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SOME 3-DIVISOR CORDIAL GRAPHS DERIVED FROM PATH

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ABSTRACT. Let G be a (p, q) graph and $2 \leq k \leq p$. Let $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a map. For each edge xy, assign the label 1 if either $f(x)$ or $f(y)$ divides the other and 0 otherwise. f is called a k-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ $i, j \in \{1, 2, ..., k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, where $x \in \{1, 2, \ldots, k\}$, $e_f(i)$ denote the number of edges labeled with i, $i \in \{0, 1\}$. A graph with a k-divisor cordial labeling is called a k-divisor cordial graph. In this paper, we obtain 3-divisor cordial graphs derived from path.

1. Introduction

Throughout this paper we have considered only simple and undirected graph. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The number of vertices and edges of a graph G are called order and size of G respectively. Graph labeling is one of the most studied subjects in graph theory. It is an assignment of integers to the elements of a graph, subject to certain constraints. In 1967, Rosa [6] initiated the study of graceful labeling. Labeled graphs serve as useful models for a broad range of applications such as X-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing [3]. In 1980, Cahit [2] introduced the cordial labeling of graphs. Hovey [5] has introduced a generalized cordial labeling, called A-cordial labeling where A is abelian. In [10],

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Varatharajan, Navanaeethakrishnan, and Nagarajan introduced a notion, called divisor cordial labeling and proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial.Bosmia and Kanani [1] proved that the graphs of the form $G \odot K_1$ where G any of the following admits a divisor cordial labeling: $K_{1,n}$, $K_{2,n}$, $K_{3,n}$, a wheel, a helm, a flower, a fan, a double fan, and a barycentric subdivision of a star.Sathish Narayanan introduced the notion of 3-divisor cordial [7]. In [8] ,[9], 3-divisor cordiality of wheel, ladder, prism, book graphs and $\overline{K_n} + 2K_2$ have been studied. In this paper, we have proved that union of a connected 3-divisor cordial graph and path P_n where $n \neq 2, 4$ is a 3-divisor cordial graph. Terms and definitions not defined here are used in the sense of Harary[4].

2. Preliminary Results

Definition 2.1. Let G be a (p, q) graph and $2 \leq k \leq p$. Let $f : V(G) \to \{1, 2, \ldots , k\}$ be a map. For each edge xy, assign the label 1 if either $f(x)$ or $f(y)$ divides the other and 0 otherwise. f is called a k-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ $i, j \in \{1, 2, ..., k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, where $x \in \{1, 2, \ldots , k\}$, $e_f(i)$ denote the number of edges labeled with i, $i \in \{0, 1\}$. A graph with a k-divisor cordial labeling is called a k-divisor cordial graph.

Definition 2.2. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Theorem 2.1. [7] The path P_n is 3-divisor cordial for all values of n.

Let $P_n : u_1u_2 \ldots u_n$ be the path. Recall the labeling of the vertices of P_n given in Theorem 2.1 [S–SN]. For $n = 1$ the vertex u_1 has the label 1. The vertices of P_2 are labeled by 1, 2. The numbers 1, 2, 3 respectively are given to the vertices u_1, u_2, u_3 of P_3. The integers 1, 2, 3, 2 respectively are assigned to the vertices u_1, u_2, u_3, u_4

of P_4. For P_5, the vertices u_1, u_2, u_3, u_4, u_5 respectively received the labels 1, 2, 3, 3, 2. If $n = 6s$ and $s > 0$, then assign the labels 1, 2, 3, 2, 3, 1 to the vertices u_1, u_2, u_3, u_4, u_5, u_6 respectively and assign the same sequence of labels to the next six consecutive vertices. Repeat the process and finally we have assigned all the vertices with the above sequence of labels. Import the same idea upto the vertex u_{6s} of P_n when $n = 6s+1$. Now, the non labeled end vertex u_{6s+1} is labeled by 1. This labeling is taken upto the vertex u_{6s+1} of P_n when $n = 6s + 2$. Then put the number 2 to u_{6s+2}. The labeling pattern for P_n when $n = 6s + 3$ is similar to that of the case $n = 6s + 2$ upto the vertex u_{6s+2}. Then the vertex u_{6s+3} is labeled by the integer 3. For $n = 6s + 4$, assign the labels to the vertices upto u_{6s+3} as in the case $n = 6s + 3$. As in case $n = 6n+3$, assign the labels to the vertices of P_{6s+5} upto the vertex u_{6s+3}. The last two non labeled vertices are now labeled by 3, 2 respectively. If g denotes the above said labeling, one can easily verify the vertex and edge conditions given in Table 1.

Table 1

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$e_f(0)$	$e_f(1)$
$6s$	$2s$	$2s$	$2s$	$3s$	$3s-1$
$6s+1$	$2s+1$	$2s$	$2s$	$3s$	$3s$
$6s+2$	$2s+1$	$2s+1$	$2s$	$3s$	$3s+1$
$6s+3$	$2s+1$	$2s+1$	$2s+1$	$3s+1$	$3s+1$
$6s+4$	$2s+1$	$2s+1$	$2s+2$	$3s+1$	$3s+2$
$6s+5$	$2s+1$	$2s+2$	$2s+2$	$3s+2$	$3s+2$

3. Main Results

With the help of Theorem 2.1, we now prove the following theorem.

Theorem 3.1. Let G be a (p,q) connected 3-divisor cordial graph and $n \neq 2,4$, then $G \cup P_n$ is a 3-divisor cordial graph.

Let f be a 3-divisor cordial labeling of G and g be a 3-divisor cordial labeling of P_n given in Theorem 2.1. Use the labeling f for G and g for P_n and relabel the vertices of P_n whenever required in several cases discussed below results a new labeling called h. In each case we produce the evidence that h satisfy the requirements of a 3-divisor cordial labeling.

Case 1. $p \equiv 0 \pmod 3$, $q \equiv 0 \pmod 2$.

Let $p = 3t$ where $t > 0$ and $q = 2r$ where $r > 0$. Then f should satisfy the vertex condition $v_f(1) = v_f(2) = v_f(3) = t$ and the edge condition $e_f(0) = e_f(1) = r$.

Subcase 1a. $n = 6s$ where $s > 0$.

In this case $v_h(1) = v_h(2) = v_h(3) = t + 2s$, $e_h(0) = r + 3s$ and $e_h(1) = r + 3s - 1$.

Subcase 1b. $n = 6s + 1$ where $s \geq 0$.

Here $v_h(1) = t + 2s + 1$, $v_h(2) = v_h(3) = t + 2s$ and $e_h(0) = e_h(1) = r + 3s$.

Subcase 1c. $n = 6s + 2$ where $s \geq 1$.

In this case we have $v_h(1) = v_h(2) = t + 2s + 1$, $v_h(3) = t + 2s$, $e_h(0) = r + 3s$ and $e_h(1) = r + 3s + 1$.

Subcase 1d. $n = 6s + 3$ where $s \geq 0$.

In this case $v_h(1) = v_h(2) = v_h(3) = t + 2s + 1$ and $e_h(0) = e_h(1) = r + 3s + 1$.

Subcase 1e. $n = 6s + 4$ where $s \geq 1$.

In this case $v_h(1) = v_h(2) = t + 2s + 1$, $v_h(3) = t + 2s + 2$, $e_h(0) = r + 3s + 1$ and $e_h(1) = r + 3s + 2$.

Subcase 1f. $n = 6s + 5$ where $s \geq 0$.

Here $v_h(1) = t + 2s + 1$, $v_h(2) = v_h(3) = t + 2s + 2$ and $e_h(0) = e_h(1) = r + 3s + 2$.

Case 2. $p \equiv 0 \pmod 3$, $q \equiv 1 \pmod 2$. Let $p = 3t$ where $t \geq 0$ and $q = 2r + 1$

where $r \geq 0$. Then f should satisfy the vertex condition given in case 1 and any one of the edge condition $e_f(0) = r$, $e_f(1) = r + 1$ (or) $e_f(0) = r + 1$, $e_f(1) = r$.

Subcase 2a. $n = 6s$ where $s > 0$.

If $e_f(0) = r$, $e_f(1) = r + 1$ then $e_h(0) = e_h(1) = 3s + r$ and if $e_f(0) = r + 1$, $e_f(1) = r$ then interchange the labels of the vertices u_{n-1}, u_n of the path P_n and we get $e_h(0) = e_h(1) = 3s + r$. In either case, we observe that $v_h(1) = v_h(2) = v_h(3) = t + 2s$.

Subcase 2b. $n = 6s + 1$ where $s \geq 0$.

In this case, we have $v_h(1) = t + 2s + 1$, $v_h(2) = v_h(3) = t + 2s$ and $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$ (or) $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r$.

Subcase 2c. $n = 6s + 2$ where $s \geq 1$.

If $e_f(0) = r+1$, $e_f(1) = r$ then $e_h(0) = e_h(1) = 3s+r+1$ and $v_h(1) = v_h(2) = t+2s+1$, $v_h(3) = t + 2s$. If $e_f(0) = r$, $e_f(1) = r + 1$ then relabel the vertex u_1 by 3. Here we note that $v_h(1) = t + 2s$, $v_h(2) = v_h(3) = t + 2s + 1$ and $e_h(0) = e_h(1) = 3s + r + 1$.

Subcase 2d. $n = 6s + 3$ where $s \geq 0$.

Here we observe that $v_h(1) = v_h(2) = v_h(3) = t + 2s + 1$ and $e_h(0) = 3s + r + 2$, $e_h(1) = 3s + r + 1$ (or) $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r + 2$.

Subcase 2e. $n = 6s + 4$ where $s \geq 1$.

If $e_f(0) = r$, $e_f(1) = r + 1$ then $e_h(0) = e_h(1) = 3s + r + 2$ and if $e_f(0) = r + 1$, $e_f(1) = r$ then interchange the labels of the vertices u_{n-1}, u_n of the path P_n and we get $e_h(0) = e_h(1) = 3s + r + 2$. In either case, we observe that $v_h(1) = v_h(3) = t + 2s + 1$, $v_h(2) = t + 2s + 2$.

Subcase 2f. $n = 6s + 5$ where $s \geq 0$.

If $e_f(0) = r$, $e_f(1) = r + 1$ then $e_h(0) = 3s + r + 2$, $e_h(1) = 3s + r + 3$ and if $e_f(0) = r + 1$, $e_f(1) = r$ then $e_h(0) = r + 3s + 3$, $e_h(1) = 3s + r + 2$. In either case, we note that $v_h(1) = t + 2s + 1$, $v_h(2) = v_h(3) = t + 2s + 2$. **Case 3.** $p \equiv 1 \pmod 3$, $q \equiv 0 \pmod 2$.

Let $p = 3t + 1$ where $t \geq 0$ and $q = 2r$ where $r > 0$. Then f should satisfy any one of the following vertex conditions

(1) $v_f(1) = t + 1$, $v_f(2) = v_f(3) = t$.
(2) $v_f(2) = t + 1$, $v_f(1) = v_f(3) = t$.
(3) $v_f(3) = t + 1$, $v_f(1) = v_f(2) = t$.

and the edge condition $e_f(0) = e_f(1) = r$.

Subcase 3a. $n = 6s$ where $s > 0$.

In this case h satisfies the edge condition $e_h(0) = r + 3s$, $e_h(1) = r + 3s - 1$ and it satisfies any one of the following vertex conditions.

(1) $v_h(1) = 2s + t + 1$, $v_h(2) = v_h(3) = t + 2s$.
(2) $v_h(2) = 2s + t + 1$, $v_h(1) = v_h(3) = t + 2s$.
(3) $v_h(3) = 2s + t + 1$, $v_h(1) = v_h(2) = t + 2s$.

Subcase 3b. $n = 6s + 1$ where $s \geq 0$.

If f satisfies vertex condition (1), then relabel the vertices u_1, u_2 by 2, 3 respectively. Then we have $v_h(1) = v_h(3) = 2s+t+1$, $v_h(2) = t+2s$. For $s = 0$, relabel the vertex u_1 by 2. Then $v_h(1) = v_h(2) = t+1$, $v_h(3) = t$. If f satisfies vertex condition (2), then $v_h(1) = v_h(2) = 2s + t + 1$, $v_h(3) = t + 2s$. Suppose f satisfies vertex condition (3), then $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = t + 2s$. In each case, $e_h(0) = e_h(1) = 3s + r$.

Subcase 3c. $n = 6s + 2$ where $s \geq 1$.

If f satisfies vertex condition (1), then relabel the vertex u_1 by 3. Then $v_h(1) = v_h(2) = v_h(3) = 2s+t+1$ and $e_h(0) = 3s+r+1$, $e_h(1) = 3s+r$. Suppose f satisfies vertex condition (2), then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$. If f satisfies vertex condition (3), then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = 3s + r$, $e_h(1) = r + 3s + 1$.

Subcase 3d. $n = 6s + 3$ where $s \geq 0$.

In this case h satisfies the edge condition $e_h(0) = e_h(1) = r + 3s + 1$ and it satisfies any one of the following vertex conditions.

(1) $v_h(1) = 2s + t + 2$, $v_h(2) = v_h(3) = t + 2s + 1$.

(2) $v_h(2) = 2s + t + 2$, $v_h(1) = v_h(3) = t + 2s + 1$.

(3) $v_h(3) = 2s + t + 2$, $v_h(1) = v_h(2) = t + 2s + 1$.

Subcase 3e. $n = 6s + 4$ where $s \geq 1$.

If f satisfies vertex condition (1), then $v_h(1) = v_h(2) = 2s + t + 2$, $v_h(3) = 2s + t + 1$ and $e_h(0) = 3s+r+2$, $e_h(1) = 3s+r+1$. Suppose f satisfies vertex condition (2), then relabel the vertex u_2 by 3, and then we have $v_h(1) = 2s+t+1$, $v_h(2) = v_h(3) = 2s+t+2$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r + 2$. If f satisfies vertex condition (3), then $v_h(1) = 2s+t+1$, $v_h(2) = v_h(3) = 2s+t+2$ and $e_h(0) = 3s+r+2$, $e_h(1) = r+3s+1$.

Subcase 3f. $n = 6s + 5$ where $s \geq 0$.

If f satisfies vertex condition (1), then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$. Suppose f satisfies vertex condition (2), then relabel the vertices u_{n-2}, u_{n-1}, u_n by 3, 2, 1 respectively. Then we have $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$. If f satisfies vertex condition (3), then relabel the vertices u_{n-1}, u_n by 2, 1 respectively. Here $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$. In each case $e_h(0) = e_h(1) = r + 3s + 2$.

Case 4. $p \equiv 1 \pmod 3$, $q \equiv 1 \pmod 2$.

Let $p = 3t + 1$ where $t \geq 0$ and $q = 2r + 1$ where $r \geq 0$. Then f should satisfy any one of the vertex conditions given in case 3 and any one of the edge conditions $e_f(0) = r$, $e_f(1) = r + 1$ (or) $e_f(0) = r + 1$, $e_f(1) = r$.

Subcase 4a. $n = 6s$ where $s > 0$.

If $e_f(0) = r$, $e_f(1) = r + 1$ then $e_h(0) = e_h(1) = r + 3s$ and if $e_f(0) = r + 1$, $e_f(1) = r$ then interchange the labels of the vertices u_{n-1}, u_n of P_n. This gives $e_h(0) = e_h(1) = r + 3s$. In this case h satisfies any one of the vertex conditions given in subcase 3a.

Subcase 4b. $n = 6s + 1$ where $s \geq 0$.

If f satisfies vertex condition (1) of case 3, then relabel the vertices u_1, u_2 by 2, 3 respectively. This gives vertex condition $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t$ and h satisfies any one of the edge conditions $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$ (or) $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r$. If $s = 0$, $v_h(1) = v_h(2) = t + 1$, $v_h(3) = t$ and h satisfies any one of the edge conditions $e_h(0) = r$, $e_h(1) = r + 1$ (or) $e_h(0) = r + 1$, $e_h(1) = r$. Suppose f satisfies vertex condition (2), then we have $v_h(1) = v_h(2) = 2s+t+1$, $v_h(3) = 2s+t$ and $e_h(0) = 3s+r$, $e_h(1) = 3s+r+1$ (or) $e_h(0) = 3s+r+1$, $e_h(1) = 3s + r$. If f satisfies vertex condition (3), then $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t$ and $e_h(0) = 3s + r$, $e_h(1) = r + 3s + 1$ (or) $e_h(0) = 3s + r + 1$, $e_h(1) = r + 3s$.

Subcase 4c. $n = 6s + 2$ where $s \geq 1$.

If f satisfies vertex condition (1) of case 3 and if $e_f(0) = r$, $e_f(1) = r+1$, then relabel the vertex u_1 by 3. If f satisfies vertex condition (1) of case 3 and if $e_f(0) = r + 1$, $e_f(1) = r$, then relabel the vertex u_{n-2} by 3. Suppose f satisfies vertex condition (2) and if $e_f(0) = r$, $e_f(1) = r + 1$, then relabel the vertices u_1, u_n by 3, 1 respectively. Suppose f satisfies vertex condition (2) and if $e_f(0) = r + 1$, $e_f(1) = r$, then relabel the vertex u_n by 3. If f satisfies vertex condition (3) of case 3 and if $e_f(0) = r$, $e_f(1) = r+1$, then interchange the labels of the vertices u_{n-2}, u_n. If f satisfies vertex condition (3) of case 3 and if $e_f(0) = r+1$, $e_f(1) = r$, assign the labels to the vertices of G and P_n as in f and g. In each of the above cases, we have the vertex condition $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and edge condition $e_h(0) = e_h(1) = 3s + r + 1$.

Subcase 4d. $n = 6s + 3$ where $s \geq 0$.

In this case h satisfies any one of the following edge conditions $e_h(0) = r + 3s + 1$, $e_h(1) = r + 3s + 2$ (or) $e_h(0) = r + 3s + 2$, $e_h(1) = r + 3s + 1$ and any one of the vertex conditions given in subcase 3d.

Subcase 4e. $n = 6s + 4$ where $s \geq 1$.

If f satisfies vertex condition (1) of case 3 and if $e_f(0) = r$, $e_f(1) = r + 1$, then $v_h(1) = v_h(2) = t + 2s + 2$, $v_h(3) = 2s + t + 1$. If f satisfies vertex condition (1) of case 3 and if $e_f(0) = r + 1$, $e_f(1) = r$, then interchange the labels of the vertices u_{n-1}, u_n. Here $v_h(1) = v_h(2) = t + 2s + 2$, $v_h(3) = 2s + t + 1$. Suppose f satisfies vertex condition (2) and if $e_f(0) = r$, $e_f(1) = r + 1$, then relabel the vertices u_1, u_n by 3, 1 respectively. In this case $v_h(1) = t + 2s + 1$, $v_h(2) = v_h(3) = 2s + t + 2$. Suppose f satisfies vertex condition (2) and if $e_f(0) = r + 1$, $e_f(1) = r$, then relabel the vertex u_n by 3. Here $v_h(1) = t + 2s + 1$, $v_h(2) = v_h(3) = 2s + t + 2$. If f satisfies vertex condition (3) of case 3 and if $e_f(0) = r$, $e_f(1) = r+1$, then $v_h(1) = t+2s+1$, $v_h(2) = v_h(3) = 2s + t + 2$. If f satisfies vertex condition (3) of case 3 and if $e_f(0) = r + 1$, $e_f(1) = r$, then relabel the vertex u_n by 1. Here $v_h(2) = t + 2s + 1$, $v_h(1) = v_h(3) = 2s + t + 2$. In each of the above cases, we have the edge condition $e_h(0) = e_h(1) = 3s + r + 2$.

Subcase 4f. $n = 6s + 5$ where $s \geq 0$.

If f satisfies vertex condition (1) of case 3, then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and h satisfies any one of the edge conditions $e_h(0) = 3s + r + 2$, $e_h(1) = 3s + r + 3$ (or) $e_h(0) = 3s + r + 3$, $e_h(1) = 3s + r + 2$. Suppose f satisfies vertex condition (2) of case 3, then relabel the vertices u_{n-3}, u_{n-2}, u_n by 3, 2, 1 respectively. Here we have $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and $e_h(0) = 3s + r + 2$, $e_h(1) = 3s + r + 3$ (or) $e_h(0) = 3s + r + 3$, $e_h(1) = 3s + r + 2$. If f satisfies vertex condition (3), then then relabel the vertices u_{n-1}, u_n by 2, 1 respectively. Then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and $e_h(0) = 3s + r + 2$, $e_h(1) = 3s + r + 3$ (or) $e_h(0) = 3s + r + 3$, $e_h(1) = 3s + r + 2$.

Case 5. $p \equiv 2 \pmod 3$, $q \equiv 0 \pmod 2$.

Let $p = 3t + 2$ where $t \geq 0$ and $q = 2r$ where $r > 0$. Then f should satisfy any one of the following vertex conditions

(1) $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$.

(2) $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$.

(3) $v_f(2) = v_f(3) = t + 1$, $v_f(1) = t$.

and the edge condition $e_f(0) = e_f(1) = r$.

Subcase 5a. $n = 6s$ where $s > 0$.

In this case h satisfies the edge condition $e_h(0) = r + 3s$, $e_h(1) = r + 3s - 1$ and it satisfies any one of the following vertex conditions.

(1) $v_h(1) = v_h(2) = 2s + t + 1$, $v_h(3) = t + 2s$.

(2) $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = t + 2s$.

(3) $v_h(2) = v_h(3) = 2s + t + 1$, $v_h(1) = t + 2s$.

Subcase 5b. $n = 6s + 1$ where $s \geq 0$.

If f satisfies vertex condition (1), then relabel the vertex u_n by 3. Then we have $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = e_h(1) = 3s + r$. If f satisfies vertex condition (2), then relabel the vertex u_n by 2. This gives $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = e_h(1) = 3s + r$. Suppose f satisfies vertex condition (3), then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = e_h(1) = 3s + r$.

Subcase 5c. $n = 6s + 2$ where $s \geq 1$.

If f satisfies vertex condition (1), then relabel the vertex u_1 by 3. Here $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t + 2$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r$. If f satisfies vertex condition (2), then $v_h(1) = 2s + t + 2$, $v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$. Suppose f satisfies vertex condition (3), then $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t + 2$ and $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$.

Subcase 5d. $n = 6s + 3$ where $s \geq 0$.

In this case h satisfies the edge condition $e_h(0) = e_h(1) = r + 3s + 1$ and it satisfies any one of the following vertex conditions.

(1) $v_h(1) = v_h(2) = 2s + t + 2$, $v_h(3) = t + 2s + 1$.

(2) $v_h(1) = v_h(3) = 2s + t + 2$, $v_h(2) = t + 2s + 1$.

(3) $v_h(2) = v_h(3) = 2s + t + 2$, $v_h(1) = t + 2s + 1$.

Subcase 5e. $n = 6s + 4$ where $s \geq 1$.

If f satisfies vertex condition (1), then relabel the vertex u_n by 3. Here $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r + 2$. If f satisfies vertex condition (2), then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and $e_h(0) = r + 3s + 2$, $e_h(1) = 3s + r + 1$. Suppose f satisfies vertex condition (3), then relabel the vertex u_n by 1. In this case $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r + 2$.

Subcase 5f. $n = 6s + 5$ where $s \geq 0$.

If f satisfies vertex condition (1), then $v_h(1) = v_h(3) = 2s + t + 2$, $v_h(2) = t + 2s + 3$ and $e_h(0) = e_h(1) = 3s + r + 2$. If f satisfies vertex condition (2), then $v_h(1) = v_h(2) = t + 2s + 2$, $v_h(3) = 2s + t + 3$ and $e_h(0) = e_h(1) = 3s + r + 2$. Suppose f satisfies vertex condition (3), then we assign the labels to the vertices of P_n, $n > 5$ as follows:

$$
\begin{aligned}
f(u_{2i-1}) &= 2, & 1 \leq i \leq \tfrac{3s+3}{2} & \quad \text{if} \quad s \equiv 1 \pmod 2 \\
& & 1 \leq i \leq \tfrac{3s+4}{2} & \quad \text{if} \quad s \equiv 0 \pmod 2 \\
f(u_{2i}) &= 3, & 1 \leq i \leq \tfrac{3s+3}{2} & \quad \text{if} \quad s \equiv 1 \pmod 2 \\
& & 1 \leq i \leq \tfrac{3s+2}{2} & \quad \text{if} \quad s \equiv 0 \pmod 2
\end{aligned}
$$

$$
\begin{aligned}
f(u_{3s+4i+1}) &= 3, & 1 \leq i \leq \tfrac{s}{2} & \quad \text{if} \quad s \equiv 0 \pmod 2 \\
& & 1 \leq i \leq \tfrac{s-1}{2} & \quad \text{if} \quad s \equiv 1 \pmod 2 \\
f(u_{3s+4i+3}) &= 2, & 1 \leq i \leq \tfrac{s-2}{2} & \quad \text{if} \quad s \equiv 0 \pmod 2 \\
& & 1 \leq i \leq \tfrac{s-1}{2} & \quad \text{if} \quad s \equiv 1 \pmod 2 \\
f(u_{3s+2i+2}) &= 1, & 1 \leq i \leq s - 1 & \\
f(u_{5s+i+1}) &= 1, & 1 \leq i \leq s + 3 &
\end{aligned}
$$

and $f(u_{6s+5}) = 3$. In this case $v_h(1) = v_h(2) = 2s + t + 2$, $v_h(3) = 2s + t + 3$ and $e_h(0) = e_h(1) = 3s + r + 2$. For $n = 5$, we use the labeling given in Figure 1.

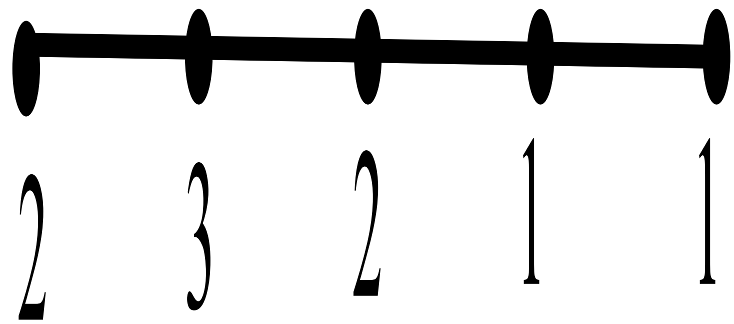

FIGURE 1

Case 6. $p \equiv 2 \pmod 3$, $q \equiv 1 \pmod 2$.

Let $p = 3t + 2$ where $t \geq 0$ and $q = 2r + 1$ where $r \geq 0$. Then f should satisfy any one of the vertex conditions given in case 5 and any one of the edge conditions $e_f(0) = r + 1$, $e_f(1) = r$ (or) $e_f(0) = r$, $e_f(1) = r + 1$.

Subcase 6a. $n = 6s$ where $s > 0$.

If f satisfies vertex condition (1) of case 5 and if $e_f(0) = r + 1$, $e_f(1) = r$, then interchange the labels of the vertices u_1, u_2. Here $v_h(1) = v_h(2) = 2s + t + 1$, $v_h(3) = 2s + t$ and if f satisfies vertex condition (1) of case 5 and if $e_f(0) = r$, $e_f(1) = r + 1$, then $v_h(1) = v_h(2) = 2s + t + 1$, $v_h(3) = 2s + t$. Suppose f satisfies vertex condition (2) of case 5 and if $e_f(0) = r + 1$, $e_f(1) = r$, then interchange the labels of the vertices u_1, u_2. Here $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t$. Suppose f satisfies vertex condition (2) of case 5 and if $e_f(0) = r$, $e_f(1) = r + 1$, then $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t$. If f satisfies vertex condition (3) of case 5 and if $e_f(0) = r + 1$, $e_f(1) = r$, then interchange the labels of the vertices u_1, u_2. In this case $v_h(2) = v_h(3) = 2s+t+1$, $v_h(1) = 2s+t$. If f satisfies vertex condition (3) of case 5 and if $e_f(0) = r$, $e_f(1) = r+1$, then $v_h(2) = v_h(3) = 2s+t+1$, $v_h(1) = 2s+t$. In each of the above cases, we have the edge condition $e_h(0) = e_h(1) = 3s + r$.

Subcase 6b. $n = 6s + 1$ where $s \geq 0$.

If f satisfies vertex condition (1) of case 5, then relabel the vertex u_n by 3. This gives $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r$ (or) $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$. If f satisfies vertex condition (2) of case 5, then relabel the vertex u_n by 2. Here also $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r$ (or) $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$. Suppose f satisfies vertex condition (3) of 5, then $v_h(1) = v_h(2) = v_h(3) = 2s + t + 1$ and $e_h(0) = 3s + r + 1$, $e_h(1) = 3s + r$ (or) $e_h(0) = 3s + r$, $e_h(1) = 3s + r + 1$.

Subcase 6c. $n = 6s + 2$ where $s \geq 1$.

If f satisfies vertex condition (1) of case 5 and if $e_f(0) = r+1$, $e_f(1) = r$, then relabel the vertex u_n by 3. Here $v_h(1) = 2s+t+2$, $v_h(2) = v_h(3) = 2s+t+1$ and if f satisfies vertex condition (1) of case 5 and if $e_f(0) = r$, $e_f(1) = r + 1$, then relabel the vertex u_1 by 3. In this case $v_h(1) = v_h(3) = 2s+t+1$, $v_h(2) = 2s+t+2$. Suppose f satisfies vertex condition (2) of case 5 and if $e_f(0) = r + 1$, $e_f(1) = r$, then $v_h(1) = 2s + t + 2$, $v_h(2) = v_h(3) = 2s + t + 1$. Suppose f satisfies vertex condition (2) of case 5 and if $e_f(0) = r$, $e_f(1) = r + 1$, then relabel u_1 by 3. Here $v_h(1) = v_h(2) = 2s + t + 1$, $v_h(3) = 2s + t + 2$. If f satisfies vertex condition (3) of case 5 and if $e_f(0) = r + 1$, $e_f(1) = r$, then $v_h(1) = v_h(3) = 2s + t + 1$, $v_h(2) = 2s + t + 2$. If f satisfies vertex condition (3) of case 5 and if $e_f(0) = r$, $e_f(1) = r + 1$, then relabel u_2 by 3. Here $v_h(1) = v_h(2) = 2s + t + 1$, $v_h(3) = 2s + t + 2$. In each of the above cases, we have the edge condition $e_h(0) = e_h(1) = 3s + r + 1$.

Subcase 6d. $n = 6s + 3$ where $s \geq 0$.

In this case h satisfies any one of the edge condition $e_h(0) = r+3s+2$, $e_h(1) = r+3s+1$ (or) $e_h(0) = r + 3s + 1$, $e_h(1) = r + 3s + 2$ and it satisfies any one of the following vertex conditions given in subcase 5d.

Subcase 6e. $n = 6s + 4$ where $s \geq 1$.

If f satisfies vertex condition (1) of case 5 and if $e_f(0) = r+1$, $e_f(1) = r$, then relabel the vertex u_n by 3 and if f satisfies vertex condition (1) of case 5 and if $e_f(0) = r$, $e_f(1) = r+1$, then relabel the vertices u_{n-2}, u_{n-1}, u_n by 3, 2, 3 respectively. Suppose f satisfies vertex condition (2) of case 5 and if $e_f(0) = r+1$, $e_f(1) = r$, then interchange the labels of the vertices u_2, u_3. Suppose f satisfies vertex condition (2) of case 5 and if $e_f(0) = r$, $e_f(1) = r + 1$, then assign the labels to the vertices of G and P_n as in f, g respectively. If f satisfies vertex condition (3) of case 5 and if $e_f(0) = r + 1$, $e_f(1) = r$, then assign the labels to the vertices of G and P_n as in f, g respectively. If f satisfies vertex condition (3) of case 5 and if $e_f(0) = r$, $e_f(1) = r+1$, then assign the labels to the vertices of P_n as given in subcase 5f. In each of the above cases, we have the vertex condition $v_h(1) = v_h(2) = v_h(3) = 2s + t + 2$ and edge condition $e_h(0) = e_h(1) = 3s + r + 2$.

Subcase 6f. $n = 6s + 5$ where $s \geq 0$.

In this case h satisfies any one of the edge condition $e_h(0) = 3s+r+2$, $e_h(1) = 3s+r+3$ (or) $e_h(0) = 3s + r + 3$, $e_h(1) = 3s + r + 2$. If f satisfies vertex condition (1) of case 5 then $v_h(1) = v_h(3) = 2s + t + 2$, $v_h(2) = t + 2s + 3$. Suppose f satisfies vertex condition (2) of case 5 then $v_h(1) = v_h(2) = 2s + t + 2$, $v_h(3) = t + 2s + 3$. If f satisfies vertex condition (3) of case 5 then assign the labels to the vertices of P_n as in subcase 5f. In this case $v_h(2) = v_h(3) = 2s + t + 2$, $v_h(1) = t + 2s + 1$.

Hence $G \cup P_n$ where $n \neq 2, 4$ is 3-divisor cordial.

Remark 1. In Theorem 3.1, subcase 4e, if $e_f(0) = r+1$, $e_f(1) = r$ and $v_f(1) = t+1$, $v_f(2) = v_f(3) = t$ then we can not create a new labeling h by relabeling the vertices of P_2. In subcase 6e, if $e_f(0) = r$, $e_f(1) = r + 1$ and $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$ then we can not create a new labeling h by relabeling the vertices of P_4.

Corollary 3.1. $P_n \cup P_n$ is 3-divisor cordial.

Figure 2 establish the existence of 3-divisor cordial labeling of $P_2 \cup P_2$ and $P_4 \cup P_4$.

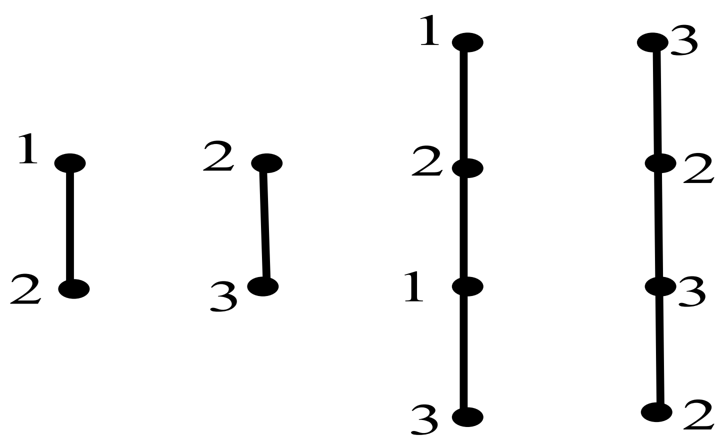

Figure 2

For $n \neq 2, 4$, the theorem follows from Theorems 2.1, 3.1.

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