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PERTURBATIONS AND NEW CHARACTERIZATIONS OF WOVEN P-FRAMES

M. S. ASGARI [(1)] AND S. HASHEMI SANATI [(2)]

Abstract. In this paper, we prove several functional-analytic properties of weaving p-frames. We first prove that Banach woven p-frames are stable under small perturbations. This is inspired by corresponding classical perturbation results for bases. We introduce new and weaker conditions that ensure the desired stability. We then present several approaches for identifying and constructing of woven p-frames. To this end, we present some conditions under which a p-frame and a finite family of operators constitute a woven p-frame.

1. Introduction

1.1. **Background on p-frames.** The concept of Banach frames was introduced in 1991 by Gröchenig [8] as an extension of the notion of frames for Hilbert spaces to the Banach space setting. Before the notion of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [6, 7] related to atomic decompositions. Aldroubi, Sung and Tang in [1] proposed the notion of p-frame which is a type of Banach frames and generalizes usual frames. Banach frames and atomic decompositions have played a fundamental role in the development of wavelet theory and Fourier analysis. Aside from their theoretical appeal, frames in

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L^p spaces and other Banach function spaces are effective tools for modeling a variety of natural signals and images. For more studies on Banach frames, p-frames, and its applications, the interested reader can refer to [5, 9, 10, 12].

Let X be a separable Banach space with dual X^*. If I is a countable set then a family $\{f_i\}_{i\in I} \subset X^*$ is a p-frame for X $(1 < p < \infty)$, if there exist constants $A, B > 0$ such that

$$A\|x\| \leq \big(\sum_{i\in I} |f_i(x)|^p\big)^{\frac{1}{p}} \leq B\|x\|, \qquad \forall x \in X. \tag{1.1}$$

The constants A and B are called the lower and upper p-frame bounds, respectively. If there exists a reconstruction operator $S : \ell^p \to X$ such that $S(\{f_i(x)\}_{i\in I}) = x$ for all $x \in X$ then $(\{f_i\}_{i\in I}, S)$ is called a Banach p-frame for X with respect to ℓ^p, where $\ell^p = \ell^p(I)$ is the space of all scalar-valued sequences that are p-summable over the set I. Also, the sequence $\{f_i\}_{i\in I}$ is called a p-Bessel sequence with bound B for X, when it satisfies at least the upper p-frame inequality for all $x \in X$. From [12, Theorem 3.1] we know that if there exists a p-frame for X, then X is reflexive. For a p-Bessel sequence $\{f_i\}_{i\in I} \subset X^*$ for X, we define two operators:

$$U : X \longrightarrow \ell^p, \quad Ux = \{f_i(x)\}_{i\in I}, \tag{1.2}$$

$$T : \ell^q \longrightarrow X^*, \quad T(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i, \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The operator U is called the analysis operator, and T is the synthesis operator. It is clear that U is a bounded operator. Moreover, if X is a reflexive Banach space. Then $\{f_i\}_{i\in I}$ is a p-frame for X if and only if T is a well defined mapping and $U^* = T$ (see [12]).

During the last ten years, some generalizations of Riesz basis to Banach spaces were done. In [1], p-Riesz bases for shift-invariant subspaces of $L^p(\mathbb{R})$ are considered.

Definition 1.1. A family $\{h_i\}_{i\in I} \subset X$ is a p-Riesz basis $(1 < p < \infty)$ for X if $\overline{\text{span}}\{h_i\}_{i\in I} = X$ and there exist constants $0 < A \leq B < \infty$ such that

$$A\Big(\sum_{i\in I}|c_i|^p\Big)^{\frac{1}{p}} \leq \Big\|\sum_{i\in I} c_i h_i\Big\|_Y \leq B\Big(\sum_{i\in I}|c_i|^p\Big)^{\frac{1}{p}} \qquad \text{for all } c \in \ell^p. \tag{1.4}$$

The numbers A, B are called lower and upper p-Riesz basis bounds. The sequence $\{h_i\}_{i\in I}$ is called a p-basic sequence in X, if it is a p-Riesz basis for its closed linear span.

1.2. **Background on woven p-frames.** Recently, Bemrose et al. [2] have introduced the notion of weaving Hilbert space frames due to some new problems arising in distributed signal processing and wireless sensor networks. Casazza and Lynch assessed the principal properties of weaving frames in [4]. Casazza, Freeman, and Lynch [3] extended the concept of weaving Hilbert space frames to the Banach space setting. Weaving frames were further studied in [14, 11, 13]. This subsection is dedicated to a brief introduction to weaving p-frames and giving some examples for it. We begin with the formal definition in full generality.

Definition 1.2. Let J be a finite subset of $\mathbb{N}$. A family of p-Bessel sequences $\big\{\{f_{ij}\}_{i\in I} : \ j \in J\big\}$ for X, $(1 < p < \infty)$ is called a woven p-frame for X if there are universal constants $0 < A \leq B < \infty$ so that for every partition $\sigma = \{\sigma_j\}_{j\in J}$ of I, the family $\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}$ is a p-frame for X with p-frame bounds A and B. In this case any family $\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}$ is called a weaving p-frame for X and the constants A, B are called the woven p-frame bounds, respectively. If every weaving is a p-Bessel sequence, it is called a woven p-Bessel sequence for X. A family of p-Bessel sequences $\big\{\{f_{ij}\}_{i\in I} : \ j \in J\big\}$ for X is said to be weakly woven if every weaving is a p-frame for X.

Example 1.1. *Let $\{g_i\}_{i\in I} \subset X^*$ be a p-frame for X with p-frame bounds $0 < C \leq D < \infty$. Let $\{\lambda_i\}_{i\in I}$ be a bounded sequence with $A < \inf_{i\in I}|\lambda_i| \leq \sup_{i\in I}|\lambda_i| < B$.*

Let J be a finite subset of $\mathbb{N}$. We define $f_{ij} = j\lambda_i g_i$ for all $j \in J$, $i \in I$. Then $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ is a woven p-frame for X. Indeed, for any partition $\sigma = \{\sigma_j\}_{j\in J}$ of I, we compute

$$\begin{aligned}(\min J)^p A^p C^p \|x\|^p &\leq (\min J)^p \sum_{i\in I} |\lambda_i g_i(x)|^p \leq \sum_{j\in J}\sum_{k\in\sigma_j} |j\lambda_k g_k(x)|^p \\ &= \sum_{j\in J}\sum_{k\in\sigma_j} |f_{kj}(x)|^p \leq (\max J)^p \sum_{j\in J}\sum_{k\in\sigma_j} |\lambda_k g_k(x)|^p \\ &= (\max J)^p \sum_{i\in I} |\lambda_i g_i(x)|^p \leq (\max J)^p B^p D^p \|x\|^p.\end{aligned}$$

Therefore, the sequence $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ is a woven p-frame for X with woven p-frame bounds $(\min J)AC$ and $(\max J)BD$.

We leave the easy proof of the proposition below to the reader.

Proposition 1.1. *Let $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ be a finite family of p-Bessel sequences for X with Bessel bounds B_j. Then $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ is a woven p-Bessel sequence with woven p-Bessel bound $\sum_{j\in J} B_j$.*

Definition 1.3. A woven p-frame $\big\{\{f_{ij}\}_{i\in I} \subset X^* : j \in J\big\}$ is called a Banach woven p-frame for X, if for any partition σ of I there exists a reconstruction operator $S_\sigma : \ell^p \to X$ such that $\big(\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}, S_\sigma\big)$ is a Banach p-frame for X with respect to ℓ^p.

Remark 1. *Let $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ be a Banach woven p-frame for X with woven p-frame bounds A and B. Then for any partition σ of I, $\|S_\sigma\|^{-1}$ and $\|U_\sigma\|$ are a choice of p-frame bounds for the Banach p-frame $\big(\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}, S_\sigma\big)$. Therefore, A and B satisfies $A \leq \|S_\sigma\|^{-1} \leq \|U_\sigma\| \leq B$.*

The following proposition establishes an equivalent characterization of the weakly woven p-frames in terms of the synthesis operator associated to weaving p-frames. This is a consequence of [5, Theorem 2.4].

Proposition 1.2. *Let X be a reflexive Banach space and $\{f_{ij}\}_{i\in I} \subset X^*$, $(j \in J)$ be a finite family of p-Bessel sequences for X. Then $\big\{\{f_{ij}\}_{i\in I} : \ j \in J\big\}$ is a weakly woven p-frame for X if and only if for every partition $\sigma = \{\sigma_j\}_{j\in J}$ of I, the synthesis operator*

$$T_\sigma : \ell^q \longrightarrow X^*, \qquad T_\sigma(\{d_i\}_{i\in I}) = \sum_{j\in J}\sum_{i\in\sigma_j} d_i f_{ij},$$

is a well defined mapping of ℓ^q onto X^*.

Definition 1.4. A finite family of p-basic sequences $\big\{\{h_{ij}\}_{i\in I} \subset X : \ j \in J\big\}$ for X is called a woven p-Riesz basis for X if there are constants $0 < A \leq B < +\infty$ so that for each partition $\sigma = \{\sigma_j\}_{j\in J}$ of I, the family $\bigcup_{j\in J}\{h_{ij}\}_{i\in\sigma_j}$ is a p-Riesz basis for X with p-Riesz basis bounds A, B respectively. In this case any family $\bigcup_{j\in J}\{h_{ij}\}_{i\in\sigma_j}$ is called a weaving p-Riesz basis for X.

Example 1.2. *Let $\{h_i\}_{i\in I} \subset X$ be a p-Riesz basis for X with p-Riesz bounds C and D, respectively. With the assumptions in Example 1.1, define h_{ij} by $h_{ij} = j\lambda_i h_i$ for all $j \in J$, $i \in I$. Then the finite family $\big\{\{h_{ij}\}_{i\in I} : \ j \in J\big\}$ is a woven p-Riesz basis for X with woven p-Riesz bounds $(\min J)AC$ and $(\max J)BD$.*

The organisation of the paper is as follows. Section 2 is devoted to demonstrating that weaving p-frames and the related concept of p-frames are stable under small perturbations. This is inspired by corresponding classical perturbation results for bases. We introduce new and weaker conditions that ensure the desired stability. In Section 3, we first prove some new weaving properties of p-frames and give sufficient conditions under which a p-frame and a finite family of operators constitute a woven p-frame.

2. Banach woven p-frames and perturbations

In this section, we study the stability of the Banach woven p-frames under small perturbations. The first result is a sufficient condition for the perturbation of a Banach woven p-frame by a bounded scalar-valued sequence.

Theorem 2.1. *Let* $\big\{\{f_{ij}\}_{i\in I}: \ j\in J\big\}$ *be a Banach woven p-frame for* X *with woven p-frame bounds* A*,* B *and let* $\big\{\{g_{ij}\}_{i\in I}: j\in J\big\}$ *be a finite family of p-Bessel sequences for* X *with Bessel bounds* B_j*. Let* $\{\lambda_{ij}\}_{i\in I, j\in J}$ *be a bounded sequence with* $\sup_{i\in I, j\in J}|\lambda_{ij}|\leq D$*. If* $D<\frac{A}{2\sum_{j\in J}B_j}$*, then the family* $\big\{\{f_{ij}+\lambda_{ij}g_{ij}\}_{i\in I}: \ j\in J\big\}$ *is a Banach woven p-frame for* X *with woven p-frame bounds* $\big(A-D\sum_{j\in J}B_j\big)$ *and* $\big(B+D\sum_{j\in J}B_j\big)$*.*

Proof. Assume that U_σ and S_σ are the analysis and reconstruction operators associated to $\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}$ and V_σ, W_σ denote the analysis operators of $\bigcup_{j\in J}\{g_{ij}\}_{i\in\sigma_j}$ and $\bigcup_{j\in J}\{f_{ij}+\lambda_{ij}g_{ij}\}_{i\in\sigma_j}$, for all partition $\sigma=\{\sigma_j\}_{j\in J}$ of I, respectively. Then for each $x\in X$ we have

$$\begin{aligned}\|W_\sigma(x)\|_{\ell^p} &= \big\|U_\sigma(x)+\bigcup_{j\in J}\{\lambda_{ij}g_{ij}\}_{i\in\sigma_j}\big\|_{\ell^p}\leq \big(\|U_\sigma(x)\|_{\ell^p}+D\|V_\sigma(x)\|_{\ell^p}\big)\\ &\leq \big(B+D\sum_{j\in J}B_j\big)\|x\|.\end{aligned}$$

This gives the required universal upper woven p-frame bound. For the woven p-frame lower inequality, we compute

$$\begin{aligned}\|W_\sigma(x)\|_{\ell^p} &= \big\|U_\sigma(x)+\bigcup_{j\in J}\{\lambda_{ij}g_{ij}\}_{i\in\sigma_j}\big\|_{\ell^p}\geq \big\|U_\sigma(x)\|_{\ell^p}-D\|V_\sigma(x)\|_{\ell^p}\\ &\geq \big(A-D\sum_{j\in J}B_j\big)\|x\|.\end{aligned}$$

Hence the family $\big\{\{f_{ij}+\lambda_{ij}g_{ij}\}_{i\in I}:\ j\in J\big\}$ is a woven p-frame for X. On the other hand, the hypotheses imply that

$$S_\sigma U_\sigma = Id_X, \qquad \text{and} \qquad \|U_\sigma - W_\sigma\| \le D\sum_{j\in J} B_j.$$

This yields

$$\|Id_X - S_\sigma W_\sigma\| \le \|S_\sigma\|\|U_\sigma - W_\sigma\| \le \|S_\sigma\| D\sum_{j\in J} B_j \le \frac{1}{2}A\|S_\sigma\| < 1.$$

Therefore, $S_\sigma W_\sigma$ is invertible. Now, if we set $S'_\sigma = (S_\sigma W_\sigma)^{-1}S_\sigma$, then $S'_\sigma W_\sigma = Id_X$. Hence $\big(\bigcup_{j\in J}\{f_{ij}+\lambda_{ij}g_{ij}\}_{i\in\sigma_j}, S'_\sigma\big)$ is a Banach p-frame for X. □

We now show that Banach woven p-frames are stable under small perturbations of the Banach woven p-frame elements.

Theorem 2.2. *Let $\big\{\{f_{ij}\}_{i\in I}:\ j\in J\big\}$ be a Banach woven p-frame for X with woven p-frame bounds A, B and let $\big\{\{g_{ij}\}_{i\in I}:\ j\in J\big\}$ be a finite family of p-Bessel sequences for X. Assume that there exist $\lambda,\mu\ge 0$ such that for every partition $\sigma=\{\sigma_j\}_{j\in J}$ of I, $x\in X$*

(*i*) $2(\lambda\|U_\sigma\|+\mu)\|S_\sigma\| < \frac{A}{B}$,

(*ii*) $\big\|\bigcup_{j\in J}\{f_{ij}(x)-g_{ij}(x)\}_{i\in\sigma_j}\big\|_{\ell^p} \le \lambda\big\|\bigcup_{j\in J}\{f_{ij}(x)\}_{i\in\sigma_j}\big\|_{\ell^p}+\mu\|x\|$,

where U_σ and S_σ are the analysis and reconstruction operators associated to $\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}$, respectively. Then the family $\big\{\{g_{ij}\}_{i\in I}:\ j\in J\big\}$ is a Banach woven p-frame for X with woven p-frame bounds $\frac{(2B-A)(\lambda A+\mu)}{A}$ and $(1+\lambda)B+\mu$.

Proof. Let V_σ denote the analysis operator of $\bigcup_{j\in J}\{g_{ij}\}_{i\in\sigma_j}$, for all partition σ of I. By the hypotheses V_σ is bounded and satisfies $\|U_\sigma(x)-V_\sigma(x)\|_{\ell^p}\le\lambda\|U_\sigma(x)\|_{\ell^p}+\mu\|x\|$, for all $x\in X$. We also have

$$\|V_\sigma(x)\|_{\ell^p} \le [(1+\lambda)\|U_\sigma\|+\mu]\|x\| \le [(1+\lambda)B+\mu]\|x\|.$$

This gives an universal upper woven p-frame bound. For the universal lower woven p-frame bound, we observe that $S_\sigma U_\sigma = Id_X$, so

$$\|Id_X - S_\sigma V_\sigma\| \leq \|S_\sigma\|\|U_\sigma - V_\sigma\| \leq (\lambda\|U_\sigma\| + \mu)\|S_\sigma\| < \frac{A}{2B} < 1.$$

This implies that $S_\sigma V_\sigma$ is invertible and

$$\|(S_\sigma V_\sigma)^{-1}\| \leq \frac{1}{1 - (\lambda\|U_\sigma\| + \mu)\|S_\sigma\|}.$$

Finally, if we set $S'_\sigma = (S_\sigma V_\sigma)^{-1} S_\sigma$, then $S'_\sigma V_\sigma = Id_X$, and

$$\|x\| \leq \|S'_\sigma\|\|V_\sigma(x)\|_{\ell^p} \leq \frac{\|S_\sigma\|}{1 - (\lambda\|U_\sigma\| + \mu)\|S_\sigma\|}\|V_\sigma(x)\|_{\ell^p}.$$

This yields

$$\|V_\sigma(x)\|_{\ell^p} \geq \frac{1 - (\lambda\|U_\sigma\| + \mu)\|S_\sigma\|}{\|S_\sigma\|}\|x\| \geq \frac{(2B - A)(\lambda A + \mu)}{A}\|x\|.$$

This shows that $\big(\bigcup_{j\in J}\{g_{ij}\}_{i\in\sigma_j}, S'_\sigma\big)$ is a Banach p-frame for X. □

The following theorem gives a precise statement of how a perturbation of the Banach woven p-frames similar to a result given by Jain et al. in [9].

Theorem 2.3. *Let $\big\{\{f_{ij}\}_{i\in I} :\ j \in J\big\}$ be a Banach woven p-frame for X with woven p-frame bounds A, B and let $\big\{\{g_{ij}\}_{i\in I} :\ j \in J\big\}$ be a family of p-Bessel sequences for X. Assume that U_σ and S_σ are the analysis and reconstruction operators associated to $\bigcup_{j\in J}\{f_{ij}\}_{i\in\sigma_j}$ and V_σ is the analysis operator associated to $\bigcup_{j\in J}\{g_{ij}\}_{i\in\sigma_j}$, for all partition $\sigma = \{\sigma_j\}_{j\in J}$ of I. If there exist constants $\lambda, \mu, \nu \geq 0$ such that*

- *(i)* $(\|U_\sigma\| + \|V_\sigma\| + 1)\sqrt{\max\{\lambda, \mu, \nu\}} < \|S_\sigma\|^{-1}$,
- *(ii)* $\|U_\sigma(x) - V_\sigma(x)\|^2_{\ell^p} \leq \lambda\|U_\sigma(x)\|^2_{\ell^p} + 2\mu\|U_\sigma(x)\|_{\ell^p}\|V_\sigma(x)\|_{\ell^p} + \nu\|V_\sigma(x)\|^2_{\ell^p}$,

for all $x \in X$, then $\big\{\{g_{ij}\}_{i\in I} :\ j \in J\big\}$ is a Banach woven p-frame for X with woven p-frame bounds

$$\frac{(\sqrt{\eta} - \eta)(1 + A)}{1 + \sqrt{\eta}} \qquad \textit{and} \qquad \frac{(1 + \sqrt{\eta})B}{(1 - \sqrt{\eta})}.$$

Proof. Let $\sigma = \{\sigma_j\}_{j\in J}$ be an arbitrary partition of I and $\eta = \max\{\lambda, \mu, \nu\}$. Then (ii) can be restated as:

$$\|U_\sigma(x) - V_\sigma(x)\|_{\ell^p} \leq \sqrt{\eta}(\|U_\sigma(x)\|_{\ell^p} + \|V_\sigma(x)\|_{\ell^p}).$$

By the hypotheses, we have

$$\begin{aligned}\|V_\sigma(x)\|_{\ell^p} &\leq \|U_\sigma(x)\|_{\ell^p} + \|U_\sigma(x) - V_\sigma(x)\|_{\ell^p}\\ &\leq (1+\sqrt{\eta})\|U_\sigma(x)\|_{\ell^p} + \sqrt{\eta}\|V_\sigma(x)\|_{\ell^p}.\end{aligned}$$

This yields

$$(1-\sqrt{\eta})\|V_\sigma(x)\|_{\ell^p} \leq (1+\sqrt{\eta})\|U_\sigma(x)\|_{\ell^p} \leq (1+\sqrt{\eta})\|U_\sigma\|\|x\|.$$

Now we observe that

$$\|V_\sigma(x)\|_{\ell^p} \leq \frac{(1+\sqrt{\eta})\|U_\sigma\|}{(1-\sqrt{\eta})}\|x\| \leq \frac{(1+\sqrt{\eta})B}{(1-\sqrt{\eta})}\|x\|.$$

This gives the required universal upper woven p-frame bound. To prove the lower woven p-frame bound, we compute

$$\begin{aligned}\|V_\sigma(x)\|_{\ell^p} &\geq \|U_\sigma(x)\|_{\ell^p} - \|U_\sigma(x) - V_\sigma(x)\|_{\ell^p}\\ &\geq (1-\sqrt{\eta})\|U_\sigma(x)\|_{\ell^p} - \sqrt{\eta}\|V_\sigma(x)\|_{\ell^p}.\end{aligned}$$

This implies that

$$(1+\sqrt{\eta})\|V_\sigma(x)\|_{\ell^p} \geq (1-\sqrt{\eta})\|U_\sigma(x)\|_{\ell^p} \geq (1-\sqrt{\eta})\|S_\sigma\|^{-1}\|x\|.$$

Now the lower woven p-frame bound follows from

$$\begin{aligned}\|V_\sigma(x)\|_{\ell^p} &\geq \frac{(1-\sqrt{\eta})\|S_\sigma\|^{-1}}{1+\sqrt{\eta}}\|x\| \geq \frac{(\sqrt{\eta}-\eta)(1+\|U_\sigma\|)}{1+\sqrt{\eta}}\|x\|\\ &\geq \frac{(\sqrt{\eta}-\eta)(1+A)}{1+\sqrt{\eta}}\|x\|.\end{aligned}$$

For the Banach woven p-frame part, we have $S_\sigma U_\sigma = Id_X$, $\|Id_X - S_\sigma V_\sigma\| < 1$, so $S_\sigma V_\sigma$ is invertible. Therefore if we put $S'_\sigma = (S_\sigma V_\sigma)^{-1} S_\sigma$, then $S'_\sigma V_\sigma = Id_X$. □

Similar to Banach woven p-frames, woven p-frames are stable under small perturbations. Specifically, we have the following.

Theorem 2.4. *Let* $\{\{f_{ij}\}_{i\in I} :\ j\in J\}$ *be a woven p-frame for* X *with woven p-frame bounds* A, B *and let* $\{\{g_{ij}\}_{i\in I} :\ j\in J\}$ *be a finite family of p-Bessel sequences for* X*. Assume that there exist constants* $0<\lambda,\mu<\dfrac{1}{2^p}$, $\eta\geq 0$ *such that for every partition* $\sigma=\{\sigma_j\}_{j\in J}$ *of* I, $x\in X$

$$\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)-g_{ij}(x)|^p\leq\lambda\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p+\mu\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p+\eta\|x\|^p.$$

Then, the family $\{\{g_{ij}\}_{i\in I} :\ j\in J\}$ *is a woven p-frame for* X *with woven p-frame bounds*

$$\frac{1}{2}\Big(\frac{(1-2^p\lambda)A^p-2^p\eta}{1+\mu}\Big)^{\frac{1}{p}}\quad \textit{and}\quad 2\Big(\frac{(1+\lambda)B^p+\eta}{1-2^p\mu}\Big)^{\frac{1}{p}}.$$

Proof. For every partition $\sigma=\{\sigma_j\}_{j\in J}$ of I, we have

$$\begin{aligned}\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p&=\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)-f_{ij}(x)+f_{ij}(x)|^p\\&\leq 2^p\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p+2^p\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)-f_{ij}(x)|^p\\&\leq 2^p(1+\lambda)\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p+2^p\mu\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p+2^p\eta\|x\|^p.\end{aligned}$$

It follows that

$$\begin{aligned}(1-2^p\mu)\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p&\leq 2^p(1+\lambda)\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p+2^p\eta\|x\|^p\\&\leq 2^p[(1+\lambda)B^p+\eta]\|x\|^p.\end{aligned}$$

Hence

$$\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p\leq\frac{2^p[(1+\lambda)B^p+\eta]}{1-2^p\mu}\|x\|^p.$$

To prove the lower woven p-frame bound, we compute

$$\begin{aligned}\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p &= \sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)-g_{ij}(x)+g_{ij}(x)|^p\\ &\le 2^p\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)-g_{ij}(x)|^p+2^p\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p.\end{aligned}$$

This yields

$$\begin{aligned}2^p\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p &\ge \sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p-2^p\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)-g_{ij}(x)|^p\\ &\ge (1-2^p\lambda)\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p-2^p\mu\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p-2^p\eta\|x\|^p.\end{aligned}$$

Employing this relation, we obtain

$$\begin{aligned}2^p(1+\mu)\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p &\ge (1-2^p\lambda)\sum_{j\in J}\sum_{i\in\sigma_j}|f_{ij}(x)|^p-2^p\eta\|x\|^p\\ &\ge [(1-2^p\lambda)A^p-2^p\eta]\|x\|^p.\end{aligned}$$

Hence

$$\sum_{j\in J}\sum_{i\in\sigma_j}|g_{ij}(x)|^p \ge \frac{[(1-2^p\lambda)A^p-2^p\eta]}{2^p(1+\mu)}\|x\|^p.$$

□

3. Constructing woven p-frames

This section is devoted to developing the construction of weaving p-frames. As a result, we construct a woven p-frame using a p-frame and a finite family of operators. The following proposition shows that adding some elements to a woven p-frame leaves a family that is still a woven p-frame.

Proposition 3.1. *Let* $\big\{\{f_{ij}\}_{i\in I} : j\in J\big\}$ *be a finite family of p-Bessel sequences for* X*. Let* $\Lambda\subset I$ *such that the family* $\big\{\{f_{ij}\}_{i\in\Lambda} : j\in J\big\}$ *be a woven p-frame. Then* $\big\{\{f_{ij}\}_{i\in I} : j\in J\big\}$ *is also a woven p-frame for* X*.*

Proof. Let $\sigma = \{\sigma_j\}_{j\in J}$ be an arbitrary partition of I. Then $\{\sigma_j \cap \Lambda \neq \emptyset\}_{j\in J}$ is a partition of Λ. Suppose that A is the woven p-frame lower bound of $\big\{\{f_{ij}\}_{i\in\Lambda} : j \in J\big\}$, then for any $x \in X$ we have

$$A^p\|x\|^p \leq \sum_{j\in J}\sum_{i\in\sigma_j\cap\Lambda\neq\emptyset} |f_{ij}(x)|^p \leq \sum_{j\in J}\sum_{i\in\sigma_j} |f_{ij}(x)|^p.$$

The upper bound condition follows from Proposition 1.1. This implies the family $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ is a woven p-frame for X. □

The following result gives a condition under which the removal of some elements of a woven p-frame leaves a family that is still a woven p-frame.

Proposition 3.2. *Let $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ be a woven p-frame for X with woven p-frame bounds A, B. Let $\Lambda \subset I$ such that the family $\big\{\{f_{ij}\}_{i\in\Lambda} : j \in J\big\}$ be a woven p-Bessel sequence for X with woven p-Bessel bound $0 < D < A$. Then $\big\{\{f_{ij}\}_{i\in I-\Lambda} : j \in J\big\}$ is also a woven p-frame for X with woven p-frame bounds $\sqrt[p]{A^p - D^p}$ and B, respectively.*

Proof. Let $\sigma = \{\sigma_j\}_{j\in J}$ be an arbitrary partition of $I - \Lambda$ and $\omega = \{\omega_j\}_{j\in J}$ be a partition of Λ. Then $\{\sigma_j \cup \omega_j\}_{j\in J}$ is a partition of I. For every $j \in J$ and $x \in X$, we have

$$\begin{aligned}(A^p - D^p)\|x\|^p &\leq \sum_{j\in J}\sum_{i\in\sigma_j\cup\omega_j} |f_{ij}(x)|^p - \sum_{j\in J}\sum_{i\in\omega_j} |f_{ij}(x)|^p \\ &= \sum_{j\in J}\sum_{i\in\sigma_j} |f_{ij}(x)|^p \leq \sum_{j\in J}\sum_{i\in\sigma_j\cup\omega_j} |f_{ij}(x)|^p \leq B^p\|x\|^p.\end{aligned}$$

□

The next theorem gives a sufficient condition for constructing a weaving p-frame.

Theorem 3.1. *Let $\{\{f_{ij}\}_{i\in I} : j \in J\}$ be a finite family of p-frames for X with p-frame bounds A_j, B_j, and there exists $M > 0$ such that for any subset $\Lambda \subset I$*

$$\sum_{i\in\Lambda} \big|f_{ij}(x) - f_{ik}(x)\big|^p \leq M \min\Big\{\sum_{i\in\Lambda} |f_{ij}(x)|^p, \sum_{i\in\Lambda} |f_{ik}(x)|^p\Big\},$$

for all $j, k \in J$ and $x \in X$. Then the family $\{\{f_{ij}\}_{i\in I} : j \in J\}$ is a woven p-frame with woven p-frame bounds $\sqrt[p]{\frac{\sum_{j\in J} A_j^p}{2^p|J|(M+1)}}$ and $\sqrt[p]{\sum_{j\in J} B_j}$.

Proof. Let $\sigma = \{\sigma_j\}_{j\in J}$ be an arbitrary partition of I. Then for $x \in X$, we have

$$\begin{aligned}
\frac{\sum_{k\in J} A_k^p}{2^p|J|(M+1)}\|x\|^p &\leq \frac{1}{2^p|J|(M+1)} \sum_{k\in J}\sum_{i\in I} |f_{ik}(x)|^p \\
&= \frac{1}{2^p|J|(M+1)} \sum_{k\in J}\sum_{j\in J}\sum_{i\in\sigma_j} |f_{ik}(x)|^p \\
&\leq \frac{1}{|J|(M+1)} \sum_{k\in J}\sum_{j\in J} \big(\sum_{i\in\sigma_j} |f_{ik}(x) - f_{ij}(x)|^p + \sum_{i\in\sigma_j} |f_{ij}(x)|^p\big) \\
&\leq \frac{1}{|J|(M+1)} \sum_{k\in J}\sum_{j\in J} \big(M\sum_{i\in\sigma_j} |f_{ij}(x)|^p + \sum_{i\in\sigma_j} |f_{ij}(x)|^p\big) \\
&= \sum_{j\in J}\sum_{i\in\sigma_j} |f_{ij}(x)|^p \leq \sum_{j\in J}\sum_{i\in I} |f_{ij}(x)|^p \leq \sum_{j\in J} B_j^p \|x\|^p.
\end{aligned}$$

Hence

$$\frac{\sum_{j\in J} A_j^p}{2^p|J|(M+1)}\|x\|^p \leq \sum_{j\in J}\sum_{i\in\sigma_j} |f_{ij}(x)|^p \leq \sum_{j\in J} B_j^p \|x\|^p.$$

Thus the proof is completed. □

The following result gives a sufficient condition for constructing a woven p-frame using the effect of invertible and bounded operators on a p-frame.

Theorem 3.2. *Let J be a finite subset of $\mathbb{N}$ and let $\{f_i\}_{i\in I}$ be a p-frame for X with p-frame bounds A, B. Let $T_j : X \to X$ be an invertible operator on X for each $j \in J$*

and there exists $M > 0$ such that for any finite subset $\Lambda \subset I$

$$\sum_{i\in\Lambda} \big|f_iT_j(x) - f_iT_k(x)\big|^p \le M \min\Big\{\sum_{i\in\Lambda}|f_iT_j(x)|^p, \sum_{i\in\Lambda}|f_iT_k(x)|^p\Big\},$$

for all $j, k \in J$ and $x \in X$. Then the family $\big\{\{f_iT_j\}_{i\in I} : j \in J\big\}$ is a woven p-frame with woven p-frame bounds $\frac{A}{2\sqrt[p]{|J|(M+1)}} \min_{j\in J} \|T_j^{-1}\|^{-1}$ and $B\sqrt[p]{|J|}\max_{j\in J}\|T_j\|$.

Proof. Let $\sigma = \{\sigma_j\}_{j\in J}$ be an arbitrary partition of I. Then for $x \in X$, we have

$$\begin{aligned}
\frac{A^p \min_{j\in J}\|T_j^{-1}\|^{-p}}{2^p|J|(M+1)}\|x\|^p &\le \frac{A^p\|T_k^{-1}\|^{-p}}{2^p|J|(M+1)}\|x\|^p \le \frac{A^p\sum_{k\in J}\|T_k^{-1}\|^{-p}}{2^p|J|(M+1)}\|x\|^p \\
&\le \frac{1}{2^p|J|(M+1)}\sum_{k\in J}\sum_{i\in I}|f_iT_k(x)|^p \\
&= \frac{1}{2^p|J|(M+1)}\sum_{k\in J}\sum_{j\in J}\sum_{i\in\sigma_j}|f_iT_k(x)|^p \\
&\le \frac{1}{|J|(M+1)}\sum_{k\in J}\sum_{j\in J}\big(\sum_{i\in\sigma_j}|f_iT_k(x) - f_iT_j(x)|^p + \sum_{i\in\sigma_j}|f_iT_j(x)|^p\big) \\
&\le \frac{1}{|J|(M+1)}\sum_{k\in J}\sum_{j\in J}\big(M\sum_{i\in\sigma_j}|f_iT_j(x)|^p + \sum_{i\in\sigma_j}|f_iT_j(x)|^p\big) \\
&= \sum_{j\in J}\sum_{i\in\sigma_j}|f_iT_j(x)|^p \le \sum_{j\in J}\sum_{i\in I}|f_iT_j(x)|^p \le \sum_{j\in J}B^p\|T_j(x)\|^p \\
&\le \sum_{j\in J}B^p\|T_j\|^p\|x\|^p \le B^p|J|\max_{j\in J}\|T_j\|^p\|x\|^p.
\end{aligned}$$

Hence

$$\frac{A^p \min_{j\in J}\|T_j^{-1}\|^{-p}}{2^p|J|(M+1)}\|x\|^p \le \sum_{j\in J}\sum_{i\in\sigma_j}|f_iT_j(x)|^p \le B^p|J|\max_{j\in J}\|T_j\|^p\|x\|^p.$$

Thus the proof is completed.. □

Remark 2. *We further observe that if $\big\{\{f_{ij}\}_{i\in I} : j \in J\big\}$ is a woven p-frame for X with universal woven p-frame bounds A, B and $T : X \to X$ is a bounded invertible operator on X, then the family $\big\{\{f_{ij}T\}_{i\in I} : j \in J\big\}$ is also a woven p-frame for X with woven p-frame bounds $A\|T^{-1}\|^{-1}$ and $B\|T\|$, respectively. This claim follows* *from the fact that composing members of a p-frame with a bounded invertible operator is also a p-frame for X.*

In the following theorem, we give a characterization of woven p-basic sequences for X.

Theorem 3.3. *Let $\big\{\{h_{ij}\}_{i\in I}:\ j\in J\big\}$ be a finite family of p-basic sequences for X with p-Riesz bounds A_j, B_j, respectively. Then the following are equivalent:*

- (i) *There are constants $0<A\leq B<\infty$ so that $\big\{\{h_{ij}\}_{i\in I}:\ j\in J\big\}$ is a woven p-basic sequence for X with woven p-basic sequence bounds A and B.*
- (ii) *There exists a constant $M>0$ such that for each partition $\sigma=\{\sigma_j\}_{j\in J}$ of I and $d\in\ell^p$,*

$$M^p\sum_{j\in J}\big\|\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p\leq\big\|\sum_{j\in J}\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p.$$

Proof. Fix the partition $\sigma=\{\sigma_j\}_{j\in J}$ of I and let $d\in\ell^p$. If (i) is satisfied, then we have

$$\begin{aligned}\sum_{j\in J}\big\|\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p&\leq B^p\sum_{j\in J}\sum_{i\in\sigma_j}|d_i|^p=B^p\sum_{i\in I}|d_i|^p\\&\leq\Big(\frac{B}{A}\Big)^p\big\|\sum_{j\in J}\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p.\end{aligned}$$

Thus also (ii) holds.

On the other hand suppose that (ii) holds. Then we have

$$\begin{aligned}\sum_{i\in I}|d_i|^p=\sum_{j\in J}\sum_{i\in\sigma_j}|d_i|^p&\leq\sum_{j\in J}\frac{1}{A_j^p}\big\|\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p\\&\leq\frac{1}{\big(\min_{j\in J}A_j\big)^p}\sum_{j\in J}\big\|\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p\\&\leq\frac{1}{M^p\big(\min_{j\in J}A_j\big)^p}\big\|\sum_{j\in J}\sum_{i\in\sigma_j}d_ih_{ij}\big\|_X^p.\end{aligned}$$

To prove the upper woven p-frame bound, we compute

$$\|\sum_{j\in J}\sum_{i\in\sigma_j} d_i h_{ij}\|_X^p \le 2^p \sum_{j\in J}\|\sum_{i\in\sigma_j} d_i h_{ij}\|_X^p \le 2^p\sum_{j\in J} B_j^p \sum_{i\in\sigma_j}|d_i|^p$$
$$\le 2^p(\max_{j\in J} B_j)^p \sum_{j\in J}\sum_{i\in\sigma_j}|d_i|^p \le 2^p(\max_{j\in J} B_j)^p\sum_{i\in I}|d_i|^p$$

□

Casazza and Lynch [4] proved a characterization theorem for woven Riesz basic sequences in Hilbert spaces. In this direction, we have the following result for woven p-basic sequences in X.

Theorem 3.4. *Let $\{u_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ be p-basic sequences in X with p-Riesz bounds A_1, B_1 and A_2, B_2 respectively. Then the following are equivalent:*

- (i) *There are constants $0 < A \le B < \infty$ so that for every $\sigma \subset I$ the family $\{u_i\}_{i\in\sigma}\cup\{h_i\}_{i\in\sigma^c}$ is a is a p-basic sequence for X with woven p-Riesz bounds A and B.*
- (ii) *There exists a constant $C > 0$ such that for every $\sigma \subset I$ and all $d \in \ell^p$*

$$C^p\|\sum_{i\in\sigma} d_i u_i\|_X^p \le \|\sum_{i\in\sigma} d_i u_i + \sum_{i\in\sigma^c} d_i h_i\|_X^p.$$

- (iii) *There exists a constant $D > 0$ such that for every $\sigma \subset I$ and all $d \in \ell^p$*

$$D^p\big(\|\sum_{i\in\sigma} d_i u_i\|_X^p + \|\sum_{i\in\sigma^c} d_i h_i\|_X^p\big) \le \|\sum_{i\in\sigma} d_i u_i + \sum_{i\in\sigma^c} d_i h_i\|_X^p.$$

- (iv) *There exists a constant $E > 0$ such that for every $\sigma \subset I$ and all $d \in \ell^p$ if $\|\sum_{i\in\sigma} d_i u_i\|_X^p = 1$, then*

$$E^p \le \|\sum_{i\in\sigma} d_i u_i + \sum_{i\in\sigma^c} d_i h_i\|_X^p.$$

Proof. Fix $\sigma \subset I$ and $d \in \ell^p$. First note that $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (iv)$ is always true.

Next to prove $(ii) \Rightarrow (iii)$ by (ii), we have

$$\begin{aligned}\|\sum_{i\in\sigma^c} d_ih_i\|_X^p &\le 2^p\|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p + 2^p\|\sum_{i\in\sigma} d_iu_i\|_X^p \\ &\le 2^p\|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p + \frac{2^p}{C^p}\|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p \\ &= \frac{2^p(1+C^p)}{C^p}\|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p.\end{aligned}$$

This yields

$$\begin{aligned}&\frac{C^p}{2^{p+1}(1+C^p)}\Big(\|\sum_{i\in\sigma} d_iu_i\|_X^p + \|\sum_{i\in\sigma^c} d_ih_i\|_X^p\Big) \\ &\qquad\le \frac{1}{2}\Big(C^p\|\sum_{i\in\sigma} d_iu_i\|_X^p + \frac{C^p}{2^p(1+C^p)}\|\sum_{i\in\sigma^c} d_ih_i\|_X^p\Big) \\ &\qquad\le \|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p.\end{aligned}$$

To prove $(iv) \Rightarrow (ii)$. This is obvious if $\sum_{i\in\sigma} d_iu_i = 0$. So assume not and by (iv), we have

$$E^p \le \frac{1}{\|\sum_{i\in\sigma} d_iu_i\|_X^p}\|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p.$$

Thus (ii) follows. These show the equivalences of (ii), (iii), and (iv).

For the implication $(i) \Rightarrow (ii)$ by (i), we have

$$\begin{aligned}\|\sum_{i\in\sigma} d_iu_i\|_X^p \le B^p\sum_{i\in\sigma}|d_i|^p &\le B^p\big(\sum_{i\in\sigma}|d_i|^p + \sum_{i\in\sigma^c}|d_i|^p\big) \\ &\le \big(\frac{B}{A}\big)^p\|\sum_{i\in\sigma} d_iu_i + \sum_{i\in\sigma^c} d_ih_i\|_X^p.\end{aligned}$$

Finally, to prove $(iii) \Rightarrow (i)$ we have

$$\begin{aligned}\sum_{i\in I}|d_i|^p &= \sum_{i\in\sigma}|d_i|^p + \sum_{i\in\sigma^c}|d_i|^p \leq \frac{1}{A_1^p}\Big\|\sum_{i\in\sigma}d_iu_i\Big\|_X^p + \frac{1}{A_2^p}\Big\|\sum_{i\in\sigma^c}d_ih_i\Big\|_X^p \\ &\leq \max\Big\{\frac{1}{A_1^p},\frac{1}{A_2^p}\Big\}\Big(\Big\|\sum_{i\in\sigma}d_iu_i\Big\|_X^p + \Big\|\sum_{i\in\sigma^c}d_ih_i\Big\|_X^p\Big) \\ &\leq \frac{1}{D^p}\max\Big\{\frac{1}{A_1^p},\frac{1}{A_2^p}\Big\}\Big\|\sum_{j\in J}\sum_{i\in\sigma_j}d_ih_{ij}\Big\|_X^p.\end{aligned}$$

This gives the required lower woven p-frame bound. To prove the upper woven p-frame bound, we compute

$$\begin{aligned}\Big\|\sum_{i\in\sigma}d_iu_i + \sum_{i\in\sigma^c}d_ih_i\Big\|_X^p &\leq 2^p\Big\|\sum_{i\in\sigma}d_iu_i\Big\|_X^p + 2^p\Big\|\sum_{i\in\sigma^c}d_ih_i\Big\|_X^p \\ &\leq 2^pB_1^p\sum_{i\in\sigma}|d_i|^p + 2^pB_2^p\sum_{i\in\sigma^c}|d_i|^p \\ &\leq 2^p\max\big\{B_1^p,B_2^p\big\}\sum_{i\in I}|d_i|^p.\end{aligned}$$

□

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(1) Department of Mathematics, Faculty of Science, Islamic Azad University, Central Tehran Branch, Tehran, Iran.

Email address: msasgari@yahoo.com moh.asgari@iauctb.ac.ir

(2) Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

Email address: shashemi61@yahoo.com