if } i \equiv 3\ (mod\ 6) \\ c & \text{if } i \equiv 4\ (mod\ 6) \\ f & \text{if } i \equiv 5\ (mod\ 6) \\ e & \text{if } i \equiv 0\ (mod\ 6); \end{cases}$$

$$g(v_i) = \begin{cases} d & \text{if } i \equiv 1 \ (mod\ 6) \\ a & \text{if } i \equiv 2 \ (mod\ 6) \\ f & \text{if } i \equiv 3 \ (mod\ 6) \\ b & \text{if } i \equiv 4 \ (mod\ 6) \\ c & \text{if } i \equiv 5 \ (mod\ 6) \\ e & \text{if } i \equiv 0 \ (mod\ 6). \end{cases}$$

Subcase (ii). $n \equiv 5\ (mod\ 6)$.

Let $n = 6k + 5$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last five vertices u_{6k+1}, u_{6k+2}, u_{6k+3}, u_{6k+4} and u_{6k+5} are labeled by a, b, f, c, e respectively and the last five vertices v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4} and v_{6k+5} are labeled by d, a, b, f, e respectively.

Subcase (iii). $n \equiv 4\ (mod\ 6)$.

Let $n = 6k + 4$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last four vertices u_{6k+1}, u_{6k+2}, u_{6k+3} and u_{6k+4} are labeled by a, f, b, e respectively and the last four vertices v_{6k+1}, v_{6k+2}, v_{6k+3} and v_{6k+4} are labeled by b, c, d, f respectively.

Subcase (iv). $n \equiv 3\ (mod\ 6)$.

Let $n = 6k + 3$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last three vertices u_{6k+1}, u_{6k+2} and u_{6k+3} are labeled by a, d, f respectively and the last three vertices v_{6k+1}, v_{6k+2} and v_{6k+3} are labeled by e, b, c respectively.

Subcase (v). $n \equiv 2\ (mod\ 6)$.

Let $n = 6k + 2$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the last two vertices u_{6k+1}, u_{6k+2} are labeled by a, b respectively and the last two vertices v_{6k+1}, v_{6k+2} are labeled by f, c respectively.

Subcase (vi). $n \equiv 1\ (mod\ 6)$.

Let $n = 6k + 1$ and $k \geq 1$. Assign the labels to the vertices u, u_i and v_i as in the Subcase (i), except that the two vertices u_{6k+1}, v_{6k+1} are labeled by f, c respectively. From Table 4, it is clear that g is a group S_3 cordial remainder labeling. □

TABLE 4

n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
3	1	1	1	2	1	1	6	6
4	1	2	1	2	1	2	8	8
5	2	2	1	2	2	2	10	10
$6k$	$2k$	$2k$	$2k$	$2k+1$	$2k$	$2k$	$12k$	$12k$
$6k+1$	$2k$	$2k$	$2k+1$	$2k+1$	$2k$	$2k+1$	$12k+2$	$12k+2$
$6k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k$	$2k+1$	$12k+4$	$12k+4$
$6k+3$	$2k+1$	$2k+1$	$2k+1$	$2k+2$	$2k+1$	$2k+1$	$12k+6$	$12k+6$
$6k+4$	$2k+1$	$2k+2$	$2k+1$	$2k+2$	$2k+1$	$2k+2$	$12k+8$	$12k+8$
$6k+5$	$2k+2$	$2k+2$	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$12k+10$	$12k+10$

Example 2.4. *A group S_3 cordial remainder labeling of sunflower graph SF_8 is shown in Figure 4.*

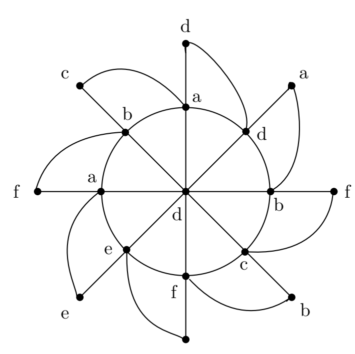

FIGURE 4

Theorem 2.6. *The triangular snake T_n is a group S_3 cordial remainder graph.*

Proof. Let T_n be a triangular snake with n blocks. Let $V(T_n) = \{u_i : 1 \leq i \leq n+1\} \bigcup \{v_i : 1 \leq i \leq n\}$ and $E(T_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}v_i : 1 \leq i \leq n\}$. Then $|V(T_n)| = 2n+1$ and $|E(T_n)| = 3n$. Define $g : V(T_n) \to S_3$ as follows:

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq n+1 \\ d & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq n+1 \\ e & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq n+1 \\ f & \text{if } i \equiv 0, 4 \ (mod\ 6) \text{ and } 1 \leq i \leq n+1 \\ b & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq n+1; \end{cases}$$

$$g(v_i) = \begin{cases} b & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ c & \text{if } i \equiv 2, 0 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ a & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ d & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ e & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq n. \end{cases}$$

From Table 5, it is clear that g is a group S_3 cordial remainder labeling. □

Example 2.5. *A group S_3 cordial remainder labeling of triangular snake T_5 is shown in Figure 5.*

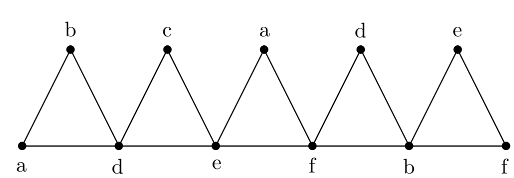

FIGURE 5

Theorem 2.7. *The quadrilateral snake Q_n is a group S_3 cordial remainder graph.*

TABLE 5

n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
1	1	1	0	1	0	0	1	2
2	1	1	1	1	1	0	3	3
3	2	1	1	1	1	1	5	4
4	2	2	1	2	1	1	6	6
5	2	2	1	2	2	2	8	7
$6k$	$2k+1$	$2k$	$2k$	$2k$	$2k$	$2k$	$9k$	$9k$
$6k+1$	$2k+1$	$2k+1$	$2k$	$2k+1$	$2k$	$2k$	$9k+1$	$9k+2$
$6k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k$	$9k+3$	$9k+3$
$6k+3$	$2k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$9k+5$	$9k+4$
$6k+4$	$2k+2$	$2k+2$	$2k+1$	$2k+2$	$2k+1$	$2k+1$	$9k+6$	$9k+6$
$6k+5$	$2k+2$	$2k+2$	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$9k+8$	$9k+7$

Proof. Let Q_n be a quadrilateral snake with $V(Q_n) = \{u_i : 1 \leq i \leq n+1\} \bigcup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(Q_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}w_i, v_iw_i : 1 \leq i \leq n\}$. Then $|V(Q_n)| = 3n+1$ and $|E(Q_n)| = 4n$. Define $g : V(Q_n) \to S_3$ as follows:

$$g(u_i) = \begin{cases} f & \text{if } i = 1 \\ d & \text{if } i \text{ is even and } 2 \leq i \leq n+1 \\ c & \text{if } i \text{ is odd and } 2 \leq i \leq n+1; \end{cases}$$

$$g(v_i) = \begin{cases} a & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ e & \text{if } i \text{ is even and } 1 \leq i \leq n; \end{cases}$$

$$g(w_i) = \begin{cases} b & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ f & \text{if } i \text{ is even and } 1 \leq i \leq n. \end{cases}$$

From Table 6, it is clear that g is a group S_3 cordial remainder labeling. $\square$

TABLE 6

n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
$2k-1$ $(k \geq 1)$	k	k	$k-1$	k	$k-1$	k	$4k-2$	$4k-2$
$2k$ $(k \geq 1)$	k	k	k	k	k	$k+1$	$4k$	$4k$

Example 2.6. *A group S_3 cordial remainder labeling of quadrilateral snake Q_5 is shown in Figure 6.*

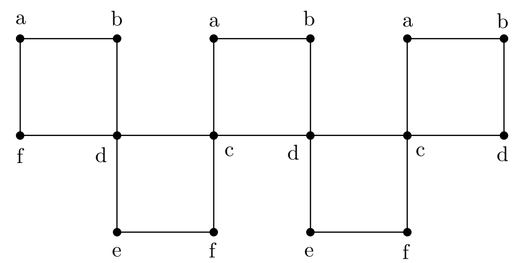

FIGURE 6

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(1) Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai-627002, Tamil Nadu, India.

Email address: lourdusamy15@gmail.com

(2) Reg. No. 17211282092013, Research Scholar, Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.

Email address: jeniferwency@gmail.com

(3) Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai-627002, Tamil Nadu, India.

Email address: patrick881990@gmail.com