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A NEW APPROACH FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS BASED ON FINITE-DIFFERENCE AND HAAR WAVELET METHODS

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Abstract. The main objective of this paper is to develop a new scheme based on finite-difference and Haar wavelet for second order diffusion equation and third order dispersive equation. Further, we have carried out the stability of the Haar wavelet. We solved four problems consisting linear diffusion equation and dispersive homogeneous and non homogeneous equation to validate the developed scheme. We have also compared our results with existing methods such as finite difference method, global extrapolation method and non polynomial spline method.

1. Introduction

Let us consider the following linear diffusion equation:

$$\frac{\partial u}{\partial \tau} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1) \tag{1.1}$$

with initial condition

$$u(x,0) = \phi(x), \tag{1.2}$$

and boundary conditions

$$u(0,\tau) = \phi_0(\tau), \quad u(1,\tau) = \phi_1(\tau), \ \ \tau > 0. \tag{1.3}$$

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The solution of (1.1) gives the temperature at a distance x from one end of a thin uniform rod after some time. Temperature changes occur through heat conduction along its length and heat transfer due to the rod is heat insulated. Numerical solution of (1.1) based on finite difference method has been proposed by G.D. Smith [1], M.K. Jain [2] and Hilberman [3]. Cubic spline method has been proposed by Sallam et al [4] and non-polynomial cubic spline method has been proposed by J. Rashidinia et al [5].

Further, we consider the following linear dispersive equation:

$$\frac{\partial u}{\partial \tau} + \mu \frac{\partial^3 u}{\partial x^3} = f(x, \tau), \quad x \in (0, 1), \tag{1.4}$$

with initial conditions

$$u(x, 0) = g(x), \tag{1.5}$$

and

$$u(0, \tau) = \phi_0(\tau), \quad u_x(0, \tau) = \phi_1(\tau), \quad u_{xx}(0, \tau) = \phi_2(\tau), \ \ \tau > 0. \tag{1.6}$$

It is well known that many physical phenomena can be described by Korteweg-de Vries equation such as model of a fluid in elastic tube, water waves and collision free hydro magnetic waves, turbulence, flow of liquids containing gas bubbles, viscous fluid, unindirectional propagation of small but finite amplitude waves, long waves in which dispersive effects are present, liquid with small bubbles, see ([6]- [9]).

Numerical solution of (1.4) has been proposed by various researchers via different approaches such as, parametric septic spline, adomain decomposition method, exponential finite difference method, pseudo spectral method, finite element method, global extrapolation method , exponential quartic spline method and heat balance integral methods see ([10]- [15]).

Lepik et al. solved integral and differential equations in [16], partial differential equations in [17] and a concise study on applications of Haar wavelet can be seen in [18].

Imran aziz et al [19] and Siraj-ul Islam et al [20] used Haar and Legendre wavelet to solve elliptic and parabolic partial differential equations respectively. Siraj-ul Islam et al [21] solved second order boundary value problems by collocation method with Haar wavelets. Imran aziz et al [22] solved delay partial differential equation using Haar wavelet finite difference method. Muhammad Ahsan et al [23], Saleem et al [24] used finite difference Haar wavelet method to solve non linear schrodinger equation and nonlinear parabolic partial differential equations, respectively. Omer Oruc ([25], [26]) used Hermite wavelet to solve 2D hyperbolic equation and long wave equation in fluid respectively. Omer Oruc et al used chebyshev wavelet for coupled Berger equation [27], nonuniform Haar wavelet for convection dominated equation and singular elliptic equation [28], KdV and coupled nonlinear schrodinger-KdV equation by Haar wavelet in [29], [30]. M. Kumar and Sapna Pandit [31] solved coupled Berger equation and Pandit and Kumar [32] solved two parameters singularly perturbed boundary value problems using Haar wavelet. A. Raza and Khan ([33]-[36]) solved neutral delay differential equation, Higher order two point boundary value problem and singularly perturbed delay difference equation using uniform and non-uniform Haar wavelet. Shah et al [37] solved singularly perturbed boundary value problem using uniform Haar wavelet. In this paper, we described Haar wavelet in section 2 and the solution of diffusion and dispersive equation using Haar wavelet and finite difference method is described in section 3. Further, the stability of Haar wavelet is described in sub-section 3.3 and the numerical illustration has been presented in section 4.

2. Haar Wavelet

The Haar Wavelet is very simple wavelet in comparison of the other wavelets such as Legendre wavelet, Hermite wavelet, Chebyshev wavelet, Laguerre wavelet and Battle-Lamare (B-spline) wavelet. The advantages of Haar wavelet are symmetry, compact support, flexibility and orthogonality. Haar wavelet allow us to approximate any square integrable function at different resolution level ([38] - [43]). The Haar wavelet family for $x \in [0,1]$ is defined as follows:

$$\mathcal{H}_i(x) = \begin{cases} 1, & \mu_1(i) \leq x < \mu_2(i), \\ -1, & \mu_2(i) \leq x < \mu_3(i), \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where i indicates the wavelet number and

$$\mu_1(i) = \frac{k}{m}, \quad \mu_2(i) = \frac{k+0.5}{m}, \quad \mu_3(i) = \frac{k+1}{m},$$

$$m = 2^j, \; j = 0, 1, 2, \ldots, J, \quad k = 0, 1, \ldots, m-1.$$

Here J indicates the level of resolution and k represents the translations parameter. Index i is calculated as $i = m + k + 1$ which is true for $i \geq 2$.

For $i = 1$, the Haar wavelet is given by

$$\mathcal{H}_1(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

The integration $\mathcal{P}_i(x)$ of Haar wavelet can be obtained as follows:

$$\mathcal{P}_i(x) = \begin{cases} x - \mu_1(i), & \mu_1(i) \leq x < \mu_2(i), \\ \mu_3(i) - x, & \mu_2(i) \leq x < \mu_3(i), \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

Furthermore, the double integration $\mathcal{Q}_i(x)$ of Haar wavelet can be obtained as follows:

$$\mathcal{Q}_i(x) = \begin{cases} \frac{1}{2}(x-\mu_1(i))^2, & \mu_1(i) \le x < \ \mu_2(i), \\ \frac{1}{4m^2} - \frac{1}{2}(\mu_3(i)-x)^2, & \mu_2(i) \le x < \mu_3(i), \\ \frac{1}{4m^2}, & \mu_3(i) \le x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

The triple integration of Haar wavelet can be obtained as follows:

$$\mathcal{R}_i(x) = \begin{cases} 0, & x < \mu_1(i), \\ \frac{1}{3!}[x-\mu_1(i)]^3, & \mu_1(i) \le x < \ \mu_2(i), \\ \frac{1}{3!}[(x-\mu_1(i))(x-\mu_2(i))^3], & \mu_2(i) \le x < \mu_3(i), \\ \frac{1}{3!}[(x-\mu_1(i))(x-\mu_2(i))^3 + (x-\mu_3(i))^3], & \mu_3(i) \le x. \end{cases} \tag{2.5}$$

The collocation grid is given as

$$X(i) = \frac{2i-1}{m}, \quad i = 1, 2, \ldots, m$$

and the time disctetization is given by

$$\tau(i) = \frac{i}{N}, \quad i = 0, 1, 2, \ldots, N.$$

3. Treatment of Higher Order Linear Partial Differential Equation

3.1. **Method for Solving Linear Diffusion Equation.** To solve linear diffusion equation we apply finite difference method to discretize time derivative and Haar wavelet to approximate space derivative.

Let us consider the following linear diffusion equation

$$\frac{\partial u}{\partial \tau} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1) \tag{3.1}$$

with initial condition

$$u(x,0) = \phi(x), \tag{3.2}$$

and boundary conditions

$$u(0,\tau) = \phi_0(\tau), \quad u(1,\tau) = \phi_1(\tau), \ \ \tau > 0, \tag{3.3}$$

To solve this problem, we assume that

$$u_{xx}(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{H}_i(x). \tag{3.4}$$

Now, integrating from 0 to x, we get

$$u_x(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{P}_i(x) + u_x(0,\tau). \tag{3.5}$$

Again integrating from 0 to x, we get

$$u(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(x) + xu_x(0,\tau) + u(0,\tau). \tag{3.6}$$

Further to find $u_x(0,\tau)$ we integrate the equation (3.5) from 0 to 1 and we get

$$u_x(0,\tau) = u(0,\tau) - u(1,\tau) - \sum_{i=1}^{N} a_i(\tau)\mathcal{C}_i(1), \tag{3.7}$$

Putting the values of $u(0,\tau)$ and $u(1,\tau)$ from (3.3), we get

$$u_x(0,\tau) = \phi_0(\tau) - \phi_1(\tau) - \sum_{i=1}^{N} a_i(\tau)\mathcal{C}_i(1). \tag{3.8}$$

Now, using equation (3.8) in equation (3.6), we get

$$u(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(x) + x(\phi_0(\tau) - \phi_1(\tau) - \sum_{i=1}^{N} a_i(\tau)\mathcal{C}_i(1)) + \phi_0(\tau). \tag{3.9}$$

$$u(x,\tau) = \sum_{i=1}^{N} a_i(\tau)(\mathcal{Q}_i(x) - x\mathcal{C}_i(1)) + x(\phi_0(\tau) - \phi_1(\tau)) + \phi_0(\tau). \tag{3.10}$$

Equation (3.10) is Haar wavelet approximate solution of the diffusion equation (3.1) with initial condition(3.2) and boundary conditions (3.3). Now, our aim is to find the unknown Haar wavelet coefficients $a_i(\tau)$, to obtain approximate solution of linear diffusion equation. Therefore, we discretize time derivative by finite-difference method as follows,

$$u_\tau(x_k, \tau_{j+1}) = \frac{u(x_k, \tau_{j+1}) - u(x_k, \tau_j)}{\triangle\tau}. \tag{3.11}$$

Using (3.10) and (3.11) in (3.1), we get,

$$\frac{u(x_k, \tau_{j+1}) - u(x_k, \tau_j)}{\triangle\tau} = \alpha \sum_{i=1}^{N} a_i(\tau_{j+1})\mathcal{H}_i(x_k). \tag{3.12}$$

Now, using the values of $u(x_k, \tau_{j+1})$ and $u(x_k, \tau_j)$ from equation (3.10) in equation (3.12), we get

$$\begin{aligned}\sum_{i=1}^{N} a_i(\tau_{j+1})(\mathcal{Q}_i(x_k) - x_k\mathcal{C}_i(1)) + x_k(\phi_0(\tau_{j+1}) - \phi_1(\tau_{j+1})) + \phi_0(\tau_{j+1}) - \sum_{i=1}^{N} a_i(\tau_j)(\mathcal{Q}_i(x_k)\\ -x_k\mathcal{C}_i(1)) + x_k(\phi_0(\tau_j) - \phi_1(\tau_j)) + \phi_0(\tau_j) = \triangle\tau\alpha \sum_{i=1}^{N} a_i(\tau_{j+1})\mathcal{H}_i(x_k).\end{aligned} \tag{3.13}$$

$$\begin{aligned}\sum_{i=1}^{N} a_i(\tau_{j+1})(\mathcal{Q}_i(x_k) - x_k\mathcal{C}_i(1) - \triangle\tau\alpha\mathcal{H}_i(x_k)) - \sum_{i=1}^{N} a_i(\tau_j)(\mathcal{Q}_i(x_k) - x_k\mathcal{C}_i(1))\\ = x_k(\phi_1(\tau_{j+1}) - \phi_0(\tau_{j+1})) - \phi_0(\tau_{j+1}) + x_k(\phi_1(\tau_j) - \phi_0(\tau_j)) - \phi_0(\tau_j).\end{aligned} \tag{3.14}$$

Now, we assume that

$\mathcal{W} = \mathcal{Q}_i(x_k) - x_k\mathcal{C}_i(1) - \triangle\tau\alpha\mathcal{H}_i(x_k)$,

$\mathcal{A} = \mathcal{Q}_i(x_k) - x_k\mathcal{C}_i(1)$

$\mathcal{V} = x_k(\phi_1(\tau_{j+1}) - \phi_0(\tau_{j+1})) - \phi_0(\tau_{j+1}) + x_k(\phi_1(\tau_j) - \phi_0(\tau_j)) - \phi_0(\tau_j)$

where the matrices $\mathcal{H}$, $\mathcal{Q}$ and $\mathcal{C}$ for eight collocation points are given as follows

$$\mathcal{H}=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1, \end{pmatrix}, \mathcal{Q}=\frac{1}{512}\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix},$$

$$\mathcal{C}=\frac{1}{64}\begin{pmatrix} 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1, \end{pmatrix}.$$

On expanding the system (3.14), we get the following matrix formulation

$$\begin{pmatrix} -\mathcal{A} & \mathcal{W} & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -\mathcal{A} & \mathcal{W} & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & -\mathcal{A} & \mathcal{W} & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{A} & \mathcal{W} & 0 & ... & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -\mathcal{A} & \mathcal{W} & 0 \\ 0 & 0 & 0 & . & . & . & 0 & -\mathcal{A} & \mathcal{W} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_N \\ a_{N+1} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ . \\ . \\ . \\ . \\ \mathcal{V}_N \\ \mathcal{V}_{N+1} \end{pmatrix} \tag{3.15}$$

where a_0, a_1, a_2 ... a_{N+1} and $\mathcal{V}_0$, $\mathcal{V}_1$,....$\mathcal{V}_{N+1}$ are vectors of size $N \times 1$. We solve system (3.15) and obtained unknown finite difference Haar wavelet coefficients $a_i(\tau_j)$ at each time level and then we put them in equation (3.10) to find the approximate finite-difference Haar wavelet solution of linear diffusion equation (3.1) with initial condition (3.2) and boundary conditions (3.3).

3.2. **Method for Solving Linear dispersive Equation.** To solve linear dispersive equation, we apply finite difference method to discretize time derivative and Haar wavelet to approximate space derivative.

Let us consider the following linear dispersive equation:

$$\frac{\partial u}{\partial \tau} + \mu \frac{\partial^3 u}{\partial x^3} = f(x, \tau), \quad x \in (0, 1) \tag{3.16}$$

with initial conditions

$$u(x, 0) = g(x), \tag{3.17}$$

and

$$u(0, \tau) = \phi_0(\tau), \quad u_x(0, \tau) = \phi_1(\tau), \quad u_{xx}(0, \tau) = \phi_2(\tau), \ \ \tau > 0, \tag{3.18}$$

To solve this problem, we assume that

$$u_{xxx}(x, \tau) = \sum_{i=1}^{N} a_i(\tau) \mathcal{H}_i(x). \tag{3.19}$$

Now, integrating from 0 to x, we get

$$u_{xx}(x, \tau) = \sum_{i=1}^{N} a_i(\tau) \mathcal{P}_i(x) + u_{xx}(0, \tau). \tag{3.20}$$

Again integrating from 0 to x, we get

$$u_x(x, \tau) = \sum_{i=1}^{N} a_i(\tau) \mathcal{Q}_i(x) + x u_{xx}(0, \tau) + u_x(0, \tau). \tag{3.21}$$

Further integrating, we get

$$u(x,\tau)=\sum_{i=1}^{N}a_i(\tau)\mathcal{R}_i(x)+\frac{x^2}{2}u_{xx}(0,\tau)+xu_x(0,\tau)+u(0,\tau). \tag{3.22}$$

Putting the values of $u(0,\tau)$, $u_x(0,\tau)$ and $u_{xx}(0,\tau)$ from equation (3.17) and (3.18) in equation (3.22), we get

$$u(x,\tau)=\sum_{i=1}^{N}a_i(\tau)\mathcal{R}_i(x)+\frac{x^2}{2}\phi_2(\tau)+x\phi_1(\tau)+\phi_0(\tau). \tag{3.23}$$

Equation (3.23) is Haar wavelet approximate solution of the diffusion equation (3.16) with initial condition (3.17) and (3.18). Now, our aim is to find the unknown Haar wavelet coefficients $a_i(\tau)$, to obtain approximate solution of linear diffusion equation. Therefore, we discretize time derivative of diffusion equation by finite-difference method as follows:

$$u_\tau(x_k,\tau_{j+1})=\frac{u(x_k,\tau_{j+1})-u(x_k,\tau_j)}{\triangle\tau}. \tag{3.24}$$

Now, using (3.19) and (3.24) in (3.16), we get,

$$\frac{u(x_k,\tau_{j+1})-u(x_k,\tau_j)}{\triangle\tau}+\mu\sum_{i=1}^{N}a_i(\tau_{j+1})\mathcal{H}_i(x_k)=f(x_k,\tau_j). \tag{3.25}$$

Using the values of $u(x_k,\tau_{j+1})$ and $u(x_k,\tau_j)$ from equation (3.23) in equation (3.25), we get

$$\sum_{i=1}^{N}a_i(\tau_{j+1})\mathcal{R}_i(x_k)+\frac{x_k^2}{2}\phi_2(\tau_{j+1})+x\phi_1(\tau_{j+1})+\phi_0(\tau_{j+1})-(\sum_{i=1}^{N}a_i(\tau_j)\mathcal{R}_i(x_k)+$$
$$\frac{x_k^2}{2}\phi_2(\tau_j)+x\phi_1(\tau_j)+\phi_0(\tau_j))+\triangle\tau\mu\sum_{i=1}^{N}a_i(\tau_{j+1})\mathcal{H}_i(x_k)=f(x_k,\tau_j). \tag{3.26}$$

On simplification, we get the following system of linear equations,

$$\sum_{i=1}^{N}a_i(\tau_{j+1})(\mathcal{R}_i(x_k)+\triangle\tau\mu\mathcal{H}_i(x_k))-\sum_{i=1}^{N}a_i(\tau_j)\mathcal{R}_i(x_k)=\triangle\tau f(x_k,\tau_j)-$$
$$(\frac{x_k^2}{2}\phi_2(\tau_{j+1})+x\phi_1(\tau_{j+1})+\phi_0(\tau_{j+1}))-(\frac{x_k^2}{2}\phi_2(\tau_j)+x\phi_1(\tau_j)+\phi_0(\tau_j)). \tag{3.27}$$

Now, we assume that

$\mathcal{W} = \mathcal{R}_i(x_k) + \triangle\tau\mu\mathcal{H}_i(x_k)$,

$\mathcal{A} = \mathcal{R}_i(x_k)$

$\mathcal{V} = \triangle\tau f(x_k,\tau_j) - (\frac{x_k^2}{2}\phi_2(\tau_{j+1}) + x\phi_1(\tau_{j+1}) + \phi_0(\tau_{j+1})) - (\frac{x_k^2}{2}\phi_2(\tau_j) + x\phi_1(\tau_j) + \phi_0(\tau_j))$

where matrix $\mathcal{H}$ is same as in section (3.1) and matrix $\mathcal{R}$ is given as follows:

$$\mathcal{R} = \frac{1}{24576}\begin{pmatrix} 1 & 27 & 125 & 343 & 729 & 1331 & 2197 & 3375 \\ 1 & 27 & 125 & 343 & 727 & 1277 & 1947 & 2689 \\ 1 & 27 & 123 & 289 & 480 & 672 & 864 & 1056 \\ 0 & 0 & 0 & 0 & 1 & 27 & 123 & 289 \\ 1 & 25 & 72 & 120 & 168 & 216 & 264 & 312 \\ 0 & 0 & 1 & 25 & 72 & 120 & 168 & 216 \\ 0 & 0 & 0 & 0 & 1 & 25 & 72 & 120 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 25 \end{pmatrix}$$

On expanding the system (3.27), we get the following matrix formulation

$$\begin{pmatrix} -\mathcal{A} & \mathcal{W} & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -\mathcal{A} & \mathcal{W} & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & -\mathcal{A} & \mathcal{W} & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{A} & \mathcal{W} & 0 & ... & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -\mathcal{A} & \mathcal{W} & 0 \\ 0 & 0 & 0 & . & . & . & 0 & -\mathcal{A} & \mathcal{W} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_N \\ a_{N+1} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ . \\ . \\ . \\ . \\ \mathcal{V}_N \\ \mathcal{V}_{N+1} \end{pmatrix} \quad (3.28)$$

where a_0, a_1, a_2 ... a_{N+1} and $\mathcal{V}_0$, $\mathcal{V}_1$,....$\mathcal{V}_{N+1}$ are vectors of size $N \times 1$. We solve system (3.28) and obtained unknown finite difference Haar wavelet coefficients $a_i(\tau_j)$ at each time level and then we put them in equation (3.23) to finding out the approximate finite-difference Haar wavelet solution of equation (3.16) with initial conditions (3.17) and (3.18).

3.3. **Stability, Convergence and Error Analysis.** In this section, we have shown the stability and convergence analysis of the developed method.
A multiresolution analysis consists of a sequence $\{V_j : j \in Z\}$ of embeded closed subspace of $L^2(R)$ that satisfy the following properties ([37]- [42]) :

(1) **Increasing:** $V_j \subset V_{j+1} : j \in Z$
(2) **Density:** $\bigcup_{j \in Z} V_j = L^2(\mathbb{R})$
(3) **Separation :**$\bigcap_{j \in Z} V_j = \{0\}$
(4) **Scaling :** $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$
(5) **Orthonormal basis:** $\exists$ a scaling function $\phi \in V_0$ such that $\{\phi_{0,k}(x) = \phi(x-k) : k \in Z\}$ is an orthonormal basis for V_0.

In our case the scaling function ϕ is Haar function which is given in equation (2.1) and hence $\mathcal{H}$ is basis function for the spaces V_j. The function given by (2.2) is considered as a scaling function of the multi-resolution analysis property (5) or father wavelet. Consider the space V_j of all functions in $L^2([0,1])$ which is generated by the Haar wavelet (2.1) i.e. $V_j = \{\mathcal{H}_{j,k} = 2^{j/2}\mathcal{H}_{j,k}(2^j x - k) : j,k \in \mathbb{Z}\}$. Obviously V_0 is the closed subspace of $L^2([0,1])$ on the intervals $[k, k+1)$ for $k \in \mathbb{Z}$ and V_1 consists the Haar function on the intervals $[\frac{k}{2}, \frac{k+1}{2})$ and so on. Clearly, the set $\{\mathcal{H}_{j,k} = 2^{j/2}\mathcal{H}_{j,k}(2^j x - k) : j,k \in \mathbb{Z}\}$ forms an orthonormal set.
Now our aim is to show the stability of the Haar wavelet by considering the following lemma.

Lemma 3.1. *Let* $f \in L^2(\mathbb{R})$ *and* $\{\mathcal{H}_{j,k} : j,k \in \mathbb{Z}\}$ *be a Haar wavelet basis of the space* $\{V_j\colon j,k \in \mathbb{Z}\}$. *Then, Haar wavelet basis* $\{\mathcal{H}_{j,k}\colon j,k \in \mathbb{Z}\}$ *satisfies the following inequality, i.e. there exists* $0 < A < B \in \mathbb{R}$ *such that*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \mathcal{H}_{j,k}\rangle|^2 \leq B\|f\|^2, \quad for \quad all \quad f \in L^2(\mathbb{R}) \tag{3.29}$$

Proof. To prove the above inequality we use Parseval identity and Fourier transform. Parseval identity for $f, g \in L^2(\mathbb{R})$ is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle \tag{3.30}$$

where $\hat{f}$ is the Fourier transform of a function f, which is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{3.31}$$

Let us assume that the Fourier transform of the Haar wavelet is $\hat{g}$, i.e. $\hat{\mathcal{H}} = \hat{g}$

$$|\hat{g}(\omega + 2k\pi)|^2 = \frac{sin^2(\omega/2 + k\pi)}{(\omega/2 + k\pi)^2}, \quad for \quad all \quad \omega \in \quad [0, 2\pi] \tag{3.32}$$

Taking summation on both sides, we get

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\omega + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \frac{sin^2(\omega/2 + k\pi)}{(\omega/2 + k\pi)^2}, \quad for \quad all \quad \omega \in [0, 2\pi] \tag{3.33}$$

since $\frac{2}{\pi} \leq \frac{sin\omega}{\omega}$.

Therefore, we have

$$\Big(\frac{2}{\pi}\Big)^2 \leq \Big(\frac{sin\omega/2}{\omega/2}\Big)^2 \tag{3.34}$$

$$\Big(\frac{2}{\pi}\Big)^2 \leq \Big(\frac{sin\omega/2}{\omega/2}\Big)^2 + \Big(\frac{sin\omega/2}{\pi - \omega/2}\Big)^2 \tag{3.35}$$

$$\Big(\frac{2}{\pi}\Big)^2 \leq \Big(\frac{sin\omega/2}{\omega/2}\Big)^2 + \Big(\frac{sin\omega/2}{\pi - \omega/2}\Big)^2 \leq \sum_{k\in\mathbb{Z}} \Big(\frac{sin\omega/2}{\omega/2 + k\pi}\Big)^2 \tag{3.36}$$

This implies

$$\left(\frac{2}{\pi}\right)^2 \leq |\hat{g}(\omega + 2k\pi)|^2 \tag{3.37}$$

Since we know that

$$|sinx| \leq |x| \quad and \quad \frac{1}{sin^2\omega} = \frac{1}{\sum_{k\in\mathbb{Z}}(\omega + 2k\pi)^2} \tag{3.38}$$

we have

$$\sum_{k\in\mathbb{Z}}\left(\frac{sin^2(\omega/2)}{(\omega/2 + k\pi)^2}\right)^2 = \sum_{k\in\mathbb{Z}}\left(\frac{sin^2(\omega/2 + k\pi)}{(\omega/2 + k\pi)^2}\right)^2 = 1 \tag{3.39}$$

Now, combining (3.37) and (3.39), we get

$$\left(\frac{2}{\pi}\right)^2 \leq \sum_{k\in\mathbb{Z}} |\hat{g}(\omega + 2k\pi)|^2 \leq 1. \tag{3.40}$$

□

Lemma 3.2. *Let $u \in L^2([0,1])$ with bounded derivative and $u(x) = \sum_{i=1}^{N} a_i \mathcal{H}_i(x)$ be the Haar wavelet series. Then, the Haar wavelet coefficient a_i satisfies the following inequality*

$$\|a_i\|^2 \leq K2^{-(3j-2)/2}, \quad where \quad |u'(x)| \leq K \tag{3.41}$$

Proof. See [31]. □

Lemma 3.3. *If $u(x)$ is the exact and $u_J(x)$ is the approximate solution of the equation (3.1) and (3.16) then error norm satisfies the following inequality*

$$\|E_J\|^2 = \|u(x) - u_J(x)\| \leq K\sqrt{C}\frac{2^{-3(2^j)-1}}{1 - 2^{-3/2}}, \quad where \quad |u'(x)| \leq K \tag{3.42}$$

Proof. See [31]. □

4. Numerical Examples

In this section, we demonstrate two examples of second order linear diffusion equation and two examples of third order linear dispersive equation to show the applicability, accuracy and efficiency of the Haar wavelet finite difference method. Further, we computed maximum absolute error and compared with finite difference method [1], global extrapolation method [13] and non polynomial spline method [14].

Problem 1. Let us consider the following parabolic PDE

$$\frac{\partial u}{\partial \tau} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1) \tag{4.1}$$

with initial condition

$$u(x,0) = x(1-x), \tag{4.2}$$

and boundary conditions

$$u(0,\tau) = 0, \quad u(1,\tau) = 0, \;\; \tau > 0. \tag{4.3}$$

The exact solution is

$$u(x,\tau) = \sum_{n=1}^{\infty} (\frac{2}{n\pi})^3 sin(n\pi x) e^{(-n^2\pi^2\alpha\tau)}. \tag{4.4}$$

Comparison of maximum absolute errors obtained by finite-difference Haar wavelet method with different resolutions level have been given in the Table 1.1. Also, surface and mesh plot of exact and finite-difference Haar wavelet solution is given in Figures 1, 2, 3 and 4.

Table 1.1 Maximum absolute errors obtained by finite-difference Haar wavelet for different values of space resolution level J and time step level N for $\alpha = 0.2$.

J/N	10	20	50	100
3	1.1348e-04	7.1893e-05	3.4722e-05	1.8636e-05
4	5.7217e-05	3.6048e-05	1.7325e-05	9.2817e-06
5	2.8615e-05	1.8037e-05	8.6843e-06	4.6559e-06
6	1.4316e-05	9.0199e-06	4.3439e-06	2.3296e-06
7	7.1583e-06	4.5102e-06	2.1719e-06	1.1650e-06
8	3.5792e-06	2.2551e-06	1.0860e-06	5.8250e-07
9	1.7896e-06	1.1276e-06	5.4299e-07	2.9125e-07
10	8.9480e-07	5.6378e-07	2.7150e-07	1.4563e-07

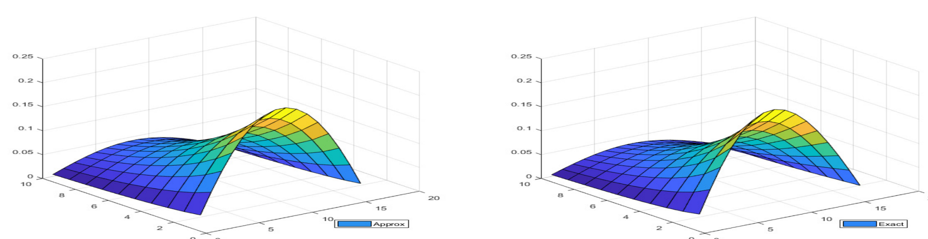

Figure 1. Surface plot of exact and finite-difference Haar wavelet solution of problem 1 with $J = 3$ and time step $N = 10$.

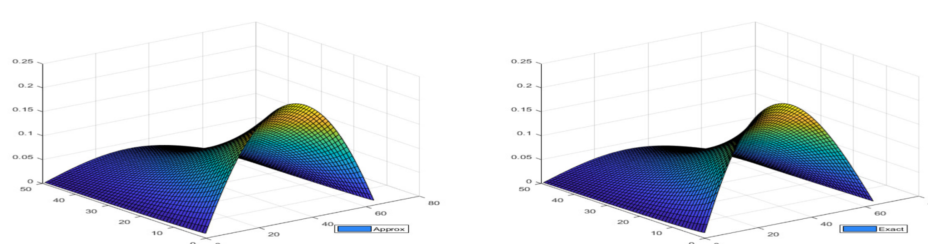

Figure 2. Surface plot of finite-difference Haar wavelet solution of problem 1 with $J = 5$ and time step $N = 50$.

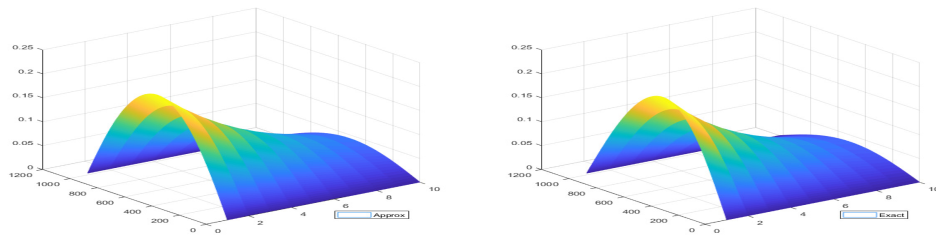

Figure 3. Mesh plot of finite-difference Haar wavelet solution of problem 1 with $J = 9$ and time step $N = 10$.

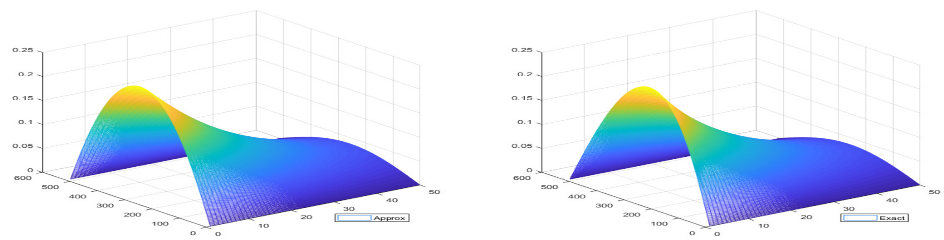

Figure 4. Mesh plot of finite-difference Haar wavelet solution of problem 1 with $J = 8$ and time step $N = 50$.

Problem 2. Let us consider the following parabolic PDE

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1) \tag{4.5}$$

with initial condition

$$u(x,0) = \begin{cases} 2x, & 0 < x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leq x < 1 \end{cases} \tag{4.6}$$

and boundary conditions

$$u(0,\tau) = 0, \quad u(1,\tau) = 0, \ \ \tau > 0. \tag{4.7}$$

The exact solution is

$$u(x,\tau) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} sin(\frac{n\pi}{2}) sin(n\pi x) e^{(-n^2\pi^2\tau)}. \tag{4.8}$$

Comparison of maximum absolute errors obtained by finite-difference Haar wavelet method with different resolutions level have been given in the Table 2.1. Also, maximum absolute error obtained by finite-difference method [1] is 3.900E(-03) when time level is divided into 100 points. Further, the surface and mesh plot of exact and finite-difference Haar wavelet solution is given in Figures 5, 6, and 7.

Table 2.1 Maximum absolute errors obtained by finite-difference Haar wavelet for different values of space resolution level J and time step level N.

J/N	10	20	50	100
3	1.7000e-03	1.1000e-03	7.2486e-04	5.0419e-04
4	8.7431e-04	5.4638e-04	3.5922e-04	2.5301e-04
5	4.3682e-04	2.7358e-04	1.8022e-04	1.2619e-04
6	2.1847e-04	1.3678e-04	9.0166e-05	6.3142e-05
7	1.0924e-04	6.8391e-05	4.5086e-05	3.1574e-05
8	5.4619e-05	3.4195e-05	2.2542e-05	1.5787e-05
9	2.7310e-05	1.7098e-05	1.1271e-05	7.8935e-06
10	1.3655e-05	8.5488e-06	5.6356e-06	3.9467e-06

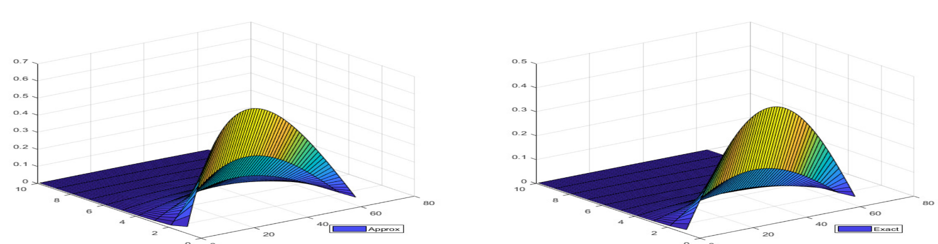

Figure 5. Surface plot of exact and finite-difference Haar wavelet solution of problem 2 with $J = 3$ and time step $N = 10$.

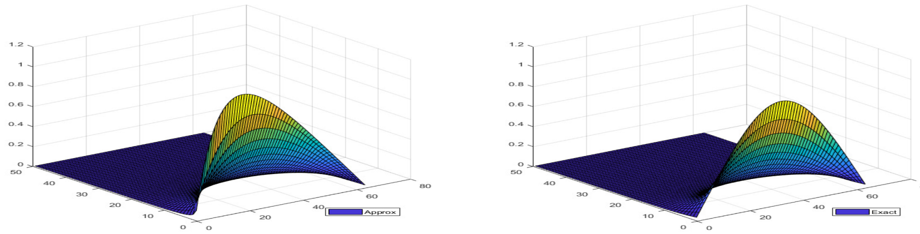

Figure 6. Surface plot of finite-difference Haar wavelet solution of problem 2 with $J = 5$ and time step $N = 50$.

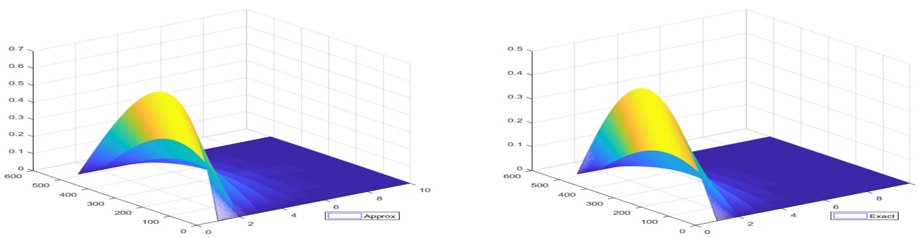

Figure 7. Mesh plot of finite-difference Haar wavelet solution of problem 2 with $J = 6$ and time step $N = 10$.

Problem 3. Let us consider the following homogeneous linear dispersive equation

$$\frac{\partial u}{\partial \tau} + \mu \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in (0,1), \quad \tau > 0, \quad \mu > 0, \tag{4.9}$$

with initial conditions,

$$u(x,0) = cosx, \quad x \in (0,1) \tag{4.10}$$

and

$$u(0,\tau) = 0, \ \frac{\partial u}{\partial x}(0,\tau) = -sin(\mu\tau), \ \frac{\partial^2 u}{\partial x^2}(0,\tau) = -cos(\mu\tau) \quad \tau \geq 0. \tag{4.11}$$

The exact solution is

$$u(x,\tau) = cos(x + \mu\tau). \tag{4.12}$$

Comparison of maximum absolute errors obtained by finite-difference Haar wavelet method with different resolutions level have been given in the Table 3.1 for $\mu = 1$. Also, maximum absolute error obtained by global extrapolation method [13] and non polynomial spline method [14] is given in Table 3.2. Further, the surface and mesh plot of exact and finite-difference Haar wavelet solution is given in Figures 8-13.

Table 3.1 Maximum absolute errors obtained by finite-difference Haar wavelet for different values of space resolution level J and time step level N for problem 3.

J/N	10	20	50	100	500	1000
3	4.1837e-05	1.0531e-05	1.5997e-06	3.5667e-07	3.3037e-08	4.9962e-08
4	2.2247e-05	5.6879e-06	9.0933e-07	2.2279e-07	7.2884e-09	5.0845e-09
5	1.1421e-05	2.9308e-06	4.7414e-07	1.1850e-07	4.5316e-09	1.8053e-09
6	5.7804e-06	1.4846e-06	2.4089e-07	6.0493e-08	2.4058e-09	1.3507e-09
7	2.9071e-06	7.4682e-07	1.2127e-07	3.0489e-08	1.2240e-09	7.3880e-10
8	1.4577e-06	3.7449e-07	6.0823e-08	1.5297e-08	6.1554e-10	3.7900e-10
9	7.2986e-07	1.8751e-07	3.0456e-08	7.6602e-09	3.0843e-10	1.9111e-10
10	3.6519e-07	9.3822e-08	1.5239e-08	3.8330e-09	1.5435e-10	9.5858e-11

Table 3.2 Maximum absolute errors obtained by global extrapolation method [13] and non polynomial spline method [14] for different values of space resolution level J and time step level N for problem 3.

J/N	50 [14]	100 [14]	100 [13]
10	—	—	1.7000e-03
20	1.0462e-06	8.4689e-07	—
40	5.8011e-07	4.8030e-07	—

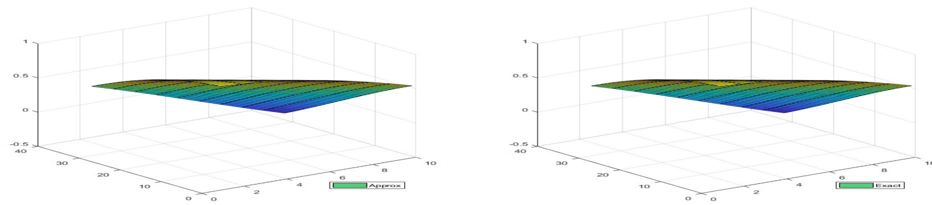

Figure 8. Surface plot of exact and finite-difference Haar wavelet solution of problem 3 with $J = 3$ and time step $N = 10$ for $\mu = 1$.

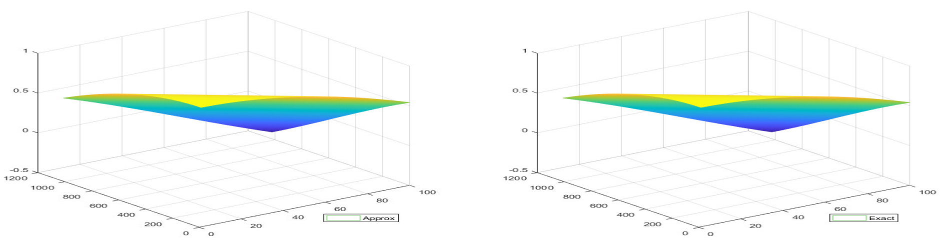

Figure 9. Mesh plot of finite-difference Haar wavelet solution of problem 3 with $J = 9$ and time step $N = 100$ for $\mu = 1$.

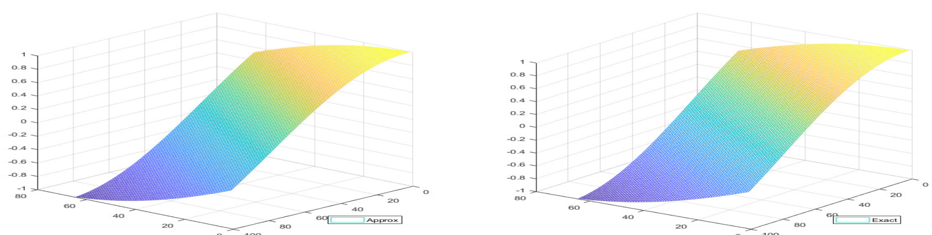

Figure 10. Mesh plot of exact and finite-difference Haar wavelet solution of problem 3 with $J = 5$ and time step $N = 100$ for $\mu = 2$.

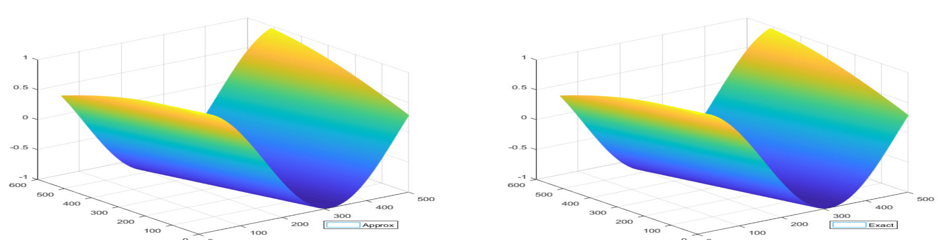

Figure 11. Mesh plot of finite-difference Haar wavelet solution of problem 3 with $J = 8$ and time step $N = 500$ for $\mu = 5$.

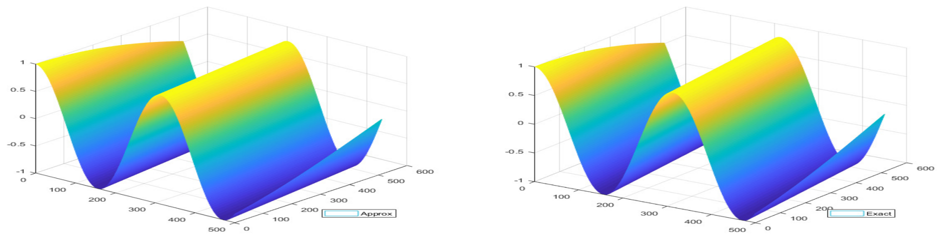

Figure 12. Mesh plot of finite-difference Haar wavelet solution of problem 3 with $J = 8$ and time step $N = 500$ for $\mu = 10$.

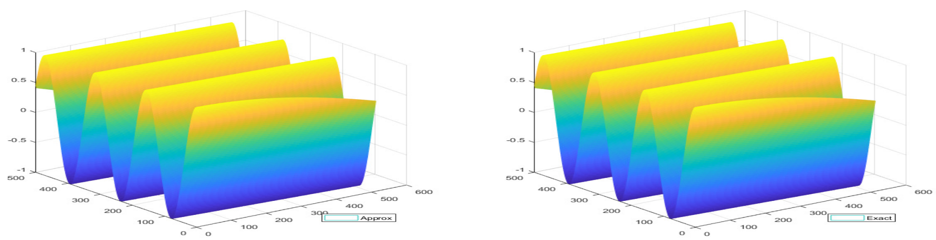

Figure 13. Mesh plot of finite-difference Haar wavelet solution of problem 3 with $J = 8$ and time step $N = 500$ for $\mu = 20$.

Problem 4. Consider the following non-homogeneous linear dispersive equation

$$\frac{\partial u}{\partial \tau} + \mu \frac{\partial^3 u}{\partial x^3} = -\pi^3 cos(\pi x) cos(\tau) - sin(\pi x) sin(\tau), \quad x \in (0,1), \quad \tau > 0, \quad \mu > 0, \tag{4.13}$$

with initial conditions,

$$u(x,0) = sin(\pi x), \quad x \in (0,1) \tag{4.14}$$

and

$$u(0,\tau) = 0, \ \frac{\partial u}{\partial x}(0,\tau) = \pi cos(\tau), \ \frac{\partial^2 u}{\partial x^2}(0,\tau) = 0, \quad \tau \geq 0. \tag{4.15}$$

The exact solution is

$$u(x,\tau) = sin(\pi x) cos(\tau). \tag{4.16}$$

Comparison of maximum absolute errors obtained by finite-difference Haar wavelet method with different resolutions level have been given in the Table 4.1 for $\mu = 1$.

Also, maximum absolute error obtained by non polynomial spline method [14] is given in Table 4.2. Further, the surface and mesh plot of exact and finite-difference Haar wavelet solution is given in Figures 14-18.

Table 4.1 Maximum absolute errors obtained by finite-difference Haar wavelet for different values of space resolution level J and time step level N for problem 4.

J/N	10	20	50	100	500	1000
3	1.5000e-03	3.6554e-04	5.7185e-05	3.0166e-06	1.1871e-06	1.3268e-06
4	7.5602e-04	1.9213e-04	3.0828e-05	4.1175e-07	1.7255e-07	1.8920e-07
5	3.8488e-04	9.8012e-05	1.5822e-05	2.1960e-07	6.0110e-08	2.5202e-08
6	1.9405e-04	4.9441e-05	7.9934e-06	1.2837e-07	3.3283e-08	1.2224e-08
7	9.7412e-05	2.4822e-05	4.0147e-06	6.6637e-08	1.7081e-08	6.3025e-09
8	4.8801e-05	1.2436e-05	2.0115e-06	3.3655e-08	8.6035e-09	3.1782e-09
9	2.4424e-05	6.2240e-06	1.0068e-06	1.6877e-08	4.3117e-09	1.5932e-09

Table 4.2 Maximum absolute errors obtained by non polynomial spline method [14] for different values of space resolution level J and time step level N for problem 4.

J/N	50 [14]	100 [14]
20	6.4058e-06	6.4058e-06
40	5.2848e-07	5.2848e-06

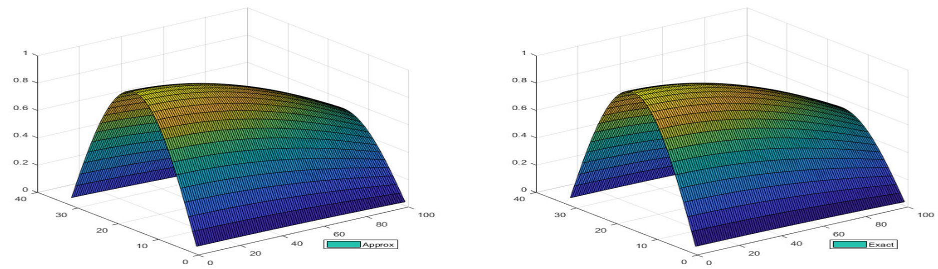

Figure 14. Surface plot of exact and finite-difference Haar wavelet solution of problem 4 with $J = 4$ and time step $N = 100$ for $\mu = 1$.

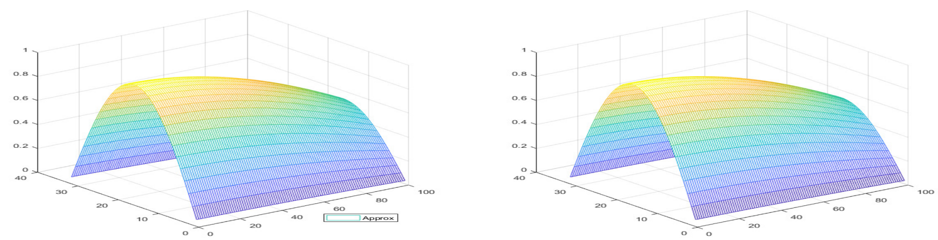

Figure 15. Mesh plot of exact and finite-difference Haar wavelet solution of problem 4 with $J = 4$ and time step $N = 100$ for $\mu = 1$.

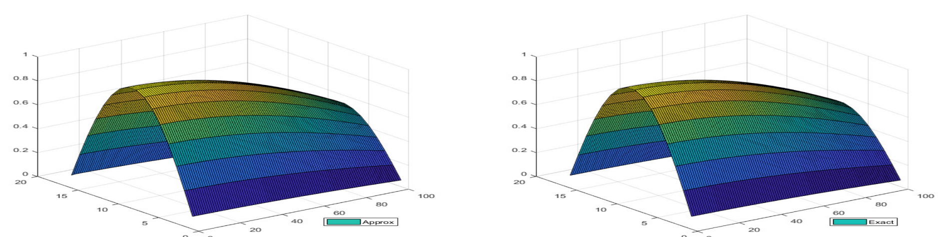

Figure 16. Surface plot of exact and finite-difference Haar wavelet solution of problem 4 with $J = 3$ and time step $N = 100$ for $\mu = 1$.

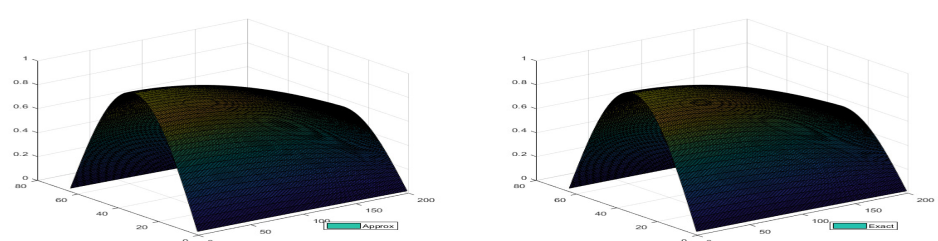

Figure 17. Surface plot of finite-difference Haar wavelet solution of problem 4 with $J = 5$ and time step $N = 200$ for $\mu = 1$.

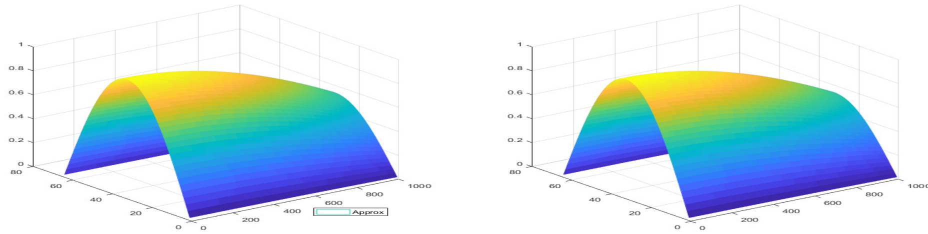

Figure 18. Mesh plot of exact and finite-difference Haar wavelet solution of problem 4 with $J = 5$ and time step $N = 1000$ for $\mu = 1$.

Conclusion

We have solved diffusion and dispersive equations using finite-difference Haar wavelet method and obtained the approximate solution. Stability and convergence of the Haar wavelet have been shown. We compared our results with the existing methods such as finite-difference method [1], global extrapolation method [13] and non polynomial spline method given in [14]. The tables $1.1 - 4.2$ clearly indicate that Haar wavelet produces better results. Further, the graphs of solved examples have been given in the figures $1 - 18$. The technique introduced here is easy to apply as well as the performance of the present method yields more accurate results. For nonlinear problem we recommend consulting the papers [22], [23], [24] and [31].

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