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MULTILINEAR STRONGLY SINGULAR CALDERÓN-ZYGMUND OPERATORS AND COMMUTATORS ON MORREY TYPE SPACES

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Abstract. In this paper, the authors establish the boundedness of multilinear strongly singular Calderón-Zygmund operators and their multilinear commutators with BMO functions or Lipschitz functions on the product of generalized Morrey spaces and weighted Morrey spaces, respectively. Moreover, the boundedness of the multilinear iterated commutators generated by the multilinear strongly singular Calderón-Zygmund operators and BMO functions on the product of generalized Morrey spaces and weighted Morrey spaces is also obtained, respectively.

1. Introduction

The boundedness of operators and their commutators is one of the important topics in harmonic analysis. And many researchers did many results about this topic.

The classical Morrey space was originally introduced by Morrey in [14] to study the local behavior of solutions of second order elliptic partial differential equations. In [13], Mizuhara not only introduced the definition of the generalized Morrey space but also discussed the boundedness of some classical operators on generalized Morrey spaces in harmonic analysis. The authors [5] defined the weighted Morrey space and studied the boundedness of the Hardy-Littlewood maximal operator, the fractional

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integral operator, and the classical Calderón-Zygmund singular integral operator on these weighted spaces.

Alvarez and Milman [1] discussed the boundedness of the strongly singular Calderón-Zygmund operator on Lebesgue spaces. In [7], Lin proved the boundedness of the strongly singular Calderón-Zygmund operator on classical Morrey spaces and generalized Morrey spaces. Moreover, Lin and Sun studied the boundedness on weighted Morrey spaces in [11]. Lin [8] established the sharp maximal pointwise estimate for the multilinear strongly singular Calderón-Zygmund operator.

The boundedness of the commutator $[b, T]$ on Morrey spaces when b is a BMO function or a Lipschitz function was discussed by Lin in [7]. Lin and Sun [11] established the boundedness of commutators generated by strongly singular Calderón-Zygmund operators and weighted BMO functions on weighted Morrey spaces. The authors gave the sharp maximal pointwise estimates for the multilinear commutators generated by multilinear strongly singular Calderón-Zygmund operators and BMO functions or Lipschitz functions in [10], respectively. Moreover, Lin and Han [9] showed the boundedness of multilinear iterated commutators generated by multilinear strongly singular Calderón-Zygmund operators on the product of weighted Lebesgue spaces.

Based on the above results, in this paper we are interested in the boundedness of multilinear strongly singular Calderón-Zygmund operators and their multilinear commutators on generalized Morrey spaces and weighted Morrey spaces.

Now we review briefly the definition of the multilinear Calderón-Zygmund operator. A systematic treatment of multilinear Calderón-Zygmund operators was discussed in [4]. Let $m \in \mathbb{N}_+$ and $K(y_0, y_1, \dots, y_m)$ be a function defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. T is an m-linear operator defined on product of test functions such that for K, the integral representation below is valid

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \dots, y_m) \prod_{j=1}^{m} f_j(y_j) dy_1 \cdots dy_m, \tag{1.1}$$

where f_j $(j = 1, \dots, m)$ are smooth functions with compact support and $x \notin \cap_{j=1}^m \mathrm{supp} f_j$. Especially, we call K a *standard m-linear Calderón-Zygmund kernel* if it satisfies the following size and smoothness conditions.

$$|K(y_0, y_1, \dots, y_m)| \le \frac{C}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}, \tag{1.2}$$

for some $C > 0$ and all $(y_0, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ away from the diagonal. And

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y_j', \dots, y_m)| \le \frac{C|y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}, \tag{1.3}$$

for some $\varepsilon > 0$, whenever $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2} \max_{0 \le k \le m} |y_j - y_k|$.

If an m-linear operator T defined by (1.1) associated with a standard m-linear Calderón-Zygmund kernel K, and satisfies either of the following two conditions for given numbers $1 \le t_1, t_2, \dots, t_m, t < \infty$ with $1/t = 1/t_1 + 1/t_2 + \dots + 1/t_m$,

(1) T maps $L^{t_1,1} \times \dots \times L^{t_m,1}$ into $L^{t,\infty}$ if $t > 1$,

(2) T maps $L^{t_1,1} \times \dots \times L^{t_m,1}$ into L^1 if $t = 1$,

where $L^{t_1,1}, \dots, L^{t_m,1}$ and $L^{t,\infty}$ are Lorentz spaces, then T is called a *standard m-linear Calderón-Zygmund operator*.

Let T be an m-linear operator defined by (1.1), given a collection of locally integrable functions $\vec{b} = (b_1, \dots, b_m)$, then the m-linear commutator of T with $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \dots, f_m) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f}),$$

where

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \dots, f_m) - T(f_1, \dots, f_{j-1}, b_j f_j, f_{j+1}, \dots, f_m).$$

The notations $\vec{b} \in BMO^m$ will stand for $b_j \in BMO(\mathbb{R}^n)$ for $j = 1, \dots, m$, and $\vec{b} \in Lip_\beta^m$ will stand for $b_j \in Lip_\beta(\mathbb{R}^n)$ for $j = 1, \dots, m$. We denote by $\|\vec{b}\|_{BMO^m} = \max_{1 \le j \le m} \|b_j\|_{BMO(\mathbb{R}^n)}$ and $\|\vec{b}\|_{Lip_\beta^m} = \max_{1 \le j \le m} \|b_j\|_{Lip_\beta(\mathbb{R}^n)}$, respectively.

Before stating our main results, let us first recall some necessary definitions and notations.

Definition 1.1. Let $T:\mathcal{S}\to\mathcal{S}'$ be a bounded linear operator. T is called a *strongly singular Calderón-Zygmund operator* if the following conditions are satisfied.

(1) T can be extended into a continuous operator from $L^2(\mathbb{R}^n)$ into itself.

(2) There exists a function $K(x,y)$ continuous away from the diagonal $\{(x,y):x=y\}$ such that

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|\le C\frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}},$$

if $2|y-z|^\alpha\le|x-z|$ for some $0<\delta\le1$ and $0<\alpha<1$. And

$$\langle Tf,g\rangle=\int\int K(x,y)f(y)g(x)dydx,$$

for $f,g\in\mathcal{S}$ with disjoint supports.

(3) For some $n(1-\alpha)/2\le\beta<n/2$, both T and its conjugate operator T^* can be extended into continuous operators from L^q to L^2, where $1/q=1/2+\beta/n$.

Definition 1.2. Let T be an m-linear operator defined by (1.1). T is called an *m-linear strongly singular Calderón-Zygmund operator* if the following conditions are satisfied.

(1) For some $\varepsilon>0$ and $0<\alpha\le1$,

$$|K(x,y_1,\dots,y_m)-K(x',y_1,\dots,y_m)|\le\frac{C|x-x'|^\varepsilon}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon/\alpha}},$$

whenever $|x-x'|^\alpha\le\frac{1}{2}\max_{1\le j\le m}|x-y_j|$.

(2) For some given numbers $1\le r_1,\dots,r_m<\infty$ with $1/r=1/r_1+\cdots+1/r_m$, T maps $L^{r_1}\times\cdots\times L^{r_m}$ into $L^{r,\infty}$.

(3) For some given numbers $1\le l_1,\dots,l_m<\infty$ with $1/l=1/l_1+\cdots+1/l_m$, T maps $L^{l_1}\times\cdots\times L^{l_m}$ into $L^{q,\infty}$, where $0<l/q\le\alpha$.

Definition 1.3. A function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the classical Morrey space $M^q_p(\mathbb{R}^n)$, $1 \le p \le q < \infty$, if

$$\|f\|_{M^q_p(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Remark 1. *It can be seen from the special case* $M^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, *with* $1 \le p < \infty$ *that Morrey space is the generalization of the Lebesgue space.*

Definition 1.4. For a general positive function φ on $\mathbb{R}^n \times \mathbb{R}^+$, the generalized Morrey space $L^{p,\varphi}$ with $1 \le p < \infty$ is defined as follows:

$$L^{p,\varphi} = \{f \in L^p_{loc}(\mathbb{R}^n), \|f\|_{L^{p,\varphi}} < +\infty\},$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{\varphi(x,r)} \int_B |f(y)|^p dy\right)^{1/p}.$$

Remark 2. *For the case* $\varphi(x,r) = r^{n(1-p/q)}$, *we have* $L^{p,\varphi} = M^q_p(\mathbb{R}^n)$, $1 \le p \le q < \infty$. *Thus, the generalized Morrey space is the generalization of the classical Morrey space.*

Definition 1.5. Let $1 \le p < \infty$, $0 < k < 1$ and ω be a weighted function. Then the weighted Morrey space $L^{p,k}(\omega)$ is defined by

$$L^{p,k}(\omega) = \{f \in L^p_{loc}(\omega) : \|f\|_{L^{p,k}(\omega)} < \infty\},$$

where

$$\|f\|_{L^{p,k}(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)^k} \int_Q |f(x)|^p \omega(x) dx\right)^{1/p},$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Definition 1.6. Let $1 \le p < \infty$ and $0 < k < 1$. Then for two weighted functions μ and ν, the weighted Morrey space $L^{p,k}(\mu,\nu)$ is defined by

$$L^{p,k}(\mu,\nu) = \{f \in L^p_{loc}(\mu) : \|f\|_{L^{p,k}(\mu,\nu)} < \infty\},$$

where

$$\|f\|_{L^{p,k}(\mu,\nu)} = \sup_Q \left(\frac{1}{\nu(Q)^k}\int_Q |f(x)|^p \mu(x)dx\right)^{1/p}.$$

Definition 1.7. Let T be an m-linear operator , $\vec{b} = (b_1, \cdots, b_m)$ is a group of locally integrable function and $\vec{f} = (f_1, \cdots, f_m)$. Then the m-linear iterated commutator generated by T and $\vec{b}$ is defined to be

$$T_{\Pi\vec{b}}(f_1, \cdots, f_m) = [b_1, [b_2, \cdots [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2]_1(\vec{f}).$$

If T is connected in the usual way to the kernel K studied in this paper, then we can write

$$\begin{aligned} &T_{\Pi\vec{b}}(f_1, \cdots, f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^m (b_j(x) - b_j(y_j)) K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \end{aligned}$$

Definition 1.8. Take positive integers j and m satisfying $1 \le j \le m$, and C_j^m be a family of all finite subsets $\phi = \{\phi(1), \cdots, \phi(j)\}$ of $\{1, \cdots, m\}$ with j different elements. If $k < l$, then $\phi(k) < \phi(l)$. For any $\phi \in C_j^m$, let $\phi' = \{1, \cdots, m\} \backslash \phi$ be the complementary sequence. In particular, $C_0^m = \varnothing$. For an m-tuple $\vec{b}$ and $\phi \in C_j^m$, the j-tuple $\vec{b}_\phi = (b_{\phi(1)}, \cdots, b_{\phi(j)})$ is a finite subset of $\vec{b} = (b_1, \cdots, b_m)$.

Let T be an m-linear operator, $\phi \in C_j^m$, and $\vec{b}_\phi = (b_{\phi(1)}, \cdots, b_{\phi(j)})$. The iterated commutator is given by

$$T_{\Pi\vec{b}_\phi}(f_1, \cdots, f_m) = [b_{\phi(1)}, [b_{\phi(2)}, \cdots [b_{\phi(j-1)}, [b_{\phi(j)}, T]_{\phi(j)}]_{\phi(j-1)} \cdots]_{\phi(2)}]_{\phi(1)}(\vec{f}).$$

It can also be written as

$$\begin{aligned} &T_{\Pi\vec{b}_\phi}(f_1,\cdots,f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} \prod_{i=1}^{j}(b_{\phi(i)}(x)-b_{\phi(i)}(y_{\phi(i)}))K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}, \end{aligned}$$

where $d\vec{y} = dy_1\cdots dy_m$. Obviously, $T_{\Pi\vec{b}_\phi} = T_{\Pi\vec{b}}$ when $\phi = \{1,2,\cdots,m\}$, and $T_{\Pi\vec{b}_\phi} = T^j_{b_j}$ when $\phi = \{j\}$.

2. Main Results

Inspired by [7], in this paper we will give the boundedness of multilinear strongly singular Calderón-Zygmund operators and their multilinear commutators with BMO functions or Lipschitz functions on the product of generalized Morrey spaces.

Theorem 2.1. *Let T be an m-linear strongly singular Calderón-Zygmund operator. Let $s = \max\{r_1,\dots,r_m,l_1,\dots,l_m\}$, where r_j and l_j are given as in Definition 1.2, $j = 1,\dots,m$. Let φ_j be a positive function on $\mathbb{R}^n\times\mathbb{R}^+$ and suppose there exists $0 < C_j < 2^n$ such that $\varphi_j(x,2r) \le C_j\varphi_j(x,r)$ for all $x \in R^n$, $r > 0$, and $\varphi^{1/p} = \prod_{j=1}^m \varphi_j^{1/p_j}$, $1/p = 1/p_1+\cdots+1/p_m$, $p > 1$. If $s < p_j < \infty$, then T can be extended into a bounded operator from $L^{p_1,\varphi_1}\times\cdots\times L^{p_m,\varphi_m}$ into $L^{p,\varphi}$.*

Remark 3. *Theorem 2.1 is the generalization of Theorem 3.1 in* [7].

Theorem 2.2. *Let T be an m-linear strongly singular Calderón-Zygmund operator and $0 < l/q < \alpha$ in (3) of Definition 1.2. Let $s_0 = \max\{r_1,\dots,r_m,l_1,\dots,l_m\}$, where r_j and l_j are given as in Definition 1.2, $j = 1,\dots,m$. Let φ_j be a positive function on $\mathbb{R}^n\times\mathbb{R}^+$ and suppose there exists $0 < C_j < 2^n$ such that $\varphi_j(x,2r) \le C_j\varphi_j(x,r)$ for all $x \in R^n, r > 0$, and $\varphi^{1/p} = \prod_{j=1}^m \varphi_j^{1/p_j}$, $1/p = 1/p_1+\cdots+1/p_m$, $s_0 < p_j < \infty$, $p > 1$. If*

$\vec{b} \in BMO^m$, *then* $T_{\vec{b}}$ *can be extended into a bounded operator from* $L^{p_1,\varphi_1} \times \cdots \times L^{p_m,\varphi_m}$ *into* $L^{p,\varphi}$.

Theorem 2.3. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator. Let* $s = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$. *Suppose* $\vec{b} \in Lip^m_\beta$, $0 < \beta < \min\{1, n/s\}$, $s < p_j < n/\beta$, $1/p = 1/p_* - \beta/n$, $1/p_1 + \cdots + 1/p_m = 1/p_*$, $p_* > 1$, $1/k_j = 1/p_j - \beta/n$. *Let* φ_j *be a positive function on* $\mathbb{R}^n \times \mathbb{R}^+$ *and suppose there exists* $0 < C_j < 2^{np_j/k_j}$ *such that* $\varphi_j(x, 2r) \leq C_j \varphi_j(x, r)$ *for all* $x \in R^n$, $r > 0$, *and* $\varphi^{1/p} = \prod_{j=1}^m \varphi_j^{1/p_j}$. *Then* $T_{\vec{b}}$ *can be extended into a bounded operator from* $L^{p_1,\varphi_1} \times \cdots \times L^{p_m,\varphi_m}$ *into* $L^{p,\varphi}$.

Remark 4. *Theorem 2.3 generalized Theorem 3.2 in* [7].

Now, let us consider the boundedness of the multilinear iterated commutators generated by the multilinear strongly singular Calderón-Zygmund operators and BMO functions on the product of generalized Morrey spaces.

Theorem 2.4. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$. *Suppose* $s_0 < p_j < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $p > 1$. *Let* φ_j *be a positive function on* $\mathbb{R}^n \times \mathbb{R}^+$ *and suppose there exists* $0 < C_j < 2^n$ *such that* $\varphi_j(x, 2r) \leq C_j \varphi_j(x, r)$ *for all* $x \in R^n$, $r > 0$, *and* $\varphi^{1/p} = \prod_{j=1}^m \varphi_j^{1/p_j}$. *If* $\vec{b} \in BMO^m$, *then* $T_{\Pi\vec{b}}$ *can be extended into a bounded operator from* $L^{p_1,\varphi_1} \times \cdots \times L^{p_m,\varphi_m}$ *into* $L^{p,\varphi}$.

Moreover, we will consider the boundedness of multilinear strongly singular Calderón-Zygmund operators and the multilinear commutators or the multilinear iterated commutators generated with BMO functions on the product of weighted Morrey spaces.

Theorem 2.5. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator. Let* $s = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2,

$j = 1, \ldots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$, $p > 1$, $\omega^{1/p} = \prod_{j=1}^m \omega_j^{1/p_j}$, $(w_1, \ldots, w_m) \in (A_{p_1/s}, \ldots, A_{p_m/s})$. *If* $s < p_j < \infty$, $0 < k < 1$, *then* T *can be extended into a bounded operator from* $L^{p_1,k}(\omega_1) \times \cdots \times L^{p_m,k}(\omega_m)$ *into* $L^{p,k}(\omega)$.

Theorem 2.6. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$, $s_0 < p_j < \infty$, $p > 1$, $\omega^{1/p} = \prod_{j=1}^m \omega_j^{1/p_j}$, $(w_1, \ldots, w_m) \in (A_{p_1/s_0}, \ldots, A_{p_m/s_0})$. *If* $\vec{b} \in BMO^m$, $0 < k < 1$, *then* $T_{\vec{b}}$ *can be extended into a bounded operator from* $L^{p_1,k}(\omega_1) \times \cdots \times L^{p_m,k}(\omega_m)$ *into* $L^{p,k}(\omega)$.

Theorem 2.7. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$, $s_0 < p_j < \infty$, $p > 1$, $\omega^{1/p} = \prod_{j=1}^m \omega_j^{1/p_j}$, $(w_1, \ldots, w_m) \in (A_{p_1/s_0}, \ldots, A_{p_m/s_0})$. *Suppose* $\vec{b} \in BMO^m$, $0 < k < 1$, *then* $T_{\Pi\vec{b}}$ *can be extended into a bounded operator from* $L^{p_1,k}(\omega_1) \times \cdots \times L^{p_m,k}(\omega_m)$ *into* $L^{p,k}(\omega)$.

Since the classical Morrey space is the special case of the generalized Morrey space, we can get the corresponding results on classical Morrey spaces as corollaries.

Corollary 2.1. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator. Let* $s = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$, $1/q = 1/q_1 + \cdots + 1/q_m$, $1/p = 1/p_1 + \cdots + 1/p_m$, $p > 1$. *If* $s < p_j \leq q_j < \infty$, *then* T *can be extended into a bounded operator from* $M_{p_1}^{q_1}(\mathbb{R}^n) \times \cdots \times M_{p_m}^{q_m}(\mathbb{R}^n)$ *into* $M_p^q(\mathbb{R}^n)$.

Corollary 2.2. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$, $1/q = 1/q_1 + \cdots + 1/q_m$,

$1/p = 1/p_1 + \cdots + 1/p_m$, $s_0 < p_j \le q_j < \infty$, $p > 1$. *If* $\vec{b} \in BMO^m$, *then* $T_{\vec{b}}$ *can be extended into a bounded operator from* $M^{q_1}_{p_1}(\mathbb{R}^n) \times \cdots \times M^{q_m}_{p_m}(\mathbb{R}^n)$ *into* $M^q_p(\mathbb{R}^n)$.

Corollary 2.3. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator. Let* $s = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$. *If* $\vec{b} \in Lip^m_\beta$, $0 < \beta < \min\{1, n/s\}$, $s < p_j \le q_j < n/\beta$, $1/p_* - 1/p = \beta/n$, $1/p_1 + \cdots + 1/p_m = 1/p_*$, $p_* > 1$, $1/q_1 + \cdots + 1/q_m - \beta/n = 1/q$, *then* $T_{\vec{b}}$ *can be extended into a bounded operator from* $M^{q_1}_{p_1}(\mathbb{R}^n) \times \cdots \times M^{q_m}_{p_m}(\mathbb{R}^n)$ *into* $M^q_p(\mathbb{R}^n)$.

Corollary 2.4. *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$, $1/q = 1/q_1 + \cdots + 1/q_m$, $1/p = 1/p_1 + \cdots + 1/p_m$, $s_0 < p_j \le q_j < \infty$, $p > 1$. *If* $\vec{b} \in BMO^m$, *then* $T_{\Pi\vec{b}}$ *can be extended into a bounded operator from* $M^{q_1}_{p_1}(\mathbb{R}^n) \times \cdots \times M^{q_m}_{p_m}(\mathbb{R}^n)$ *into* $M^q_p(\mathbb{R}^n)$.

3. Necessary Lemmas

Lemma 3.1. ([8]) *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $s = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$. *If* $0 < \delta < 1/m$, *then*

$$M^\sharp_\delta(T(\vec{f}))(x) \le C \prod_{j=1}^m M_s(f_j)(x),$$

for all m*-tuples* $\vec{f} = (f_1, \ldots, f_m)$ *of bounded measurable functions with compact support.*

Lemma 3.2. ([10]) *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \ldots, m$. *If* $\vec{b} \in BMO^m$,

$0 < \delta < 1/m$, $\delta < t < \infty$ *and* $s_0 < s < \infty$, *then*

$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \le C\|\vec{b}\|_{BMO^m}\bigg(M_t(T(\vec{f}))(x) + \prod_{j=1}^m M_s(f_j)(x)\bigg),$$

for all m*-tuples* $\vec{f} = (f_1, \dots, f_m)$ *of bounded measurable functions with compact support.*

Lemma 3.3. ([10]) *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $s_0 = \max\{r_1, \dots, r_m, l_1, \dots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \dots, m$. *If* $\vec{b} \in Lip_\beta^m$, $0 < \beta < 1$ *and* $0 < \delta < 1/m$, *then*

$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \le C\|\vec{b}\|_{Lip_\beta^m}\sum_{j=1}^m\bigg(M_{\beta,\delta}(T(\vec{f}))(x) + M_{\beta,s_0}(f_j)(x)\prod_{i=1, i\neq j}^m M_{s_0}(f_i)(x)\bigg),$$

for all m*-tuples* $\vec{f} = (f_1, \dots, f_m)$ *of bounded measurable functions with compact support.*

Lemma 3.4. ([9]) *Let* T *be an* m*-linear strongly singular Calderón-Zygmund operator and* $0 < l/q < \alpha$ *in* (3) *of Definition* 1.2. *Let* $s_0 = \max\{r_1, \dots, r_m, l_1, \dots, l_m\}$, *where* r_j *and* l_j *are given as in Definition* 1.2, $j = 1, \dots, m$. *If* $\vec{b} \in BMO^m$, $0 < \delta < 1/m$, $\delta < t < \infty$ *and* $s_0 < s < \infty$, *then*

$$\begin{aligned} M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))(x) \;\le\; & C\prod_{j=1}^m \|b_j\|_{BMO}\left(\prod_{k=1}^m M_s(f_k)(x) + M_t(T(\vec{f}))(x)\right) \\ & + C\sum_{j=1}^{m-1}\sum_{\psi\in C_j^m}\prod_{i=1}^j \|b_{\psi(i)}\|_{BMO} M_t(T_{\Pi\vec{b}_{\psi'}}(\vec{f})(x)), \end{aligned}$$

for all m*-tuples* $\vec{f} = (f_1, \dots, f_m)$ *of bounded measurable functions with compact support.*

Lemma 3.5. ([12]) *Let* φ *be a positive function on* $\mathbb{R}^n \times \mathbb{R}^+$ *and suppose there exists* $0 < C_0 < 2^n$ *such that*

$$\varphi(x, 2r) \le C_0\varphi(x, r) \quad for \quad all \quad x \in \mathbb{R}^n, r > 0. \tag{3.1}$$

If $1 < p < \infty$, *then*

$$\|Mf\|_{L^{p,\varphi}} \le C\|f\|_{L^{p,\varphi}},$$

where C *is independent of* f.

Lemma 3.6. ([7]) *Let* φ_1 *be a positive function on* $\mathbb{R}^n \times \mathbb{R}^+$. *Suppose* $0 < \alpha < n$, $1 < l < p_1 < n/\alpha$, $1/p_2 = 1/p_1 - \alpha/n$ *and* $\varphi_2^{1/p_2} = \varphi_1^{1/p_1}$. *If there exists* $0 < C_1 < 2^{np_1/p_2}$ *such that* (1) *holds for* φ_1 *and* C_1, *then*

$$\|M_{\alpha,l}f\|_{L^{p_2,\varphi_2}} \le C\|f\|_{L^{p_1,\varphi_1}},$$

where C *is independent of* f.

Lemma 3.7. ([15]) *Let* $0 < \delta < 1$, $1 < p < \infty$ *and* $0 < k < 1$. *If* $\mu, \nu \in A_\infty$, *then we have*

$$\|M_\delta(f)\|_{L^{p,k}(\mu,\nu)} \le C\|M_\delta^\sharp(f)\|_{L^{p,k}(\mu,\nu)},$$

for all functions f such that the left hand side is finite. In particular, when $\mu = \nu = \omega$ *and* $\omega \in A_\infty$, *we have*

$$\|M_\delta(f)\|_{L^{p,k}(\omega)} \le C\|M_\delta^\sharp(f)\|_{L^{p,k}(\omega)},$$

for all functions f such that the left hand side is finite.

Lemma 3.8. ([5]) *If* $1 < p < \infty$, $0 < k < 1$, *and* $\omega \in A_p$, *then* M *is bounded on* $L^{p,k}(\omega)$.

Lemma 3.9. ([3]) *For* $(w_1, \dots, w_m) \in (A_{p_1}, \dots, A_{p_m})$ *with* $1 \le p_1, \dots, p_m < \infty$, *and for* $0 < \theta_1, \dots, \theta_m < 1$ *such that* $\theta_1 + \cdots + \theta_m = 1$, *we have* $w_1^{\theta_1} \cdots w_m^{\theta_m} \in A_{\max\{p_1,\dots,p_m\}}$.

Lemma 3.10. ([6]) *Let* $0 < \delta, p < \infty$ *and* $w \in A_\infty$. *Then there exists a constant* $C > 0$ *depending only on the* A_∞ *constant of* ω *such that*

$$\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p w(x)dx \le C \int_{\mathbb{R}^n} [M_\delta^\sharp(f)(x)]^p w(x)dx,$$

for every function f such that the left-hand side is finite.

Lemma 3.11. *Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and suppose there exists $0 < C_0 < 2^n$ such that $\varphi(x, 2r) \leq C_0\varphi(x, r)$, for all $x \in R^n$, $r > 0$. If $\delta > 0$, $1 < p < \infty$, then*

$$\|M_\delta(f)\|_{L^{p,\varphi}} \leq C\|M_\delta^\sharp(f)\|_{L^{p,\varphi}},$$

where C is independent of f.

Proof. For any ball $B = B(x, r) \subset \mathbb{R}^n$, since $M\chi_B \in A_1$ in [2], $M\chi_B \leq 1$ and $M\chi_B(y) \leq \frac{r^n}{(|y-x|-r)^n}$ if $y \in B^c$, by Lemma 3.10, we have

$$\begin{aligned}
&\int_B |M_\delta(f)(y)|^p dy \\
&\leq \int_{R^n} |M_\delta(f)(y)|^p M\chi_B(y)dy \leq \int_{R^n} |M_\delta^\sharp(f)(y)|^p M\chi_B(y)dy \\
&\leq C\left(\int_B |M_\delta^\sharp(f)(y)|^p M\chi_B(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\backslash 2^kB} |M_\delta^\sharp(f)(y)|^p M\chi_B(y)dy\right) \\
&\leq C\left(\int_B |M_\delta^\sharp(f)(y)|^p dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B} \frac{r^n}{(|y-x|-r)^n}|M_\delta^\sharp(f)(y)|^p dy\right) \\
&\leq C\left(\|M_\delta^\sharp(f)\|_{L^{p,\varphi}}^p\varphi(x,r) + \sum_{k=0}^{\infty}\left(\frac{r}{2^{k-1}r}\right)^n \|M_\delta^\sharp(f)\|_{L^{p,\varphi}}^p\varphi(x,2^{k+1}r)\right) \\
&\leq C\left(\|M_\delta^\sharp(f)\|_{L^{p,\varphi}}^p\varphi(x,r) + \sum_{k=0}^{\infty}2^{-kn}\|M_\delta^\sharp(f)\|_{L^{p,\varphi}}^p C_0^{k+1}\varphi(x,r)\right) \\
&\leq C\|M_\delta^\sharp(f)\|_{L^{p,\varphi}}^p\varphi(x,r)\sum_{k=0}^{\infty}\left(\frac{C_0^{\frac{1}{n}}}{2}\right)^{kn} \\
&\leq C\|M_\delta^\sharp(f)\|_{L^{p,\varphi}}^p\varphi(x,r).
\end{aligned}$$

Thus,

$$\|M_\delta(f)\|_{L^{p,\varphi}} = \sup_{x\in R^n, r>0}\left(\frac{1}{\varphi(x,r)}\int_B |M_\delta(f)(y)|^p dy\right)^{\frac{1}{p}} \leq C\|M_\delta^\sharp(f)\|_{L^{p,\varphi}},$$

which completes the proof of Lemma 3.11.

Lemma 3.12. *Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and suppose there exists $0 < C_0 < 2^n$ such that $\varphi(x, 2r) \le C_0\varphi(x, r)$, for all $x \in \mathbb{R}^n$, $r > 0$. If $0 < \beta < n$, $0 < \delta < 1 < t < p < n/\beta$, then*

$$\|M_{\beta,\delta}(f)\|_{L^{p,\varphi}} \le C\|M_{\beta,t}(f)\|_{L^{p,\varphi}},$$

where C is independent of f.

Proof. For any ball $B = B(x, r) \subset \mathbb{R}^n$, we have

$$\begin{aligned}
&\int_B |M_{\beta,\delta}(f)(y)|^p dy \\
&\le \int_{R^n} |M_{\beta,\delta}(f)(y)|^p M\chi_B(y)dy \le \int_{R^n} |M_{\beta,t}(f)(y)|^p M\chi_B(y)dy \\
&\le C\left(\int_B |M_{\beta,t}(f)(y)|^p M\chi_B(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^k B} |M_{\beta,t}(f)(y)|^p M\chi_B(y)dy\right) \\
&\le C\left(\int_B |M_{\beta,t}(f)(y)|^p dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B} \frac{r^n}{(|y-x|-r)^n}|M_{\beta,t}(f)(y)|^p dy\right) \\
&\le C\left(\|M_{\beta,t}(f)\|^p_{L^{p,\varphi}}\varphi(x,r) + \sum_{k=0}^{\infty}\left(\frac{r}{2^{k-1}r}\right)^n \|M_{\beta,t}(f)\|^p_{L^{p,\varphi}} C_0^{k+1}\varphi(x,r)\right) \\
&\le C\|M_{\beta,t}(f)\|^p_{L^{p,\varphi}}\varphi(x,r)\sum_{k=0}^{\infty}\left(\frac{C_0^{\frac{1}{n}}}{2}\right)^{kn} \\
&\le C\|M_{\beta,t}(f)\|^p_{L^{p,\varphi}}\varphi(x,r).
\end{aligned}$$

Thus,

$$\|M_{\beta,\delta}(f)\|_{L^{p,\varphi}} = \sup_{x\in R^n, r>0}\left(\frac{1}{\varphi(x,r)}\int_B |M_{\beta,\delta}(f)(y)|^p dy\right)^{1/p} \le C\|M_{\beta,t}(f)\|_{L^{p,\varphi}},$$

which completes the proof of Lemma 3.12.

Lemma 3.13. *Let φ_j be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$. If $1 \le p_j < \infty$, $f_j \in L^{p_j,\varphi_j}$, $j = 1, \cdots, m$, and $\varphi^{1/p} = \prod_{j=1}^m \varphi_j^{1/p_j}$, $1/p_1 + \cdots + 1/p_m = 1/p$, $p > 1$, then*

$$\left\| \prod_{j=1}^m f_j \right\|_{L^{p,\varphi}} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}},$$

where C is independent of f_j.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
\left\| \prod_{j=1}^m f_j \right\|_{L^{p,\varphi}} &= \sup_{x\in R^n, r>0} \left(\frac{1}{\varphi(x,r)} \int_{B(x,r)} \left| \prod_{j=1}^m f_j(y) \right|^p dy \right)^{\frac{1}{p}} \\
&= \sup_{x\in R^n, r>0} \varphi(x,r)^{-\frac{1}{p}} \left(\int_{B(x,r)} \left| \prod_{j=1}^m f_j(y) \right|^p dy \right)^{\frac{1}{p}} \\
&\le C \sup_{x\in R^n, r>0} \varphi(x,r)^{-\frac{1}{p}} \prod_{j=1}^m \left(\int_{B(x,r)} |f_j(y)|^{p_j} dy \right)^{\frac{1}{p_j}} \\
&= C \sup_{x\in R^n, r>0} \prod_{j=1}^m \left(\frac{1}{\varphi_j(x,r)} \int_{B(x,r)} |f_j(y)|^{p_j} dy \right)^{\frac{1}{p_j}} \\
&\le C \prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}},
\end{aligned}$$

which completes the proof of Lemma 3.13.

Lemma 3.14. *If $1 \le p_j < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, $p > 1$, $0 < k < 1$, ω_j is a weight function, and $\omega^{\frac{1}{p}} = \prod_{j=1}^m \omega_j^{\frac{1}{p_j}}$, $f_j \in L^{p_j,k}(\omega_j)$, $j = 1, \cdots, m$, then*

$$\left\| \prod_{j=1}^m f_j \right\|_{L^{p,k}(\omega)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j,k}(\omega_j)},$$

where C is independent of f_j.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
\Big\|\prod_{j=1}^m f_j\Big\|_{L^{p,k}(\omega)} &= \sup_Q \left(\frac{1}{\omega(Q)^k}\int_Q \Big|\prod_{j=1}^m f_j(x)\Big|^p \omega(x)dx\right)^{\frac{1}{p}} \\
&\le C\sup_Q \frac{1}{\omega(Q)^{\frac{k}{p}}}\prod_{j=1}^m \left(\int_Q |f_j(x)|^{p_j}\omega_j(x)dx\right)^{\frac{1}{p_j}} \\
&= C\sup_Q \prod_{j=1}^m \left(\frac{1}{\omega_j(Q)^k}\int_Q |f_j(x)|^{p_j}\omega_j(x)dx\right)^{\frac{1}{p_j}} \\
&\le C\prod_{j=1}^m \|f_j\|_{L^{p_j,k}(\omega_j)},
\end{aligned}$$

which completes the proof of Lemma 3.14.

4. Proof of Main Results

Now we are able to prove our main results.

Proof of Theorem 2.1. Let $C_0 = \max\{C_1, \cdots, C_m\}$, we have

(4.1)
$$\varphi(x,2r)^{\frac{1}{p}} = \prod_{j=1}^m \varphi_j(x,2r)^{\frac{1}{p_j}} \le \prod_{j=1}^m C_j^{\frac{1}{p_j}}\varphi_j(x,r)^{\frac{1}{p_j}} \le C_0^{\frac{1}{p}}\prod_{j=1}^m \varphi_j(x,r)^{\frac{1}{p_j}} = C_0^{\frac{1}{p}}\varphi(x,r)^{\frac{1}{p}},$$

thus $\varphi(x,2r) \le C_0\varphi(x,r)$ and $0 < C_0 < 2^n$. Take $0 < \delta < \frac{1}{m}$, from Lemma 3.11, Lemma 3.1, Lemma 3.13 and Lemma 3.5, we have

$$\begin{aligned}
\|T(\vec{f})\|_{L^{p,\varphi}} &\le \|M_\delta(T(\vec{f}))\|_{L^{p,\varphi}} \le C\|M_\delta^\sharp(T(\vec{f}))\|_{L^{p,\varphi}} \\
&\le C\Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^{p,\varphi}} \le C\prod_{j=1}^m \|M_s(f_j)\|_{L^{p_j,\varphi_j}} \\
&= C\prod_{j=1}^m \|M(|f_j|^s)\|_{L^{p_j/s,\varphi_j}}^{1/s} \le C\prod_{j=1}^m \||f_j|^s\|_{L^{p_j/s,\varphi_j}}^{1/s} \\
&= C\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Take δ and t such that $0 < \delta < \frac{1}{m}$, $\delta < t < p$. Since $s_0 < \min_{1\le j\le m}\{p_j\}$, there exists an s such that $s_0 < s < \min_{1\le j\le m}\{p_j\}$. From (4.1), Lemma 3.11 and Lemma 3.2, we have

$$\begin{aligned}\|T_{\vec{b}}(\vec{f})\|_{L^{p,\varphi}} &\le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^{p,\varphi}} \le C\|M_\delta^\sharp(T_{\vec{b}}(\vec{f}))\|_{L^{p,\varphi}}\\ &\le C\|\vec{b}\|_{BMO^m}\left(\|M_t(T(\vec{f}))\|_{L^{p,\varphi}} + \left\|\prod_{j=1}^m M_s(f_j)\right\|_{L^{p,\varphi}}\right).\end{aligned}$$

Applying Lemma 3.5, and Theorem 2.1, we have

$$\begin{aligned}\|M_t(T(\vec{f}))\|_{L^{p,\varphi}} &= \|M(|T(\vec{f})|^t)\|_{L^{p/t,\varphi}}^{1/t} \le C\||T(\vec{f})|^t\|_{L^{p/t,\varphi}}^{1/t}\\ &= C\|T(\vec{f})\|_{L^{p,\varphi}} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.\end{aligned}$$

Applying Lemma 3.13, and Lemma 3.5, we have

$$\begin{aligned}\left\|\prod_{j=1}^m M_s(f_j)\right\|_{L^{p,\varphi}} &\le C\prod_{j=1}^m \|M_s(f_j)\|_{L^{p_j,\varphi_j}} = C\prod_{j=1}^m \|M(|f_j|^s)\|_{L^{p_j/s,\varphi_j}}^{1/s}\\ &\le C\prod_{j=1}^m \||f_j|^s\|_{L^{p_j/s,\varphi_j}}^{1/s} = C\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.\end{aligned}$$

In conclusion,

$$\|T_{\vec{b}}(\vec{f})\|_{L^{p,\varphi}} \le C\|\vec{b}\|_{BMO^m}\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Since $\frac{p_*}{p} = 1 - \frac{\beta}{n}p_*$, $\frac{p_j}{k_j} = 1 - \frac{\beta}{n}p_j$, and $p_* < p_j$ for $j = 1, \cdots, m$, thus $\frac{p_*}{p} > \max_{1\le j\le m}\{\frac{p_j}{k_j}\}$. Let $C_0 = \max\{C_1, \cdots, C_m\}$, then $0 < C_0 < \max_{1\le j\le m}\{2^{\frac{np_j}{k_j}}\}$. We have

$$\begin{aligned}\varphi(x,2r)^{\frac{1}{p}} &= \prod_{j=1}^m \varphi_j(x,2r)^{\frac{1}{p_j}} \le \prod_{j=1}^m C_j^{\frac{1}{p_j}}\varphi_j(x,r)^{\frac{1}{p_j}}\\ &\le C_0^{\frac{1}{p_*}}\prod_{j=1}^m \varphi_j(x,r)^{\frac{1}{p_j}} = C_0^{\frac{1}{p_*}}\varphi(x,r)^{\frac{1}{p}},\end{aligned}$$

thus,

$$\varphi(x,2r) \le C_0^{\frac{p}{p_*}} \varphi(x,r) < 2^n \varphi(x,r).$$

Take $0<\delta<\frac{1}{m}$, from Lemma 3.11 and Lemma 3.3, we have

$$\begin{aligned}\|T_{\vec{b}}(\vec{f})\|_{L^{p,\varphi}} &\le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^{p,\varphi}} \le C\|M_\delta^\sharp(T_{\vec{b}}(\vec{f}))\|_{L^{p,\varphi}} \\ &\le C\|\vec{b}\|_{Lip_\beta^m} \sum_{j=1}^m \left(\|M_{\beta,\delta}(T(\vec{f}))\|_{L^{p,\varphi}} + \left\| M_{\beta,s}(f_j) \prod_{i=1,i\neq j}^m M_s(f_i) \right\|_{L^{p,\varphi}} \right).\end{aligned}$$

Let $\varphi_* = \varphi^{\frac{p_*}{p}}$, then $\varphi_*(x,2r) \le C_0\varphi_*(x,r)$ and $0<C_0<2^{np_*/p}$. We can choose an t such that $1<t<p_*<\frac{n}{\beta}$. Applying Lemma 3.12, Lemma 3.6 and Theorem 2.1, we have

$$\|M_{\beta,\delta}(T(\vec{f}))\|_{L^{p,\varphi}} \le C\|M_{\beta,t}(T(\vec{f}))\|_{L^{p,\varphi}} \le C\|T(\vec{f})\|_{L^{p_*,\varphi_*}} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.$$

For every $j=1,\cdots,m$, $\frac{1}{k_j}=\frac{1}{p_j}-\frac{\beta}{n}$, $\frac{1}{p}=\frac{1}{p_*}-\frac{\beta}{n}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}-\frac{\beta}{n}=\frac{1}{k_j}+\sum_{i=1,i\neq j}^m \frac{1}{p_i}$. Let $\varphi_j^* = \varphi_j^{k_j/p_j}$, by Lemma 3.13, Lemma 3.6 and Lemma 3.5, we have

$$\begin{aligned}\left\| M_{\beta,s}(f_j) \prod_{i=1,i\neq j}^m M_s(f_i) \right\|_{L^{p,\varphi}} &\le C\|M_{\beta,s}(f_j)\|_{L^{k_j,\varphi_j^*}} \prod_{i=1,i\neq j}^m \|M_s(f_i)\|_{L^{p_i,\varphi_i}} \\ &\le C\|f_j\|_{L^{p_j,\varphi_j}} \prod_{i=1,i\neq j}^m \|M(|f_i|^s)\|_{L^{p_i/s,\varphi_i}}^{1/s} \\ &\le C\|f_j\|_{L^{p_j,\varphi_j}} \prod_{i=1,i\neq j}^m \||f_i|^s\|_{L^{p_i/s,\varphi_i}}^{1/s} \\ &= C\|f_j\|_{L^{p_j,\varphi_j}} \prod_{i=1,i\neq j}^m \|f_i\|_{L^{p_i,\varphi_i}} \\ &= C\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.\end{aligned}$$

In conclusion,

$$\|T_{\vec{b}}(\vec{f})\|_{L^{p,\varphi}} \le C\|\vec{b}\|_{Lip_\beta^m} \prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. From (4.1) we can get $\varphi(x,2r)\le C_0\varphi(x,r)$, where $0<C_0<2^n$. Since $s_0<\min_{1\le j\le m}\{p_j\}$, there exists an s such that $s_0<s<\min_{1\le j\le m}\{p_j\}$. Take $\delta,\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m$ satisfying $0<\delta<\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_m<\frac{1}{m}$. Applying Lemma 3.11 and Lemma 3.1, we can get

$$\|M_{\varepsilon_j}(T(\vec{f}))\|_{L^{p,\varphi}}\le C\|M^{\sharp}_{\varepsilon_j}(T(\vec{f}))\|_{L^{p,\varphi}}\le C\Big\|\prod_{i=1}^m M_{s_0}(f_i)\Big\|_{L^{p,\varphi}},\ j=1,\cdots,m.$$

From Lemma 3.11 and Lemma 3.4, we have

$$\begin{aligned}\|T_{\Pi\vec{b}}(\vec{f})\|_{L^{p,\varphi}} &\le \|M_\delta(T_{\Pi\vec{b}}(\vec{f}))\|_{L^{p,\varphi}}\le C\|M^{\sharp}_\delta(T_{\Pi\vec{b}}(\vec{f}))\|_{L^{p,\varphi}}\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\left(\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,\varphi}}+\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^{p,\varphi}}\right)\\ &\quad +C\sum_{j=1}^{m-1}\sum_{\phi\in C_j^m}\prod_{i=1}^j\|b_{\phi(i)}\|_{BMO}\|M_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,\varphi}}\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\left(\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,\varphi}}+\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^{p,\varphi}}\right)\\ &\quad +C\sum_{j=1}^{m-1}\sum_{\phi\in C_j^m}\prod_{i=1}^j\|b_{\phi(i)}\|_{BMO}\|M^{\sharp}_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,\varphi}}.\end{aligned}$$

In order to reduce the dimension of the BMO function in the commutator, we apply Lemma 3.4 again to $\|M^{\sharp}_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,\varphi}}$.

Let $\phi=\{\phi(1),\cdots,\phi(j)\}$ and $\phi'=\{\phi(j+1),\cdots,\phi(m)\}$, $B_h=\{\phi_1:$ any finite subset of ϕ' with different elements$\}$, and $\phi_1'=\phi'-\phi_1$. It follows from Lemma 3.4 that

$$\begin{aligned}\|M^{\sharp}_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,\varphi}} &\le C\prod_{l=j+1}^m\|b_{\phi(l)}\|_{BMO}\left(\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,\varphi}}+\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^{p,\varphi}}\right)\\ &\quad +C\sum_{h=1}^{m-j-1}\sum_{\phi_1\in B_h}\prod_{i=1}^h\|b_{\phi_1(i)}\|_{BMO}\|M_{\varepsilon_2}(T_{\Pi\vec{b}_{\phi_1'}}(\vec{f}))\|_{L^{p,\varphi}}.\end{aligned}$$

By putting the formula above into $\|M_\delta^\sharp\big(T_{\Pi\vec{b}}(\vec{f})\big)\|_{L^{p,\varphi}}$, we can reduce the dimension of BMO functions. Repeating the process above and using Lemma 3.2, we can get

$$\begin{aligned}\|M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))\|_{L^{p,\varphi}} &\le C\prod_{j=1}^m \|b_j\|_{BMO}\Big(G_{m+1}(m,n)\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,\varphi}}\\ &\quad+G_1(m,n)\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^{p,\varphi}}+G_2(m,n)\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^{p,\varphi}}\\ &\quad+\cdots+G_m(m,n)\|M_{\varepsilon_m}(T(\vec{f}))\|_{L^{p,\varphi}}\Big),\end{aligned}$$

where $G_1(m,n), G_2(m,n), \cdots, G_{m+1}(m,n)$ are finite real numbers related to m and n. Then, by Lemma 3.13 and Lemma 3.5, we get

$$\begin{aligned}\|T_{\Pi\vec{b}}(\vec{f})\|_{L^{p,\varphi}} &\le C\prod_{j=1}^m \|b_j\|_{BMO}\Big(G_{m+1}(m,n)\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,\varphi}}\\ &\quad+G_{m+2}(m,n)\Big\|\prod_{j=1}^m M_{s_0}(f_j)\Big\|_{L^{p,\varphi}}\Big)\\ &\le C\prod_{j=1}^m \|b_j\|_{BMO}\Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^{p,\varphi}}\\ &\le C\prod_{j=1}^m \|b_j\|_{BMO}\prod_{j=1}^m \|M_s(f_j)\|_{L^{p_j,\varphi_j}}\\ &= C\prod_{j=1}^m \|b_j\|_{BMO}\prod_{j=1}^m \|M(|f_j|^s)\|^{1/s}_{L^{p_j/s,\varphi_j}}\\ &\le C\prod_{j=1}^m \|b_j\|_{BMO}\prod_{j=1}^m \||f_j|^s\|^{1/s}_{L^{p_j/s,\varphi_j}}\\ &= C\prod_{j=1}^m \|b_j\|_{BMO}\prod_{j=1}^m \|f_j\|_{L^{p_j,\varphi_j}}.\end{aligned}$$

This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5. We have $\omega \in A_{\max\{\frac{p_1}{s},\cdots,\frac{p_m}{s}\}} \subset A_\infty$ from Lemma 3.9. Take $0<\delta<\frac{1}{m}$, applying Lemma 3.7, Lemma 3.1, Lemma 3.14 and Lemma 3.8, we can get

$$
\begin{aligned}
\|T(\vec{f})\|_{L^{p,k}(\omega)} &\leq \|M_\delta(T(\vec{f}))\|_{L^{p,k}(\omega)} \leq C\|M_\delta^\sharp(T(\vec{f}))\|_{L^{p,k}(\omega)} \\
&\leq C\Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^{p,k}(\omega)} \leq C\prod_{j=1}^m \Big\|M_s(f_j)\Big\|_{L^{p_j,k}(\omega_j)} \\
&= C\prod_{j=1}^m \|M(|f_j|^s)\|^{1/s}_{L^{p_j/s,k}(\omega_j)} \leq C\prod_{j=1}^m \||f_j|^s\|^{1/s}_{L^{p_j/s,k}(\omega_j)} \\
&= C\prod_{j=1}^m \|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned}
$$

This completes the proof of Theorem 2.5. □

Proof of Theorem 2.6. We have $\omega \in A_{\max\{\frac{p_1}{s_0},\cdots,\frac{p_m}{s_0}\}} \subset A_\infty$ from Lemma 3.9. For any $j = 1, \cdots, m$, since $\omega_j \in A_{p_j/s_0}$, there exists a t_j satisfying $1 < t_j < p_j/s_0$ and $\omega_j \in A_{t_j}$. Since $s_0 < p_j/t_j$, there exists an s_j satisfying $s_0 < s_j < p_j/t_j < p_j$. Let $s = \min_{1\leq j\leq m} s_j$, then we have $s_0 < s < p_j$. Since $t_j < p_j/s_j \leq p_j/s$, then $\omega_j \in A_{t_j} \subset A_{p_j/s}$, $j = 1, \cdots, m$. Choose δ and t such that $0 < \delta < t < \frac{1}{m}$, by Lemma 3.7 and Lemma 3.2,

$$
\begin{aligned}
\|T_{\vec{b}}(\vec{f})\|_{L^{p,k}(\omega)} &\leq \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^{p,k}(\omega)} \leq C\|M_\delta^\sharp(T_{\vec{b}}(\vec{f}))\|_{L^{p,k}(\omega)} \\
&\leq C\|\vec{b}\|_{BMO^m}\left(\|M_t(T(\vec{f}))\|_{L^{p,k}(\omega)} + \Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^{p,k}(\omega)}\right).
\end{aligned}
$$

Applying Lemma 3.7, Lemma 3.1, Lemma 3.14 and Lemma 3.8, we have

$$
\begin{aligned}
\|M_t(T(\vec{f}))\|_{L^{p,k}(\omega)} &\leq C\|M_t^\sharp(T(\vec{f}))\|_{L^{p,k}(\omega)} \leq C\Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^{p,k}(\omega)} \\
&\leq C\prod_{j=1}^m \|M_s(f_j)\|_{L^{p_j,k}(\omega_j)} = C\prod_{j=1}^m \|M(|f_j|^s)\|^{1/s}_{L^{p_j/s,k}(\omega_j)} \\
&\leq C\prod_{j=1}^m \||f_j|^s\|^{1/s}_{L^{p_j/s,k}(\omega_j)} = C\prod_{j=1}^m \|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned}
$$

Applying Lemma 3.14 and Lemma 3.8, we have

$$\begin{aligned}\Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^{p,k}(\omega)} &\le C\prod_{j=1}^m \|M_s(f_j)\|_{L^{p_j,k}(\omega_j)} = C\prod_{j=1}^m \|M(|f_j|^s)\|^{1/s}_{L^{p_j/s,k}(\omega_j)}\\ &\le C\prod_{j=1}^m \||f_j|^s\|^{1/s}_{L^{p_j/s,k}(\omega_j)} = C\prod_{j=1}^m \|f_j\|_{L^{p_j,k}(\omega_j)}.\end{aligned}$$

In conclusion,

$$\|T_{\vec{b}}(\vec{f})\|_{L^{p,k}(\omega)} \le C\|\vec{b}\|_{BMO^m}\prod_{j=1}^m \|f_j\|_{L^{p_j,k}(\omega_j)}.$$

This completes the proof of Theorem 2.6. □

Proof of Theorem 2.7. For any $j = 1, \cdots, m$, since $\omega_j \in A_{p_j/s_0}$, there exists l_j satisfying $1 < l_j < p_j/s_0$ and $\omega_j \in A_{l_j}$. Since $s_0 < p_j/l_j$, there exists an s_j satisfying $s_0 < s_j < p_j/l_j < p_j$. Let $s = \min_{1\le j\le m} s_j$, then we have $s_0 < s < p_j$. Since $l_j < p_j/s_j \le p_j/s$, then $\omega_j \in A_{l_j} \subset A_{p_j/s}$, $j = 1, \cdots, m$. We have $\omega \in A_{\max\{\frac{p_1}{s_0},\cdots,\frac{p_m}{s_0}\}} \subset A_\infty$ from Lemma 3.9. Take $\delta, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ satisfying $0 < \delta < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m < \frac{1}{m}$. Applying Lemma 3.7 and Lemma 3.1, we can get

$$\|M_{\varepsilon_j}(T(\vec{f}))\|_{L^{p,k}(\omega)} \le C\|M^\sharp_{\varepsilon_j}(T(\vec{f}))\|_{L^{p,k}(\omega)} \le C\Big\|\prod_{i=1}^m M_{s_0}(f_i)\Big\|_{L^{p,k}(\omega)},\ j = 1, \cdots, m.$$

From Lemma 3.7 and Lemma 3.4, we have

$$\begin{aligned}\|T_{\Pi\vec{b}}(\vec{f})\|_{L^{p,k}(\omega)} &\le \|M_\delta(T_{\Pi\vec{b}}(\vec{f}))\|_{L^{p,k}(\omega)} \le C\|M^\sharp_\delta(T_{\Pi\vec{b}}(\vec{f}))\|_{L^{p,k}(\omega)}\\ &\le C\prod_{j=1}^m \|b_j\|_{BMO}\left(\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,k}(\omega)} + \|M_{\varepsilon_1}(T(\vec{f}))\|_{L^{p,k}(\omega)}\right)\\ &\quad + C\sum_{j=1}^{m-1}\sum_{\phi\in C_j^m}\prod_{i=1}^j \|b_{\phi(i)}\|_{BMO}\|M_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,k}(\omega)}\\ &\le C\prod_{j=1}^m \|b_j\|_{BMO}\left(\Big\|\prod_{l=1}^m M_s(f_l)\Big\|_{L^{p,k}(\omega)} + \|M_{\varepsilon_1}(T(\vec{f}))\|_{L^{p,k}(\omega)}\right)\\ &\quad + C\sum_{j=1}^{m-1}\sum_{\phi\in C_j^m}\prod_{i=1}^j \|b_{\phi(i)}\|_{BMO}\|M^\sharp_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,k}(\omega)}.\end{aligned}$$

In order to reduce the dimension of the BMO function in the commutator, we apply Lemma 3.4 again to $\|M^{\sharp}_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,k}(\omega)}$. Let $\phi=\{\phi(1),\cdots,\phi(j)\}$ and $\phi'=\{\phi(j+1),\cdots,\phi(m)\}$, $B_h=\{\phi_1:$ any finite subset of ϕ' with different elements$\}$, and $\phi'_1=\phi'-\phi_1$. From Lemma 3.4 we can get

$$\begin{aligned}
&\|M^{\sharp}_{\varepsilon_1}(T_{\Pi\vec{b}_{\phi'}}(\vec{f}))\|_{L^{p,k}(\omega)}\\
&\leq C\prod_{r=j+1}^{m}\|b_{\phi(r)}\|_{BMO}\left(\left\|\prod_{r=1}^{m}M_s(f_r)\right\|_{L^{p,k}(\omega)}+\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^{p,k}(\omega)}\right)\\
&+C\sum_{h=1}^{m-j-1}\sum_{\phi_1\in B_h}\prod_{i=1}^{h}\|b_{\phi_1(i)}\|_{BMO}\|M_{\varepsilon_2}(T_{\Pi\vec{b}_{\phi'_1}}(\vec{f}))\|_{L^{p,k}(\omega)}.
\end{aligned}$$

By putting the formula above into $\|M^{\sharp}_{\delta}\big(T_{\Pi\vec{b}}(\vec{f})\big)\|_{L^{p,k}(\omega)}$, we can reduce the dimension of BMO functions. Repeating the process above and using Lemma 3.2, we can get

$$\begin{aligned}
\|M^{\sharp}_{\delta}(T_{\Pi\vec{b}}(\vec{f}))\|_{L^{p,k}(\omega)} \leq\ & C\prod_{j=1}^{m}\|b_j\|_{BMO}\Big(G_{m+1}(m,n)\left\|\prod_{l=1}^{m}M_s(f_l)\right\|_{L^{p,k}(\omega)}\\
&+G_1(m,n)\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^{p,k}(\omega)}+G_2(m,n)\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^{p,k}(\omega)}\\
&+\cdots+G_m(m,n)\|M_{\varepsilon_m}(T(\vec{f}))\|_{L^{p,k}(\omega)}\Big).
\end{aligned}$$

By Lemma 3.14 and Lemma 3.8, we have

$$\begin{aligned}
\|T_{\Pi\vec{b}}(\vec{f})\|_{L^{p,k}(\omega)} \leq\ & C\prod_{j=1}^{m}\|b_j\|_{BMO}\Big(G_{m+1}(m,n)\left\|\prod_{l=1}^{m}M_s(f_l)\right\|_{L^{p,k}(\omega)}\\
&+G_{m+2}(m,n)\left\|\prod_{j=1}^{m}M_{s_0}(f_j)\right\|_{L^{p,k}(\omega)}\Big)\\
\leq\ & C\prod_{j=1}^{m}\|b_j\|_{BMO}\left\|\prod_{j=1}^{m}M_s(f_j)\right\|_{L^{p,k}(\omega)}\\
\leq\ & C\prod_{j=1}^{m}\|b_j\|_{BMO}\prod_{j=1}^{m}\|M_s(f_j)\|_{L^{p_j,k}(\omega_j)}
\end{aligned}$$

$$\begin{aligned}
&= C\prod_{j=1}^{m}\|b_j\|_{BMO}\prod_{j=1}^{m}\|M(|f_j|^s)\|^{1/s}_{L^{p_j/s,k}(\omega_j)}\\
&\leq C\prod_{j=1}^{m}\|b_j\|_{BMO}\prod_{j=1}^{m}\||f_j|^s\|^{1/s}_{L^{p_j/s,k}(\omega_j)}\\
&= C\prod_{j=1}^{m}\|b_j\|_{BMO}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned}$$

This completes the proof of Theorem 2.7. □

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