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APPROXIMATIVE RECONSTRUCTION PROPERTY IN BANACH SPACES

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Abstract. Casazza and Christensen in [5] introduced and studied the reconstruction property in Banach spaces. In this paper, we defined approximative reconstruction property(ARP) in a Banach space and give examples for the existence of ARP. A necessary and sufficient condition for a Banach space to have an approximative reconstruction property is given. Finally, we give some Paley-Wiener type perturbations results concerning the approximative reconstruction property in Banach spaces.

1. Introduction

D.Gabor [14], in 1946, introduced a fundamental approach to signal decomposition in terms of elementary signals. Duffin and Schaeffer [11] in 1952, while addressing some deep problem in non-harmonic Fourier series,abstracted Gabor's method to define frames for Hilbert spaces. Coifman and Weiss in [9] introduced the notion of atomic decomposition for certain function spaces. Later, Feichtinger and Gröchenig [12, 13] extended the notion of Hilbert frames to Banach spaces. This concept was further generalized by Gröchenig [15] who defined the notion of Banach frames for Banach spaces. Casazza, Han and Larson [4] also carried out a study of atomic decompositions and Banach frames. Recently, various generalization of frames in

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Banach spaces have been introduced and studied [1, 3, 10, 17]. Han and Larson [16] defined a Schauder frame for a Banach space E to be an inner direct summand (i.e. a compression) of a Schauder basis of E. Schauder frames were further studied in [18, 19, 20, 23]. The reconstruction property in Banach spaces was introduced and studied by Casazza and Christensen in [5] and further studied in [8, 24, 25]. The reconstruction property is an important tool in several areas of mathematics and engineering. As the perturbation result of Paley and Wiener preserves reconstruction property, it becomes more important from an application point of view. The reconstruction property is also used to study some geometrical properties of Banach spaces. In fact, it is related to bounded approximation property as observed in [2]. In the present paper, we shall extend this study further and introduce approximative reconstruction property in Banach spaces.

2. Approximative Reconstruction Property

We begin this section with the following definition of approximative reconstruction property.

Definition 2.1. Let E be a separable Banach space. A sequence $\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \subset E^*$ (where $\{m_n\}$ is an increasing sequence of positive integers) has *approximative reconstruction property*(ARP) for E with respect to a sequence $\{f_n\} \subset E$ if

$$f = \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) f_i, \quad \forall f \in E.$$

We will say that the pair $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ has the approximative reconstruction property for E. More precisely we say that $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ is an approximative reconstruction system for E or E has approximative reconstruction system.

Regarding the existence of approximative reconstruction property, we have the following example.

Example 2.1. *Let $E = c_0$ and $\{e_n\}$ be the sequence of standard unit vectors in E . Define a sequence $\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,n \\ n\in\mathbb{N}}} \subset E^*$ by*

$$f^*_{n,i}(f) = \xi_i, i = 1, 2, ..., n, \quad n \in \mathbb{N}, \quad f \in E.$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f^*_{n,i}(f)e_i = \lim_{n\to\infty} \sum_{i=1}^{n} \xi_i e_i = f.$$

Hence $(\{e_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,n \\ n\in\mathbb{N}}})$ has approximative reconstruction property for E.

Remark 1. (1) If a pair of sequence $(\{f_n\}, \{f^*_n\})$ has reconstruction property for E. Write $f_{n,i} = f_i, \quad i = 1, 2, ..., n.$ Then the pair of a sequence $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}})$ (where $\{m_n\}$ is an increasing sequence of positive integer) has approximative reconstruction property for E. Indeed, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f^*_{n,i}(f)f_i = \lim_{n\to\infty} \sum_{i=1}^{n} f^*_i(f)f_i = f, f \in E.$$

(2) Let $E = c_0$ so $E^* = \ell^1$ and $\{e_n\}$ and $\{e^*_n\}$ be sequence of standard unit vector in E and E^* respectively. Define $\{f_n\} \subset E$ and $\{f^*_n\} \subset E^*$ by

$$f_1 = e_1, f_2 = e_1, f_i = e_{i-1}, n = 3, 4, \dots \text{ and } f^*_1 = e^*_1, f^*_2 = e^*_1, f^*_i = e^*_{n-1}, n \geq 3.$$

Then $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f^*_i(f)f_i \neq f$.

So the pair $(\{f_n\}, \{f^*_n\})$ does not have a reconstruction property for E. Now define

$$f^*_{1,1} = \frac{f^*_1}{2}, \quad f^*_{2,1} = \frac{f^*_1}{2}, \quad f^*_{2,2} = f^*_2, \text{and} \quad f^*_{n,i} = f^*_i, \text{ for all } \ i \geq 3.$$

Note that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f^*_{n,i}(f)f_i = f, \quad \text{for all} \quad f \in E.$$

Thus the pair $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,n \\ n\in\mathbb{N}}})$ has the approximative reconstruction property for E.

In the following result, we give a duality type result:

Lemma 2.1. *If* $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}})$ *has approximative reconstruction property for* E*, then* $(\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,n \\ n\in\mathbb{N}}}, \{f_n\})$ *has approximative reconstruction property for* E^* *in* w^**-topology.*

Proof. For any $f \in E$ and $\phi \in E^*$, we have

$$\begin{aligned}\phi(f) &= \phi\Big(\lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(f)f_i\Big)\\ &= \Big(\lim_{n\to\infty}\sum_{i=1}^{m_n} \phi(f_i)f^*_{n,i}(f)\Big)\\ &= \Big(\lim_{n\to\infty}\sum_{i=1}^{m_n} \phi(f_i)f^*_{n,i}\Big)(f)\end{aligned}$$

and $\phi = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n} \phi(f_i)f^*_{n,i}$ with convergence in w^* topology. □

Next, we prove that a Banach space E has ARP, then the image of E under a continuous linear projection also have ARP.

Proposition 2.1. *Let* E *be a Banach space and let* $P : E \to E$ *be a continuous linear projection. If* $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}})$ *has approximative reconstruction property for* E *then* $(P(\{f_n\}), P^*(\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}))$ *has approximative reconstruction property for* $P(E)$.

Proof. Since $P^*(f^*_n)(f) = f^*_n(P(f)) = f^*_n(f)$, for all $f \in P(E)$ and $n \in \mathbb{N}$.S we have

$$\begin{aligned}f = P(f) &= P\Big(\lim_{n\to\infty}\sum_{i=1}^{n} f^*_{n,i}(f)f_i\Big)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f^*_{n,i}(f)Pf_i\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} P^*f^*_{n,i}(f)Pf_i.\end{aligned}$$

Hence $(P(\{f_n\}), P^*(\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}))$ has approximative reconstruction property for $P(E)$. □

Definition 2.2. Let E be a Banach space and let $\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \subset E^*$ has approximative reconstruction property for E with respect to a sequence $\{f_n\} \subset E$ where$\{m_n\}$ is an increasing sequence of positive integers. Let E_d be Banach space of scalar valued sequences with basis $\{e_i\}$. We define bounded linear operators $S : E_d \to E$ and $T : E \to E_d$ by

$$S(\{\alpha_i\}) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i f_i \quad \text{and} \quad T(f) = \{f^*_{n,i}(f)\}_{\substack{i=1,2,\dots m_n\\ n\in\mathbb{N}}}.$$

We call S the associated approximative reconstruction operator and T the associated approximative analysis operator.

Next, we prove a necessary and sufficient condition for the existence of an approximative reconstruction property for a Banach space.

Theorem 2.1. *Let E be a Banach space. Then the following statements are equivalent:*
(a)E has approximative reconstruction property.
(b) E is isomorphic to a complemented subspace of a Banach space with a basis.

Proof. $(a) \Rightarrow (b)$ For each $f \in E$, we have

$$\begin{aligned} SoT(f) &= S(\{f^*_{n,i}(f)\}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) f_i \\ &= f. \end{aligned}$$

Then $SoT = I$. Also, S is surjective and T is an isomorphism from E onto $T(E)$. Moreover, since $ToS(E_d) = T(E)$ and $ToSoToS = ToS$, we conclude that ToS is a projection from E_d onto $T(E)$.

$(b) \Rightarrow (a)$ Suppose that E is isomorphic to a complemented subspace of a Banach space E_d having a basis $\{e_i\}$ with the biorthogonal functionals $\{e_i^*\}$. let $T : E \to E_d$ be the isomorphic operator onto $T(E)$ and $P : E_d \to E_d$ be the projection of E_d onto $T(E)$. Then, we have

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{m_n} T^* e_i^*(f) T^{-1} o P(e_i) &= \lim_{n\to\infty}\sum_{i=1}^{m_n} e_i^*(T(f)) T^{-1} o P(e_i) \\
&= T^{-1} o P\Big(\lim_{n\to\infty}\sum_{i=1}^{m_n} e_i^*(T(f)) e_i\Big) \\
&= T^{-1} o P o T(f) \\
&= T^{-1} o T(f) \\
&= f \quad \text{for all} \quad f \in E.
\end{aligned}$$

Hence $(T^{-1}oP(e_i), T^*(e_i^*))$ has approximative reconstruction property for E. □

In the following result, we give a necessary condition for ARP. More precisely, we have

Theorem 2.2. *Let E be a separable Banach space. Let $\{f_{n,i}^*\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \subset E^*$ (where $\{m_n\}$ is an increasing sequence of positive integers) has approximative reconstruction property for E with respect to a sequence $\{f_n\} \subset E$. Let $\{p_n\} \subseteq L(E, E)$ be defined by*

$$p_n(f) = \sum_{i=1}^{m_n} f_{n,i}^*(f) f_i, \quad f \in E \tag{2.1}$$

Then

$$[f_{n,i}^*]_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} = \Big[\bigcup_{n=1}^{\infty} p_n^*(E^*)\Big]$$

and

$$f^* = w^* - \lim_{n\to\infty} p_n^*(f^*), f^* \in E^*.$$

Proof. For each $f \in E$, we have

$$\begin{aligned}(p_n^*(f^*))(f) &= f^*(p_n(f)) \\ &= f^*\Big(\sum_{i=1}^{m_n} f_{n,i}^*(f)f_i\Big),\ f \in E,\ f^* \in E^*, n \in \mathbb{N} \\ &= \Big(\sum_{i=1}^{m_n} f^*(f_i)f_{n,i}^*\Big)(f).\end{aligned}$$

This gives

$$p_n^*(f^*) = \sum_{i=1}^{m_n} f^*(f_i)f_{n,i}^*, \quad \text{for all} \quad f^* \in E^*.$$

Hence

$$[f_{n,i}^*]_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}} = \Big[\bigcup_{n=1}^{\infty} p_n^*(E^*)\Big].$$

Also, we have

$$f^*(f) = \lim_{n\to\infty} f^*(p_n(f)) = \lim_{n\to\infty} p_n^*(f^*)(f),\ \text{for all } f \in E,\ f^* \in E^*.$$

□

We prove the following result related to the existence of approximative reconstruction property in the product space of two Banach spaces.

Theorem 2.3. *Let E and F be two Banach spaces having approximative reconstruction property. Then the product space $E \times F$ also has an approximative reconstruction property.*

Proof. Let $(\{f^*_{n,i}\},\{f_i\})$ and $(\{g^*_{n,i}\},\{g_i\})$ has reconstruction property for E and F respectively. Suppose $h=(f,g)\in E\times F$ be arbitrary, where $f\in E$,$g\in F$. Define $\{h_n\}\subset E\times F$ and $h^*_{n,i}\subset E^*\times F^*$ by

$$\begin{cases} h_{2n}=(0,g_n) \\ h_{2n-1}=(f_n,0), n\in\mathbb{N} \end{cases}$$

$$\begin{cases} h^*_{2n,i}(f,g)=g^*_{n,i}(g) \\ h^*_{2n-1,i}(f)=f^*_{n,i}(f). \end{cases}$$

Then

$$\begin{aligned} \lim_{n\to\infty}\sum_{i=1}^{m_n} h^*_{n,i}(h)h_i &= \lim_{n\to\infty}\sum_{i=1}^{m_n} h^*_{n,i}(f,g)h_i \\ &= \Big(\lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(f)f_i, \lim_{n\to\infty}\sum_{i=1}^{m_n} g^*_{n,i}(g)g_i\Big) \\ &= h, \text{for all} \quad f\in E\times F. \end{aligned}$$

Thus,$\{h^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ has the approximative reconstruction property for $E\times F$ with respect to $\{h_n\}$. □

If E and F are two Banach spaces and T a bounded linear operator from E onto F. In the following result, we prove that if E has ARP, then F also have ARP.

Theorem 2.4. *Let E and F be two Banach spaces.Let T be a bounded linear operator from E onto F possessing inverse.Let E has approximative reconstruction property then F has approximative reconstruction property.*

Proof. Let $\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}\subset E^*$ (where $\{m_n\}$ is an increasing sequence of positive integer) has approximative reconstruction property for E with respect to a sequence $\{f_n\}\subset E$. Suppose $Tf_n=g_n, n\in\mathbb{N}$. Let $g\in F$, then there exists $f\in E$ such that

$Tf = g$. Let

$$f = \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) f_i, \quad \forall f \in E.$$

Then

$$\begin{aligned} g = Tf &= \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) T f_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(T^{-1}g) g_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} (T^{-1})^* f^*_{n,i}(g) g_i, \quad \text{for all } g \in F. \end{aligned}$$

Hence F has approximative reconstruction property. □

3. Perturbation of Approximative reconstruction property

Perturbation theory is an important tool in various areas of mathematics, applied mathematics, and engineering [5, 6, 7]. The pioneers who initiated this theory of perturbation were Paley and Wiener [21, 22]. The basic idea of Paley and Wiener was that a system that is sufficiently close to an orthonormal system (basis) in a Hilbert space is also form an orthonormal system (basis). Since then, a number of generalization of this type of perturbation in the context of Hilbert frames and frames in Banach spaces have been done [8, 6, 7]. In this section we study perturbation of approximative reconstruction property and obtained some Paley Weiner type results.

Theorem 3.1. *Let* $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}})$ *be an approximative reconstruction property for E. Let* $\{g_n\}$ *be a sequence in E which satisfies* $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{m_n} \|f_n - g_n\| \|f^*_{n,i}\| < 1$. *Then there exists a sequence* $\{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ *in* E^* *such that* $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}})$ *has approximative reconstruction property for E.*

Proof. Suppose that $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ has approximative reconstruction property for E. Define the operator $L : E \to\ E$ by

$$Lf = \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) g_i.$$

Then L is well defined bounded linear operator. Fix $f \in E$ then

$$\begin{aligned}\|(I - L)f\| &= \|f - Lf\| \\ &= \| \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) f_i - \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) g_n\| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{m_n} \|f_n - g_n\| \|f^*_{n,i}\| < 1.\end{aligned}$$

Then L is invertible operator. Choose $g^*_{n,i} = (L^{-1})^* f^*_{n,i}$, for all $i = 1, 2, ..., m_n, n \in \mathbb{N}$. Then we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{m_n} g^*_{n,i}(f) g_i &= \lim_{n\to\infty} \sum_{i=1}^{m_n} (L^{-1})^* f^*_{n,i}(f) g_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(L^{-1}f) g_i \\ &= L(L^{-1}f) = f, \text{for all } f \in E.\end{aligned}$$

Hence $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ has approximative reconstruction property for E. □

Theorem 3.2. *Let* $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ *be a reconstruction property for* E. *Let* $\{g_n\}$ *be a sequence in* E *which satisfies* $\sum_{n=1}^{\infty} \dfrac{\|f_n - g_n\|}{\|f_n\|} < \dfrac{1}{K}$, *where* $K = \sup_{f\in S_E} \sup_{m\leq n} \| \sum_{i=m_q}^{m_p} f^*_{n,i}(f) f_i\|$. *Then there exists a sequence* $\{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ *in* E^* *such that the pair* $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ *has approximative reconstruction property for* E.

Proof. Proof is on the same line as Theorem 3.1 □

Theorem 3.3. *Let* $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ *has approximative reconstruction property for* E. *Let* $\{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ *be a sequence in* E^* *which satisfies* $\lim_{n\to\infty} \sum_{i=1}^{m_n} \|(\|f^*_{n,i} - g^*_{n,i})(f)\| f_n\| <$

$\epsilon\|f\|$, $\epsilon \in (0,1)$. *Then there exists a sequence* $\{g_n\}$ *in* E *such that the pair* $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ *has approximative reconstruction property for* E.

Proof. Let $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ has approximative reconstruction property for E. Define

$$Lf = \lim_{n\to\infty}\sum_{i=1}^{m_n}(f^*_{n,i} - g^*_{n,i})(f)f_i.$$

Then L is well defined bounded linear operator. Indeed, we have

$$\begin{aligned}\|Lf\| &= \|\lim_{n\to\infty}\sum_{i=1}^{m_n}(f^*_{n,i} - g^*_{n,i})(f)f_i\| \\ &\leq \lim_{n\to\infty}\sum_{i=1}^{m_n}\|(\|f^*_{n,i} - g^*_{n,i})(f)\|\|f_n\| \\ &\leq \epsilon\|f\|.\end{aligned}$$

Thus $T = I - L$ is a continuous invertible operator. Put $g_n = T^{-1}f_n$. Then, we have

$$\begin{aligned}f &= TT^{-1}f \\ &= T^{-1}f - \lim_{n\to\infty}\sum_{i=1}^{m_n}(f^*_{n,i} - g^*_{n,i})(T^{-1}f)f_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n} g^*_{n,i}(f)g_i, \forall f \in E.\end{aligned}$$

Hence $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ has approximative reconstruction property for E. □

Finally, we prove a Paley-Wiener type perturbation result for an approximative reconstruction property.

Theorem 3.4. *Let* $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ *has approximative reconstruction property for* E. *Let* $\lambda \in (0,1)$ *and* $\{g_n\}$ *be a sequence in* E *such that*

$$\|\lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(f_i - g_i)\| \leq \lambda\|f\| \quad \textit{for all} \quad f \in E.$$

If there exists $U \in B(E,E)$ *such that* $U(f_n) = g_n$ *for all* $n \in \mathbb{N}$*, then there exists a sequence* $\{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ *in* E^* *such that the pair* $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ *has reconstruction property for* E*.*

Proof. Let $(\{f_n\}, \{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})$ has approximative reconstruction property for E.Define the operator $L : E \to E$ by

$$Lf = \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) g_i.$$

Then L is well defined bounded linear operator. Indeed for all $p > q$, $p, q \in \mathbb{N}$, we have

$$\begin{aligned}
&\Big\| \sum_{i=1}^{m_p} f^*_{n,i}(f) g_i - \sum_{i=1}^{m_q} f^*_{n,i}(f) g_i \Big\| \\
&= \Big\| U\Big(\sum_{i=1}^{m_p} f^*_{n,i}(f) f_i - \sum_{i=1}^{m_q} f^*_{n,i}(f) f_i \Big) \Big\| \\
&\leq \|U\| \Big\| \sum_{i=1}^{m_p} f^*_{n,i}(f) f_i - \sum_{i=1}^{m_q} f^*_{n,i}(f) f_i \Big\| \to 0 \qquad \text{as} \quad p, q \to \infty.
\end{aligned}$$

Fix $f \in E$. Then we have

$$\begin{aligned}
\|(I - L)f\| &= \|f - Lf\| \\
&= \| \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) f_i - \lim_{n\to\infty} \sum_{i=1}^{m_n} f^*_{n,i}(f) g_i \| \\
&\leq \lambda \|f\|.
\end{aligned}$$

Therefore $\|I-L\| \leq 1$. Thus L in an invertible operator. Write $g^*_{n,i} = (L^{-1})^* f^*_{n,i}$, for all $i = 1, 2, ..., m_n,\ n \in \mathbb{N}$. Then we have

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{m_n} g^*_{n,i}(f)g_i &= \lim_{n\to\infty}\sum_{i=1}^{m_n}(L^{-1})^* f^*_{n,i}(f)g_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(L^{-1}f)g_i \\ &= L(L^{-1}f) = f, \text{for all} f \in E.\end{aligned}$$

Hence $(\{g_n\}, \{g^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}})$ has approximative reconstruction property for E. □

Finally, we prove the following result which gives a type of representation of the conjugate T^* of a given operator T.

Proposition 3.1. *Let a sequence* $\{f^*_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ *in* E^* *has the reconstruction property for a Banach space* E *with respect to* $\{f_n\}$ *and let* $\{g_n\}$ *be a sequence in* E*. Let* $T \in B(E^*, E^*)$ *be given by*

$$T(f^*) = \lim_{n\to\infty}\sum_{i=1}^{m_n} f^*(f_i - g_i)f^*_{n,i}$$

and let $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}(f^*_{n,i}(f)(f_i - g_i)$ *converges in* E*. Then*

$$T^*\pi_f = \lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(f)(f_i - g_i)$$

where π_f *is given by* $\pi(f^*) = f^*(f)$*, for all* $f \in E^*$.

Proof. Define $L : E \to\ E$ by

$$Lf = \lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(f)(f_i - g_i).$$

Since $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n} f^*_{n,i}(f)(f_i - g_i)$ converges in E, L is well defined bounded linear operator.

Again

$$\begin{aligned} T^*_{\pi(f)}(f) &= (\pi_f)T^*(f) \\ &= T(f^*(f)) \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n} f^*(f_i - g_i)f^*_{n,i}(f) \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n} f^*_{n,i}(f)(f_i - g_i)(f^*). \end{aligned}$$

□

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