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A NEW VIEW ON FUZZY $\mathscr{F}^*$-STRUCTURE HOMOTOPY AND ITS $\mathscr{F}^*$-FUNDAMENTAL GROUP

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Abstract. In this paper, the concept of fuzzy $\mathscr{F}^*$-structure isomorphisms between $\mathscr{F}^*$-fundamental groups are studied. Also it is shown that, for every fuzzy $\mathscr{F}^*$-structure continuous function, there is an induced fuzzy $\mathscr{F}^*$-structure group homomorphism between their $\mathscr{F}^*$-fundamental groups. Further in fuzzy $\mathscr{F}^*$-path connected space, all the $\mathscr{F}^*$-fundamental groups $\pi_1((X,\Im),x_\lambda)$ are fuzzy $\mathscr{F}^*$-isomorphic. Also in fuzzy $\mathscr{F}^*$-path connected space, the $\mathscr{F}^*$-fundamental group $\pi_1((X,\Im),x_\lambda)$ is independent of the fuzzy base point x_λ up to fuzzy $\mathscr{F}^*$-structure isomorphism of groups.

1. Introduction

Zadeh [18] innovated the concept of a fuzzy set in 1965 and Chang [1] gave a note to the fuzzy topological space which provided a natural framework. There are many different approaches to define fuzzy homotopy concept of fuzzy topological spaces. The main problem is to introduce the appropriate definition of fuzzy unit interval. In [15], [16], Salleh and Tap defined fuzzy topology of the unit interval. A fuzzy set in this topology is fuzzy open if only and only if the support of this fuzzy set belongs to usual topology of unit interval. On the basis of this, fuzzy path and

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their production is being introduced and proved that the set of fuzzy path (fuzzy loop) creates groupoid (group). In [14], following the studies of unit interval [6], [17], Salleh defined fuzzy homotopy and proved homotopy invariance of singular homology group of fuzzy topological spaces that have been determined in study [14]. In [8], Klawonn, definition of homotopy groups of fuzzy topological spaces elucidates the definition of homotopy groups of ordinary topological spaces on the basis of Huber's [5] categorical approach. For this, Kan's approach of homotopy. This approach is based on embeddings of complete semi-simplicial complexes into the topological space. Cuvalcioglu and Citil [2] introduced another definition of fuzzy homotopy for the fuzzy sets.

In this paper, first section contains some basic definitions and propositions necessary for the paper. In the second section, fuzzy $\mathscr{F}^*$-loop and fuzzy $\mathscr{F}^*$-loop homotopy are defined. This section contains an equivalence relation defined on a certain collection of fuzzy $\mathscr{F}^*$-loops in a fuzzy $\mathscr{F}^*$-structure space. Further fuzzy $\mathscr{F}^*$-path connected spaces is introduced stating an important property that the $\mathscr{F}^*$-fundamental group $\pi_1((X,\mathfrak{I});x_\lambda)$ is independent of the fuzzy base point x_λ up to fuzzy $\mathscr{F}^*$-structure isomorphism of groups. Also it is proved that, for every fuzzy $\mathscr{F}^*$-structure continuous function, there is an induced fuzzy $\mathscr{F}^*$-structure group homomorphism between their $\mathscr{F}^*$-fundamental groups. Some properties of the $\mathfrak{F}^*$ fundamental group of fuzzy $\mathscr{F}^*$-structure space $(X,\mathfrak{I})$ are established.

2. Preliminaries

In this section, some basic concepts of fuzzy homotopy have been recalled. Also some related results and propositions are studied from various research articles. Some definitions and preliminary results are presented in this section in our form. Throughout this paper, the set of all fuzzy points of X will be denoted by $\mathcal{FP}(X)$. Also the closed unit interval $[0,1]$ will be denoted by I and the set of all fuzzy points of I will be denoted by $\mathcal{FP}(I)$.

Definition 2.1 ([10], Definition 3.1, p. 42)**.** Let (X,τ) be a fuzzy topological space. A fuzzy set $\mu \in I^X$ is called fuzzy irreducible if $\mu \neq 0_X$ and for all fuzzy closed sets $\gamma, \delta \in I^X$ with $\mu \leq (\gamma \vee \delta)$, it follows that either $\mu \leq \gamma$ or $\mu \leq \delta$.

Remark 1 ([10], Remark 3.1, p. 42). Let (X,τ) be a fuzzy topological space. Any $\lambda \in I^X$ is said to be fuzzy irreducible closed if it is both fuzzy irreducible and fuzzy closed.

Definition 2.2 ([10], Definition 3.2, p. 42)**.** Let (X,τ) be a fuzzy topological space and let $\alpha \in I^X$ be a fuzzy open set in (X,τ). Then the collection $\mathfrak{F} = \{\sigma \in I^X : \alpha\, \mathfrak{q}\, \sigma$ and $1-\sigma$ is a fuzzy irreducible closed set in (X,τ) $\}$. Then the collection $\mathfrak{F}$ which is finer than the fuzzy topology τ on X is said to be a $\mathfrak{F}$-structure on X. A nonempty set X with a $\mathfrak{F}$-structure denoted by $(X,\mathfrak{F})$ is said to be fuzzy $\mathfrak{F}$-structure space. Each member of $\mathfrak{F}$ is said to be fuzzy $\mathfrak{F}$-structure open set and the complement of each fuzzy $\mathfrak{F}$-structure open set is said to be fuzzy $\mathfrak{F}$-structure closed. A $\mathfrak{F}$-structure on a nonempty set X together with 0_X is said to be fuzzy $\mathfrak{F}^*$-structure. Then $(X,\mathfrak{F}^*)$ is called a fuzzy $\mathfrak{F}^*$-structure space generated by τ.

Definition 2.3. [13] Let $(X,\mathfrak{F}^*)$ be a fuzzy $\mathfrak{F}^*$-structure space. If $Y \subset X$ and χ_Y is the characteristic function of Y, then the collection $\mathfrak{F}^*_Y = \{\ \lambda|_Y = \lambda \wedge \chi_Y : \lambda \in \mathfrak{F}^*\ \}$ is a fuzzy $\mathfrak{F}^*$-structure on Y, called the fuzzy $\mathfrak{F}^*$ subspace structure and the pair $(Y,\mathfrak{F}^*_Y)$ is called a fuzzy $\mathfrak{F}^*$-structure subspace of $(X,\mathfrak{F}^*)$.

Definition 2.4 ([10], Definition 3.5, p. 42)**.** Let $(X_1,\mathfrak{F}^*_1)$ and $(X_2,\mathfrak{F}^*_2)$ be any two fuzzy $\mathfrak{F}^*$-structure spaces. A function $f : (X_1,\mathfrak{F}^*_1) \to (X_2,\mathfrak{F}^*_2)$ is said to be a fuzzy $\mathfrak{F}^*$-structure continuous function if for each fuzzy $\mathfrak{F}^*$-structure open set $\lambda \in I^{X_2}$, $f^{-1}(\lambda)$ is a fuzzy $\mathfrak{F}^*$-structure open set in $(X_1,\mathfrak{F}^*_1)$.

Definition 2.5 ([10], Definition 4.1, p. 43). Let (X, τ) be any topological space and $(X, \mathfrak{F}^*)$ be fuzzy $\mathfrak{F}^*$-structure space. Let U be the subset of X and χ_U be the fuzzy characteristic function of U. Then the fuzzy $\widetilde{\xi}$-structure introduced by τ is $\mathfrak{F}^*(\tau) = \{\chi_U : U \in \tau\}$ and the pair $(X, \mathfrak{F}^*(\tau))$ is said to be fuzzy $\widetilde{\xi}$-structure space introduced by (X, τ).

Note 2.1. *Let I be the unit interval. Let ζ be an Euclidean topology on I, then $(I, \mathfrak{F}^*(\zeta))$ is a fuzzy $\mathfrak{F}^*$-structure space introduced by the (usual) topological space (I, ζ).*

Definition 2.6 ([10], Definition 6.1, p. 51). Let $\pi_1((X, \mathfrak{F}^*), x_\lambda)$ and $\pi_1((X, \mathfrak{F}^*), x'_\mu)$ be any two fuzzy $\mathfrak{F}^*$-fundamental groups of $(X, \mathfrak{F}^*)$ at x_λ and x'_μ respectively. A function $f : \pi_1((X_1, {\mathfrak{F}_1}^*), x_\lambda) \to \pi_2((Y, {\mathfrak{F}_2}^*), x'_\mu)$ is said to be fuzzy $\mathfrak{F}^*$ structure homomorphism if $f([\theta] \circ [\eta]) = f([\theta]) \circ f([\eta])$ for all $[\theta], [\eta] \in \pi_1((X_1, {\mathfrak{F}_1}^*), x_\lambda)$. Moreover the fuzzy $\mathfrak{F}^*$ structure homomorphism is said to be a fuzzy $\mathfrak{F}^*$ structure isomorphism if it is bijective.

Proposition 2.1. *([10], Proposition 5.3, p. 47). Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space and let $x_\lambda \in \mathscr{F}\mathscr{P}(X)$. Let $\eta_0, \eta_1, \theta_0, \theta_1 \in \Upsilon((X, \mathfrak{F}^*), x_\lambda)$ be any fuzzy $\mathfrak{F}^*$-loops in $(X, \mathfrak{F}^*)$. If $\eta_0 \cong_{\mathfrak{P}} \eta_1$ and $\theta_0 \cong_{\mathfrak{P}} \theta_1$, then $\eta_1 * \theta_1 \cong_{\mathfrak{P}} \eta_0 * \theta_0$.*

Proposition 2.2. *([10], Proposition 5.4, p. 48). Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space. Let $[\alpha], [\beta], [\gamma] \in \pi_1((X, \mathfrak{F}^*), x_\lambda)$ where x_λ is a fuzzy point in X. Then $([\alpha] \circ [\beta]) \circ [\gamma] = [\alpha] \circ ([\beta] \circ [\gamma])$.*

Proposition 2.3. *([10], Proposition 5.5, p. 48). Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space and let $(I, \mathfrak{F}^*(\zeta))$ be any fuzzy $\mathfrak{F}^*$-structure space introduced by (I, ζ). Also let $e : (I, \mathfrak{F}^*(\zeta)) \to (X, \mathfrak{F}^*)$ be the fuzzy $\mathfrak{F}^*$-path defined by $e(t_\zeta) = x_\lambda$ for each t_ζ in $(I, \mathfrak{F}^*(\zeta))$ and x_λ is fuzzy point in X. Then $[\alpha] \circ [e] = [e] \circ [\alpha] = [\alpha]$ for each $[\alpha] \in \pi_1((X, \mathfrak{F}^*), x_\lambda)$.*

Proposition 2.4. *([10], Proposition 5.7, p. 49). Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space and $x_\lambda)$ be fuzzy point in X. Let $[\alpha] \in \pi_1((X, \mathfrak{F}^*), x_\lambda)$. There exists $[\overline{\alpha}] \in \pi_1((X, \mathfrak{F}^*), x_\lambda)$ such that $[\alpha] \circ [\overline{\alpha}] = [\overline{\alpha}] \circ [\alpha] = [e]$.*

Remark 2. From Proposition 2.2, Proposition 2.3, Proposition 2.4, it is seen that $\pi_1((X, \mathfrak{F}^*), x_\lambda)$ forms a group under an operation (namely : multiplication).It is called fuzzy $\mathfrak{F}^*$-fundamental group of $(X, \mathfrak{F}^*)$ based at x_λ.

Definition 2.7 ([10], Definition 5.1, p. 45)**.** Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space and let $\alpha : (I, \mathfrak{F}^*(\zeta)) \to (X, \mathfrak{F}^*)$ be a fuzzy $\mathfrak{F}^*$-path. If the initial point equals the terminal point (i.e) $\alpha(0) = \alpha(1) = x_\lambda$, then the fuzzy $\mathfrak{F}^*$-path is called as fuzzy $\mathfrak{F}^*$-loop at x_λ.

Notation 2.1. *Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space. We denote the collection of all fuzzy $\mathfrak{F}^*$-loops in $(X, \mathfrak{F}^*)$ by $\Upsilon((X, \mathfrak{F}^*), x_\lambda)$. Then the fuzzy point x_λ is called fuzzy base point of $(X, \mathfrak{F}^*)$. If $\alpha \in \Upsilon((X, \mathfrak{F}^*), x_\lambda)$, then $[\alpha]$ denote the fuzzy $\mathfrak{F}^*$-structure path homotopy equivalence class that contains α and $\pi_1((X, \mathfrak{F}^*), x_\lambda)$ denote the set of all fuzzy $\mathfrak{F}^*$-structure path homotopy equivalence classes on $\Upsilon((X, \mathfrak{F}^*), x_\lambda)$. Also we define an operation on $\pi_1((X, \mathfrak{F}^*), x_\lambda)$ by $[\alpha] \circ [\beta] = [\alpha * \beta]$.*

Definition 2.8 ([10], Definition 5.3, p. 46)**.** Let $(I, \mathfrak{F}^*(\zeta))$ and $(I, \mathfrak{F}^*(\omega))$ be any two fuzzy $\mathfrak{F}^*$-structure space introduced by (I, ζ) and (I, ω) respectively. Let $(X, \mathfrak{F}^*)$ be any fuzzy $\mathfrak{F}^*$-structure space. Any two fuzzy $\mathfrak{F}^*$-paths η and θ in $(X, \mathfrak{F}^*)$ from x_λ to x'_μ are said to be a fuzzy $\mathfrak{F}^*$-structure path homotopy (denoted by, $\eta \cong_p \theta$) if there exists fuzzy $\mathfrak{F}^*$-structure continuous function $G : (I, \mathfrak{F}^*(\zeta)) \times (I, \mathfrak{F}^*(\omega)) \to (X, \mathfrak{F}^*)$ such that

$$G(0, t_\omega) = x_\lambda \text{ and } G(1, t_\omega) = x'_\mu \text{ for all } t_\omega \in (I, \mathfrak{F}^*(\omega))$$
$$G(t_\zeta, 0) = \eta(t_\zeta) \text{ and } G(t_\zeta, 1) = \theta(t_\zeta) \text{ for all } t_\zeta \in (I, \mathfrak{F}^*(\zeta))$$

3. $\mathscr{F}^*$-Fundamental Group and its Properties

Throughout this paper, the $\mathscr{F}^*$-fundamental group of fuzzy $\mathscr{F}^*$-structure space $(X, \mathfrak{I})$ at x_λ will be denoted by $\pi_1((X, \mathfrak{I}), x_\lambda)$. In this section, fuzzy $\mathscr{F}^*$-structure isomorphism between $\mathscr{F}^*$-fundamental groups are studied. Also it is proved that, for every fuzzy $\mathscr{F}^*$-structure continuous function, there is a induced fuzzy $\mathscr{F}^*$-structure group homomorphism between their $\mathscr{F}^*$-fundamental groups. Some properties related with $\mathscr{F}^*$-fundamental group are also discussed.

Definition 3.1. Let $(I, \mathfrak{I}(\zeta))$ and $(I, \mathfrak{I}(\omega))$ be any two fuzzy $\mathscr{F}^*$-structure spaces introduced by (I, ζ) and (I, ω) respectively. Let $(X, \mathfrak{I})$ be any fuzzy $\mathscr{F}^*$-structure space and $x_\lambda \in \mathcal{FP}(X)$. Any two fuzzy $\mathscr{F}^*$-loops l_1 and l_2 in $(X, \mathfrak{I})$ at x_λ are said to be a fuzzy $\mathscr{F}^*$-loop homotopy at x_λ (denoted by, $l_1 \cong_{\mathfrak{L}} l_2$) if there exists fuzzy $\mathscr{F}^*$-structure continuous function $\mathscr{G} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ such that

$$\mathscr{G}(0, t_\varpi) = x_\lambda \text{ and } \mathscr{G}(1, t_\varpi) = x_\lambda \text{ for all } t_\varpi \in \mathcal{FP}(I) \text{ in } (I, \mathfrak{I}(\omega)),$$
$$\mathscr{G}(t_\varepsilon, 0) = l_1(t_\varepsilon) \text{ and } \mathscr{G}(t_\varepsilon, 1) = l_2(t_\varepsilon) \text{ for all } t_\varepsilon \in \mathcal{FP}(I) \text{ in } (I, \mathfrak{I}(\zeta)).$$

Proposition 3.1. *Let $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-structure spaces. Let Y and Z be any two subsets of X_1 and $(Y, \mathfrak{I}_Y)$, $(Z, \mathfrak{I}_Z)$ be two fuzzy $\mathscr{F}^*$-structure subspaces of $(X_1, \mathfrak{I}_1)$. Let $1_{X_1} = (\chi_Y \vee \chi_Z)$, where χ_Y and χ_Z are the characteristic functions of Y and Z and also fuzzy $\mathscr{F}^*$-structure closed sets in $(X_1, \mathfrak{I}_1)$. Let $\varphi : (Y, \mathfrak{I}_Y) \to (X_2, \mathfrak{I}_2)$ and $\psi : (Z, \mathfrak{I}_Z) \to (X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-structure continuous functions. If $\varphi|_{Y \cap Z} = \psi|_{Y \cap Z}$, then $\phi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$ defined by*

$$\phi(x) = \begin{cases} \varphi(x), & x \in Y, \\ \\ \psi(x), & x \in Z, \end{cases}$$

is a fuzzy $\mathscr{F}^*$-structure continuous function.

Proof. Let $\sigma \in I^{X_2}$ be any fuzzy $\mathscr{F}^*$-structure closed set in $(X_2, \mathfrak{I}_2)$. Let $\chi_Y, \chi_Z \in I^{X_1}$ be any two fuzzy $\mathscr{F}^*$-structure closed sets in $(X_1, \mathfrak{I}_1)$. Then,

$$\phi^{-1}(\sigma) = \phi^{-1}(\sigma) \wedge 1_{X_1} = \phi^{-1}(\sigma) \wedge (\chi_Y \vee \chi_Z)$$

$$= (\phi^{-1}(\sigma) \wedge \chi_Y) \vee (\phi^{-1}(\sigma) \wedge \chi_Z) = \varphi^{-1}(\sigma) \vee \psi^{-1}(\sigma)$$

Since φ and ψ are fuzzy $\mathscr{F}^*$-structure continuous functions, $\varphi^{-1}(\sigma)$ and $\psi^{-1}(\sigma)$ are fuzzy $\mathscr{F}^*$- structure closed sets in $(Y, \mathfrak{I}_Y)$ and $(Z, \mathfrak{I}_Z)$ respectively. Thus $\phi^{-1}(\sigma)$ is a fuzzy $\mathscr{F}^*$- structure closed set in $(X_1, \mathfrak{I}_1)$. Hence ϕ is a fuzzy $\mathscr{F}^*$-structure continuous function. □

Proposition 3.2. *Let $(I, \mathfrak{I}(\zeta))$ and $(I, \mathfrak{I}(\omega))$ be any two fuzzy $\mathscr{F}^*$-structure spaces introduced by (I, ζ) and (I, ω) respectively. Let $(X, \mathfrak{I})$ be any fuzzy $\mathscr{F}^*$-structure space and let l_1 and l_2 be any two fuzzy $\mathscr{F}^*$-loops in $(X, \mathfrak{I})$ such that $\mathrm{l}_1 \cong_{\mathfrak{L}} \mathrm{l}_2$. Then the relation "$\cong_{\mathfrak{L}}$" is an equivalence relation.*

Proof. Let $(X, \mathfrak{I})$ be a fuzzy $\mathscr{F}^*$-structure space and $(I, \mathfrak{I}(\zeta))$ and $(I, \mathfrak{I}(\omega))$ be any two fuzzy $\mathscr{F}^*$-structure spaces introduced by (I, ζ) and (I, ω) respectively.

(i) Reflexive: Let $l : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ be any fuzzy $\mathscr{F}^*$-loop. Define a function $\mathcal{G} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ such that $\mathcal{G}(t_\varepsilon, t) = l(t_\varepsilon)$ for any fuzzy point $t_\varepsilon \in \mathcal{FP}(I)$ and $t \in I$. Then by Proposition 3.1 , $\mathcal{G}$ is a fuzzy $\mathscr{F}^*$-structure continuous function, $\mathcal{G}(t_\varepsilon, 0) = l(t_\varepsilon)$ and $\mathcal{G}(t_\varepsilon, 1) = l(t_\varepsilon)$. Hence $l \cong_{\mathfrak{L}} l$.

(ii) Symmetric: Let $l_1, l_2 : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ be any two fuzzy $\mathscr{F}^*$-loops. If $l_1 \cong_{\mathfrak{L}} l_2$, then there exists a fuzzy $\mathscr{F}^*$-structure continuous function $\mathcal{G} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ such that $\mathcal{G}(t_\varepsilon, 0) = l_1(t_\varepsilon)$ and $\mathcal{G}(t_\varepsilon, 1) = l_2(t_\varepsilon)$ for each fuzzy point $t_\varepsilon \in \mathcal{FP}(I)$ in $(I, \mathfrak{I}(\zeta))$. Let $\mathcal{H} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ be such that $\mathcal{H}(t_\varepsilon, t) = \mathcal{G}(t_\varepsilon, 1 - t)$, for all $t_\varepsilon \in \mathcal{FP}(I)$ and $t \in I$. By Proposition 3.1, $\mathcal{H}$ is a fuzzy $\mathscr{F}^*$-structure continuous function. Now $\mathcal{H}(t_\varepsilon, 0) = \mathcal{G}(t_\varepsilon, 1) = l_2(t_\varepsilon)$ and $\mathcal{H}(t_\varepsilon, 1) = \mathcal{G}(t_\varepsilon, 0) = l_1(t_\varepsilon)$. Therefore, $l_2 \cong_{\mathfrak{L}} l_1$.

(iii) Transitive: Suppose $l_1, l_2, l_3 : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ are three fuzzy $\mathscr{F}^*$-loops. Let $l_1 \cong_{\mathfrak{L}} l_2$ and $l_2 \cong_{\mathfrak{L}} l_3$. Since $l_1 \cong_{\mathfrak{L}} l_2$, then there is a fuzzy $\mathscr{F}^*$-structure continuous function $\mathcal{H} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ such that $\mathcal{H}(t_\varepsilon, 0) = l_1(t_\varepsilon)$ and $\mathcal{H}(t_\varepsilon, 1) = l_2(t_\varepsilon)$, for all $t_\varepsilon \in \mathcal{FP}(I)$ in $(I, \mathfrak{I}(\zeta))$. Similarly, $l_2 \cong_{\mathfrak{L}} l_3$ which implies there exists $\mathcal{G} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ such that $\mathcal{G}(t_\varepsilon, 0) = l_2(t_\varepsilon)$ and $\mathcal{G}(t_\varepsilon, 1) = l_3(t_\varepsilon)$, for all $t_\varepsilon \in \mathcal{FP}(I)$ in $(I, \mathfrak{I}(\zeta))$. Let $\mathcal{P} : (I, \mathfrak{I}(\zeta)) \times (I, \mathfrak{I}(\omega)) \to (X, \mathfrak{I})$ be defined by

$$\mathcal{P}(t_\varepsilon, t) = \begin{cases} \mathcal{H}(t_\varepsilon, 2t), & 0 \le t \le 1/2 \\ \mathcal{G}(t_\varepsilon, 2t-1)), & 1/2 \le t \le 1 \end{cases}$$

for all $t_\varepsilon \in \mathcal{FP}(I)$ in $(I, \mathfrak{I}(\zeta))$ and $t \in I$. Since $\mathcal{H}$ and $\mathcal{G}$ are fuzzy $\mathscr{F}^*$-structure continuous functions and by Proposition 3.1, $\mathcal{P}$ is a fuzzy $\mathscr{F}^*$-structure continuous function. Now $\mathcal{P}(t_\varepsilon, 0) = \mathcal{H}(t_\varepsilon, 0) = l_1(t_\varepsilon)$ and $\mathcal{P}(t_\varepsilon, 1) = \mathcal{G}(t_\varepsilon, 1) = l_3(t_\varepsilon)$. Therefore, $l_1 \cong_{\mathfrak{L}} l_3$.

Thus the relation "$\cong_{\mathfrak{L}}$" is an equivalence relation. □

Remark 3. Let $(X, \mathfrak{I})$ be any fuzzy $\mathscr{F}^*$-structure space and let $x_\lambda, y_\mu \in \mathcal{FP}(X)$. Let β be any fuzzy $\mathscr{F}^*$-path joining x_λ with y_μ. Also let $l_{x_\lambda}, l_{y_\mu} : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ be any two fuzzy $\mathscr{F}^*$-loops defined such that

$$l_{x_\lambda}(t_\varrho) = x_\lambda \text{ for all } t_\varrho \in \mathcal{FP}(I) \text{ in } (I, \mathfrak{I}(\zeta))$$
$$l_{y_\mu}(t_\varrho) = y_\mu \text{ for all } t_\varrho \in \mathcal{FP}(I) \text{ in } (I, \mathfrak{I}(\zeta))$$

Then

(i) $[l_{x_\lambda}] \circ [\beta] = [l_{x_\lambda} * \beta] = [\beta]$. {On replacing $[e]$ by $[l_{x_\lambda}]$ in the proof of Proposition 2.3}

(ii) $[\beta] \circ [l_{y_\mu}] = [\beta * l_{y_\mu}] = [\beta]$. {On replacing $[e]$ by $[l_{y_\mu}]$ in the proof of Proposition 2.3}

(iii) $[\beta] \circ [\overline{\beta}] = [\beta * \overline{\beta}] = [l_{x_\lambda}]$. where $\overline{\beta}(t) = \beta(1-t)$ {On replacing $[e]$ by $[l_{x_\lambda}]$ in the proof of Proposition 2.4}

(iv) $[\overline{\beta}] \circ [\beta] = [\overline{\beta} * \beta] = [l_{y_\mu}]$. {On replacing $[e]$ by $[l_{y_\mu}]$ in the proof of Proposition 2.4}

(v) If $l_1 \cong_{\mathfrak{L}} l_2$, then $(\bar{\beta} * l_1) * \beta \cong_{\mathfrak{L}} \bar{\beta} * (l_1) * \beta)$

Therefore, l_{x_λ} serves as the left identity and l_{y_μ} serves as the right identity for any $[l\,]$.

Definition 3.2. Let $(X, \mathfrak{I})$ be a fuzzy $\mathscr{F}^*$-structure space and let $(I, \mathfrak{I}(\zeta))$ be any fuzzy $\mathscr{F}^*$-structure space introduced by (I, ζ). Let $x_\lambda, y_\mu \in \mathcal{FP}(X)$ be any two fuzzy points. A fuzzy $\mathscr{F}^*$-structure space $(X, \mathfrak{I})$ is said to be a fuzzy $\mathscr{F}^*$-path connected space if for each fuzzy points $x_\lambda, y_\mu \in \mathcal{FP}(X)$, there exists a fuzzy $\mathscr{F}^*$-path $\beta : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ such that $\beta(0) = x_\lambda$ and $\beta(1) = y_\mu$.

Remark 4. The following proposition 3.3 proves that if $(X, \mathfrak{I})$ is a fuzzy $\mathscr{F}^*$-path connected space, all the $\mathscr{F}^*$-fundamental groups $\pi_1((X, \mathfrak{I}), x_\lambda)$ are fuzzy $\mathscr{F}^*$-isomorphic.

Proposition 3.3. *Let $(X, \mathfrak{I})$ be a fuzzy $\mathscr{F}^*$-path connected space and let $x_\lambda, y_\mu \in \mathcal{FP}(X)$ be any two fuzzy points. If $\pi_1((X, \mathfrak{I}), x_\lambda)$ and $\pi_1((X, \mathfrak{I}), y_\mu)$ are two $\mathscr{F}^*$-fundamental groups of $(X, \mathfrak{I})$ at x_λ and y_μ respectively, then there is a fuzzy $\mathscr{F}^*$-structure isomorphism $\widehat{\gamma}$ from $\pi_1((X, \mathfrak{I}), x_\lambda)$ onto $\pi_1((X, \mathfrak{I}), y_\mu)$.*

Proof. Let $(I, \mathfrak{I}(\zeta))$ be any fuzzy $\mathscr{F}^*$-structure space introduced by (I, ζ). Let β be a fuzzy $\mathscr{F}^*$-path, that is, $\beta : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ be such that $\beta(0) = x_\lambda$ and $\beta(1) = y_\mu$ and $\overline{\beta}$ is the inverse fuzzy $\mathscr{F}^*$-path of β, that is, $\overline{\beta}(t) = \beta(1-t)$ for each $t \in I$. If l is fuzzy $\mathscr{F}^*$ loop based at x_λ, then $\overline{\beta} * l * \beta$ is a fuzzy $\mathscr{F}^*$ loop based at y_μ. Hence let us define a function $\widehat{\gamma}_\beta : \pi_1((X, \mathfrak{I}), x_\lambda) \to \pi_1((X, \mathfrak{I}), y_\mu)$ such that $\widehat{\gamma}_\beta([l\,]) = [\overline{\beta} * l * \beta]$ where $[l\,] \in \pi_1((X, \mathfrak{I}), x_\lambda)$.

To prove $\widehat{\gamma}_\beta$ is well defined. Suppose l_1 and l_2 are any two fuzzy $\mathscr{F}^*$-loops at x_λ. Let $l_1 \cong_{\mathfrak{L}} l_2$,that is l_1 is fuzzy $\mathscr{F}^*$-loop homotopic to l_2 at x_λ. Thus

$$\begin{aligned} l_1 &\cong_{\mathfrak{L}} l_2 \\ \overline{\beta} * l_1 * \beta &\cong_{\mathfrak{L}} \overline{\beta} * l_2 * \beta \ \{By\ Remark\ 3\ (iv)\} \\ [\overline{\beta} * l_1 * \beta] &= [\overline{\beta} * l_2 * \beta] \ \{By\ using\ Proposition\ 2.2\} \\ \widehat{\gamma}_\beta([l_1]) &= \widehat{\gamma}_\beta([l_2]) \end{aligned}$$

Thus, $\widehat{\gamma}_\beta$ is well-defined.

To show that $\widehat{\gamma}_\beta$ and $\widehat{\gamma}_{\overline{\beta}}$ are fuzzy $\mathscr{F}^*$-structure homomorphisms.

$$\begin{aligned} \widehat{\gamma}_\beta([l_1] \circ [l_2]) = \widehat{\gamma}_\beta[l_1 * l_2] &= [\overline{\beta} * (l_1 * l_2) * \beta] \\ &= [(\overline{\beta} * l_1) * (l_2 * \beta)]\{By\ Notation\ 2.1\} \\ &= [\overline{\beta} * l_1] \circ [l_2 * \beta] = [\overline{\beta} * l_1 * l_{x_\lambda}] \circ [l_2 * \beta] \\ &= [\overline{\beta} * l_1 * \beta * \overline{\beta}] \circ [l_2 * \beta] \ \{by\ Remark\ 3(iii)\} \\ &= [(\overline{\beta} * l_1 * \beta) * (\overline{\beta} * l_2 * \beta)] \ \{By\ using\ Notation\ 2.1\} \\ &= [\overline{\beta} * l_1 * \beta] \circ [\overline{\beta} * l_2 * \beta]\widehat{\gamma}_\beta([l_1] \circ [l_2]) \\ &= \widehat{\gamma}_\beta([l_1]) \circ \widehat{\gamma}_\beta([l_2]). \end{aligned}$$

Hence, $\widehat{\gamma}_\beta$ is a fuzzy $\mathscr{F}^*$-structure homomorphism.

Similarly, if β is replaced with $\overline{\beta}$, then $\widehat{\gamma}_{\overline{\beta}} : \pi_1((X, \mathfrak{I}), y_\mu) \to \pi_1((X, \mathfrak{I}), x_\lambda)$ is defined as $\widehat{\gamma}_{\overline{\beta}}([l\,]) = [\beta * l * \overline{\beta}]$. Thus $\widehat{\gamma}_{\overline{\beta}}([l_1] \circ [l_2]) = \widehat{\gamma}_{\overline{\beta}}([l_1]) \circ \widehat{\gamma}_{\overline{\beta}}([l_2])$. Hence, $\widehat{\gamma}_{\overline{\beta}}$ is also a fuzzy $\mathscr{F}^*$-structure homomorphism.

Further for each $[l\,] \in \pi_1((X,\mathfrak{I}),x_\lambda)$,

$$\begin{aligned}(\widehat{\gamma}_{\overline{\beta}} \circ \widehat{\gamma}_\beta)([l\,]) = \widehat{\gamma}_{\overline{\beta}}(\widehat{\gamma}_\beta([l\,])) = \widehat{\gamma}_{\overline{\beta}}[\overline{\beta} * l\, * \beta] &= [\beta * (\overline{\beta} * l\, * \beta) * \overline{\beta}] \\ &= [(\beta * \overline{\beta}) * l\, * (\beta * \overline{\beta})]\ \{by\ Proposition\ 2.2\} \\ &= [l_{x_\lambda} * l\, * l_{x_\lambda}]\ \{by\ Remark\ 3(iii)\} \\ &= [l\,].\end{aligned}$$

Hence $\widehat{\gamma}_{\overline{\beta}} \circ \widehat{\gamma}_\beta$ is an identity function on $\pi_1((X,\mathfrak{I}),x_\lambda)$. Similarly, $\widehat{\gamma}_\beta \circ \widehat{\gamma}_{\overline{\beta}}$ is also an identity function on $\pi_1((X,\mathfrak{I}),x_\lambda)$.

Therefore, $\widehat{\gamma}_\beta$ is a fuzzy $\mathscr{F}^*$-structure isomorphism. □

Remark 5. From Proposition 3.3, it is clear that for a fuzzy $\mathscr{F}^*$-path connected space $(X,\mathfrak{I})$, the $\mathscr{F}^*$-fundamental group $\pi_1((X,\mathfrak{I}),x_\lambda)$ is independent of the fuzzy base point x_λ up to fuzzy $\mathscr{F}^*$-structure isomorphism of groups.

Proposition 3.4. *Let $(X,\mathfrak{I})$ be a fuzzy $\mathscr{F}^*$-structure space and let $x_\lambda, y_\mu \in \mathcal{FP}(X)$ be any two fuzzy points. Let β_1 and β_2 be any two fuzzy $\mathscr{F}^*$-paths joining two fuzzy points x_λ and y_μ in $(X,\mathfrak{I})$. If $\beta_1 \cong_{\mathfrak{P}} \beta_2$, then the fuzzy $\mathscr{F}^*$-structure isomorphisms $\widehat{\gamma}_{\beta_1}$ and $\widehat{\gamma}_{\beta_2}$ are identical.*

Proof. Let $\beta_1 \cong_{\mathfrak{P}} \beta_2$. Then it follows that the inverse fuzzy $\mathscr{F}^*$-paths $\overline{\beta_1}$ and $\overline{\beta_2}$ are also fuzzy $\mathscr{F}^*$-path homotopic (i.e.,)$\overline{\beta_1} \cong_{\mathfrak{P}} \overline{\beta_2}$. For any fuzzy $\mathscr{F}^*$-loop l based at x_λ,

$$\overline{\beta_1} \cong_{\mathfrak{P}} \overline{\beta_2}$$

$$\overline{\beta_1} * l \cong_{\mathfrak{P}} \overline{\beta_2} * l$$

Therefore from $\beta_1 \cong_{\mathfrak{P}} \beta_2$ and $\overline{\beta_1} * l \cong_{\mathfrak{P}} \overline{\beta_2} * l$, by Proposition 2.1, we get

$$\overline{\beta_1} * l * \beta_1 \cong_{\mathfrak{P}} \overline{\beta_2} * l * \beta_2$$

$$[\overline{\beta_1} * l * \beta_1] = [\overline{\beta_2} * l * \beta_2]$$

$$\widehat{\gamma}_{\beta_1}([l\,]) = \widehat{\gamma}_{\beta_2}([l\,])$$

Therefore, the fuzzy $\mathscr{F}^*$-structure isomorphisms $\widehat{\gamma}_{\beta_1}$ and $\widehat{\gamma}_{\beta_2}$ are identical. □

Proposition 3.5. *Let $(X, \mathfrak{I})$ be a fuzzy $\mathscr{F}^*$-path connected space and let $x_\lambda, y_\mu \in \mathcal{FP}(X)$ be any two fuzzy points. Then $\pi_1((X, \mathfrak{I}), x_\lambda)$ is abelian if and only if for each pair of fuzzy $\mathscr{F}^*$-paths β_1 and β_2 joining x_λ with y_μ, $\widehat{\gamma}_{\beta_1}([l\,]) = \widehat{\gamma}_{\beta_2}([l\,])$.*

Proof. Let $\pi_1((X, \mathfrak{I}), x_\lambda)$ be a $\mathscr{F}^*$-fundamental group at x_λ. Suppose $\pi_1((X, \mathfrak{I}), x_\lambda)$ is abelian. If $\overline{\beta_2}$ is a fuzzy $\mathscr{F}^*$-path from y_μ to x_λ, by Remark 3 (iii), $(\beta_1 * \overline{\beta_2})$ is a fuzzy $\mathscr{F}^*$-loop based at x_λ. Then for each $[l\,] \in \pi_1((X, \mathfrak{I}), x_\lambda)$,

$$[\beta_1 * \overline{\beta_2}] \circ [l\,] = [l\,] \circ [\beta_1 * \overline{\beta_2}]$$

$$[\beta_1 * \overline{\beta_2} * l\,] = [l * \beta_1 * \overline{\beta_2}]$$

$$[\beta_1 * \overline{\beta_2} * l\,] \circ [\beta_2] = [l * \beta_1 * \overline{\beta_2}] \circ [\beta_2]$$

$$[\beta_1 * \overline{\beta_2} * l * \beta_2] = [l * \beta_1 * \overline{\beta_2} * \beta_2]$$

$$[\beta_1 * \overline{\beta_2} * l * \beta_2] = [l * \beta_1 * l_{y_\mu}] \ \{by\ Remark\ 3(iv)\}$$

$$[\beta_1 * \overline{\beta_2} * l * \beta_2] = [l * \beta_1] \ \{by\ Remark\ 3(ii)\}$$

$$[\overline{\beta_1}] \circ [\beta_1 * \overline{\beta_2} * l * \beta_2] = [\overline{\beta_1}] \circ [l * \beta_1]$$

$$[\beta_1^{-1} * \beta_1 * \overline{\beta_2} * l * \beta_2] = [\overline{\beta_1} * l * \beta_1]$$

$$[l_{y_\mu} * \overline{\beta_2} * l * \beta_2] = [\overline{\beta_1} * l * \beta_1] \ \{by\ Remark\ 3(iv)\}$$

$$[\overline{\beta_2} * l * \beta_2] = [\overline{\beta_1} * l * \beta_1] \ \{by\ Remark\ 3(ii)\}$$

$$\widehat{\gamma}_{\beta_1}([l\,]) = \widehat{\gamma}_{\beta_2}([l\,]).$$

Conversely, suppose $[l_1], [l_2] \in \pi_1((X,\mathfrak{I}), x_\lambda)$. Let β be a fuzzy $\mathscr{F}^*$-path in $(X,\mathfrak{I})$ joining x_λ with y_μ. Then $l_2 * \beta$ is also a fuzzy $\mathscr{F}^*$-path in $(X,\mathfrak{I})$ joining x_λ with y_μ. Hence by given hypothesis, $\widehat{\gamma}_{l_2*\beta}([l_1]) = \widehat{\gamma}_\beta([l_1])$. Thus

$$\widehat{\gamma}_{l_2*\beta}([l_1]) = \widehat{\gamma}_\beta([l_1])$$

$$[\overline{(l_2 * \beta)} * l_1 * (l_2 * \beta)] = [\overline{\beta} * l_1 * \beta]$$

$$[\overline{\beta} * \overline{l_2} * l_1 * (l_2 * \beta)] = [\overline{\beta} * l_1 * \beta]$$

Multiplying by $[\beta]$ on left and $[\overline{\beta}]$ on right:

$$[\beta] \circ [\overline{\beta} * \overline{l_2} * l_1 * l_2 * \beta] \circ [\overline{\beta}] = [\beta] \circ [\overline{\beta} * l_1 * \beta] \circ [\overline{\beta}]$$

$$[(\beta * \overline{\beta}) * \overline{l_2} * l_1 * l_2 * (\beta * \overline{\beta})] = [(\beta * \overline{\beta}) * l_1 * (\beta * \overline{\beta})]$$

$$[l_{x_\lambda} * \overline{l_2} * l_1 * l_2 * l_{x_\lambda}] = [l_{x_\lambda} * l_1 * l_{x_\lambda}] \ \{by\ Remark\ 3(iii)\}$$

$$[\overline{l_2} * l_1 * l_2] = [l_1]$$

Similarly, we get $[\overline{l_1} * l_2 * l_1] = [l_2]$.

To prove $\pi_1((X,\mathfrak{I}), x_\lambda)$ is abelian, it is enough to prove that, for $[l_1], [l_2] \in \pi_1((X,\mathfrak{I}), x_\lambda)$,

$$[l_1] \circ [l_2] = [l_2] \circ [l_1].$$

Let

$$[l_1] \circ [l_2] = [\overline{l_2} * l_1 * l_2] \circ [\overline{l_1} * l_2 * l_1] = [(\overline{l_2} * l_1 * l_2) * (\overline{l_1} * l_2 * l_1)] \ \{By\ Notation\ 2.1\}$$

$$= [\overline{l_2} * l_1 * (l_2 * \overline{l_1}) * l_2 * l_1] \ \{By\ using\ Associative\ Property\}$$

$$= [\overline{l_2} * l_1 * (\overline{l_1} * l_2) * l_2 * l_1] = [\overline{l_2} * ((\overline{l_1} * l_2) * l_1) * l_2 * l_1]$$

$$= [((\overline{l_1} * l_2) * l_1) * \overline{l_2} * (l_2 * l_1)] = [((\overline{l_1} * l_2) * l_1) * (\overline{l_2} * (l_1 * l_2))]$$

$$= [\overline{l_1} * l_2 * l_1] \circ [\overline{l_2} * l_1 * l_2][l_1] \circ [l_2] = [l_2] \circ [l_1]$$

Therefore, $\mathscr{F}^*$-fundamental group $\pi_1((X,\mathfrak{I}), x_\lambda)$ is abelian. $\square$

Note 3.1. *For each fuzzy $\mathscr{F}^*$-structure space $(X, \mathfrak{I})$, $\pi_1((X, \mathfrak{I}), x_\lambda)$ is the $\mathscr{F}^*$-fundamental group associated with $(X, \mathfrak{I})$. Next we show that, for every fuzzy $\mathscr{F}^*$-structure continuous function $\psi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$, there is a induced fuzzy $\mathscr{F}^*$-structure group homomorphism $\psi_\sharp$ between their $\mathscr{F}^*$-fundamental groups.*

Proposition 3.6. *Let $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-structure spaces and let $\psi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$ be a fuzzy $\mathscr{F}^*$-structure continuous function. If $\psi_\sharp : \pi_1((X_1, \mathfrak{I}_1), x_\lambda) \to \pi_1((X_2, \mathfrak{I}_2), y_\mu)$, then $\psi_\sharp$ is induced fuzzy $\mathscr{F}^*$-structure group homomorphism.*

Proof. Let $x_\lambda \in \mathcal{FP}(X_1)$ and $y_\mu \in \mathcal{FP}(X_2)$ be any two fuzzy points. Let $\pi_1((X_1, \mathfrak{I}_1), x_\lambda)$, $\pi_1((X_2, \mathfrak{I}_2), y_\mu)$ be any two $\mathscr{F}^*$-fundamental groups. Let $l_1, l_2 : (I, \mathfrak{I}(\zeta)) \to (X_1, \mathfrak{I}_1)$ be any two fuzzy $\mathscr{F}^*$-loops in $(X_1, \mathfrak{I}_1)$ based at x_λ. Thus $\psi \circ l_1$ is a fuzzy $\mathscr{F}^*$-loop in $(X_2, \mathfrak{I}_2)$ based at y_μ. Moreover, if $l_1 \cong_{\mathfrak{L}} l_2$ at x_λ and $\mathscr{H}$ is a fuzzy $\mathscr{F}^*$-loop homotopy between l_1 and l_2, then $\psi \circ l_1 \cong_{\mathfrak{L}} \psi \circ l_2$ at y_μ and $\psi \circ \mathscr{H}$ is a fuzzy $\mathscr{F}^*$-loop homotopy from $\psi \circ l_1$ to $\psi \circ l_2$ based at y_μ. Hence let us define a function

$$\psi_\sharp : \pi_1((X_1, \mathfrak{I}_1), x_\lambda) \to \pi_1((X_2, \mathfrak{I}_2), y_\mu)$$

such that $\psi_\sharp([l\,]) = [\psi \circ l]$.

To prove $\psi_\sharp$ is fuzzy $\mathscr{F}^*$-structure homomorphism. Let l_1, l_2 be any two fuzzy $\mathscr{F}^*$-loops based at x_λ. Let

$(\psi \circ (l_1 * l_2))(t)_\varpi = \psi((l_1 * l_2)(t)_\varpi)$

$$= \begin{cases} \psi(l_1(2t) = y_\mu, & \text{if } 0 \leq t \leq \frac{1}{2} \\ \psi(l_2(2t-1) = y_\mu, & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$= \begin{cases} (\psi \circ l_1)(2t) = y_\mu, & \text{if } 0 \leq t \leq \frac{1}{2} \\ \\ (\psi \circ l_2)(2t-1) = y_\mu, & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$= ((\psi \circ l_1) * (\psi \circ l_2))(t)_\varpi$$

for each $t_\varpi \in \mathcal{FP}(I)$. Hence

$$\psi \circ (l_1 * l_2) = (\psi \circ l_1) * (\psi \circ l_2) \tag{3.1}$$

Also

$$\begin{aligned} \psi_\sharp([l_1] \circ [l_2]) &= \psi_\sharp([l_1 * l_2]) \\ &= [\psi \circ (l_1 * l_2)] \ \{by\ definition\ of\ \psi_\sharp\} \\ &= [(\psi \circ l_1) * (\psi \circ l_2)] \ \{by\ Equation\ 3.1\ \} \\ &= \psi_\sharp[l_1] \circ \psi_\sharp[l_2]. \end{aligned}$$

Therefore, $\psi_\sharp$ is a fuzzy $\mathscr{F}^*$-structure group homomorphism. □

Proposition 3.7. *Let $(X_1, \mathfrak{I}_1)$, $(X_2, \mathfrak{I}_2)$ and $(X_3, \mathfrak{I}_3)$ be any three fuzzy $\mathscr{F}^*$-structure spaces. Then the following properties holds:*

(i) *If $\varphi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$ and $\psi : (X_2, \mathfrak{I}_2) \to (X_3, \mathfrak{I}_3)$ are fuzzy $\mathscr{F}^*$-structure continuous functions, then $(\psi \circ \varphi)_\sharp = \psi_\sharp \circ \varphi_\sharp$.*

(ii) *If $\phi : (X, \mathfrak{I}) \to (X, \mathfrak{I})$ is an identity function, then $\phi_\sharp : \pi_1((X, \mathfrak{I}), x_\lambda) \to \pi_1((X, \mathfrak{I}), x_\lambda)$ is also an identity function for all $x_\lambda \in \mathcal{FP}(X)$.*

Proof. (i) Since $\varphi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$, $\varphi_\sharp : \pi_1((X_1, \mathfrak{I}_1), x_\lambda) \to \pi_1((X_2, \mathfrak{I}_2), z_\delta)$ where $x_\lambda \in \mathcal{FP}(X_1)$, $z_\delta \in \mathcal{FP}(X_2)$ and $y_\mu \in \mathcal{FP}(X_3)$. Also since $\psi : (X_2, \mathfrak{I}_2) \to (X_3, \mathfrak{I}_3)$, $\psi_\sharp : \pi_1((X_2, \mathfrak{I}_2), z_\delta) \to \pi_1((X_3, \mathfrak{I}_3), y_\mu)$. Further if

$\psi \circ \varphi : (X_1, \mathfrak{I}_1) \to (X_3, \mathfrak{I}_3)$, then $(\psi \circ \varphi)_\sharp : \pi_1((X_1, \mathfrak{I}_1), x_\lambda) \to \pi_1((X_3, \mathfrak{I}_3), y_\mu)$. Thus for each $[l\,] \in \pi_1((X_1, \mathfrak{I}_1), x_\lambda)$,

$$\begin{aligned} (\psi \circ \varphi)_\sharp([l\,]) &= [(\psi \circ \varphi) \circ l\,] \\ &= [\psi \circ (\varphi \circ l)\,] \\ &= \psi_\sharp([\varphi \circ l\,])\ \{by\ definition\ of\ \psi_\sharp\} \\ &= \psi_\sharp(\varphi_\sharp([l\,])) \\ &= (\psi_\sharp \circ \varphi_\sharp)[l\,]. \end{aligned}$$

Therefore, $(\psi \circ \varphi)_\sharp = \psi_\sharp \circ \varphi_\sharp$.

(ii) Let $\phi : (X, \mathfrak{I}) \to (X, \mathfrak{I})$ be an identity function. So $\phi_\sharp : \pi_1((X, \mathfrak{I}), x_\lambda) \to \pi_1((X, \mathfrak{I}), x_\lambda)$. Then for each fuzzy $\mathscr{F}^*$-loop $l : (I, \mathfrak{I}(\zeta)) \to (X, \mathfrak{I})$ based at x_λ, $\phi \circ l = l$. Thus for each $[l\,] \in \pi_1((X_1, \mathfrak{I}_1), x_\lambda)$, $\phi_\sharp[l\,] = [\phi \circ l\,] = [l\,]$. Hence $\phi_\sharp$ is also an identity function on $\pi_1((X, \mathfrak{I}), x_\lambda)$.

□

Definition 3.3. Let $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-structure spaces and also let $\varphi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$ be any fuzzy $\mathscr{F}^*$-structure continuous function. A function $\varphi_\sharp : \pi_1((X_1, \mathfrak{I}_1), x_\lambda) \to \pi_1((X_2, \mathfrak{I}_2), y_\mu)$ is said to be an induced fuzzy $\mathscr{F}^*$-structure homomorphism if $\varphi_\sharp([l\,]) = [\varphi \circ l]$ where $[l\,] \in \pi_1((X_1, \mathfrak{I}_1), x_\lambda)$.

Proposition 3.8. *Let $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-structure spaces. Let $\psi, \varphi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-structure continuous functions where $\mathcal{Q}$ is fuzzy $\mathscr{F}^*$-structure homotopy between ψ and φ. Let $(I, \mathfrak{I}(\zeta))$ be any fuzzy $\mathscr{F}^*$-structure space introduced by (I, ζ). Also if $\beta : (I, \mathfrak{I}(\zeta)) \to (X_2, \mathfrak{I}_2)$ is a fuzzy $\mathscr{F}^*$-path joining $\psi(x_\lambda)$ with $\varphi(x_\lambda)$ defined by $\beta(t) = \mathcal{Q}(x_\lambda, t)$ where $t \in I$ and $x_\lambda \in \mathcal{FP}(X_1)$, then the following triangle of induced fuzzy $\mathscr{F}^*$-structure homomorphisms is commutative.*

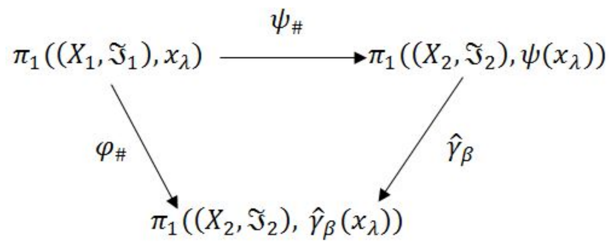

FIGURE 1

Proof. Let $\mathcal{Q}$ be a fuzzy $\mathscr{F}^*$-homotopy between ψ and φ, that is, $\mathcal{Q}:\psi\cong\varphi$ defined as $\mathcal{Q}(x_\lambda,0)=\psi(x_\lambda)$, $\mathcal{Q}(x_\lambda,1)=\varphi(x_\lambda)$ where $x_\lambda\in\mathcal{FP}(X_1)$ is a fuzzy point and let l be a fuzzy $\mathscr{F}^*$-loop in $(X_1,\Im_1)$ based at x_λ, that is, $l(0)=x_\lambda=l(1)$.

It is known that $\psi:(X_1,\Im_1)\to(X_2,\Im_2)$ be any fuzzy $\mathscr{F}^*$-structure continuous function. Thus $\psi_\sharp:\pi_1((X_1,\Im_1),x_\lambda)\to\pi_1((X_2,\Im_2),\psi(x_\lambda))$ be such that $\psi_\sharp[l\,]=[\varphi\circ l]$ where $[l\,]\in\pi_1((X_1,\Im_1),x_\lambda)$ and if $\varphi:(X_1,\Im_1)\to(X_2,\Im_2)$ be any fuzzy $\mathscr{F}^*$-structure continuous function. Thus $\varphi_\sharp:\pi_1((X_1,\Im_1),x_\lambda)\to\pi_1((X_2,\Im_2),\widehat{\gamma}_\beta(x_\lambda))$ be such that $\varphi_\sharp[l\,]=[\varphi\circ l]$ where $[l\,]\in\pi_1((X_1,\Im_1),x_\lambda)$. Also $\widehat{\gamma}_\beta:\pi_1((X_2,\Im_2),\psi(x_\lambda))\to\pi_1((X_2,\Im_2),\widehat{\gamma}_\beta(x_\lambda))$ be such that $\widehat{\gamma}_\beta([l\,])=[\overline{\beta}*l*\beta]$.

It is known that $\varphi_\sharp[l\,]=[\varphi\circ l]$ and $(\widehat{\gamma}_\beta\circ\psi_\sharp)[l\,]=\widehat{\gamma}_\beta[\psi_\sharp\circ l]=[\overline{\beta}*(\psi\circ l)*\beta]$.

Hence it is sufficient to prove that the fuzzy $\mathscr{F}^*$-loops $\overline{\beta}*(\psi\circ l)*\beta$ and $\varphi\circ l$ based at $\varphi(x_\lambda)$ are equivalent. Define a fuzzy $\mathscr{F}^*$-structure continuous function $\mathscr{G}:(I,\Im(\zeta))\times(I,\Im(\zeta))\to(X_2,\Im_2)$ defined by $\mathscr{G}(t,s)=\mathcal{Q}(l(t),s)$ where $t,s\in I$.

Also $\varepsilon:(I,\Im(\zeta))\times(I,\Im(\zeta))\to(I,\Im(\zeta))\times(I,\Im(\zeta))$ is defined for four fuzzy $\mathscr{F}^*$-paths along the edges which are as follows:

$$\varepsilon_1:(0,0)\to(1,0)\ defined\ by\ \varepsilon_1(t)=(t,0)$$

$$\varepsilon_2:(0,1)\to(1,1)\ defined\ by\ \varepsilon_2(t)=(t,1)$$

$$\varepsilon_3:(0,0)\to(0,1)\ defined\ by\ \varepsilon_3(t)=(0,t)$$

$$\varepsilon_4:(1,0)\to(1,1)\ defined\ by\ \varepsilon_4(t)=(1,t)$$

Since $\beta : (I, \Im(\zeta)) \to (X_2, \Im_2)$ is a fuzzy $\mathscr{F}^*$-path joining $\psi(x_\lambda)$ with $\varphi(x_\lambda)$ defined

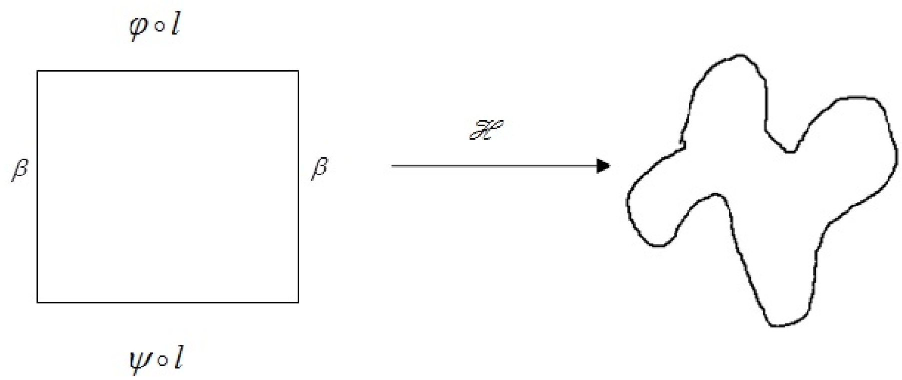

Figure 2

by $\beta(t) = \mathcal{Q}(x_\lambda, t)$ where $t \in I$. Thus

$$(\mathrm{G} \circ \varepsilon_2)(t) = (\varphi \circ l)(t)$$

$$(\mathscr{G} \circ \varepsilon_3)(t) = \mathscr{G}(\varepsilon_3(t)) = \mathscr{G}(0, t) = \mathcal{Q}(l(0), t)$$

$$(\mathscr{G} \circ \varepsilon_3)(t)$$

$$= \beta(t)$$

$$(\mathscr{G} \circ \varepsilon_4)(t) = \mathscr{G}(\varepsilon_4(t)) = \mathscr{G}(1, t) = \mathcal{Q}(\mathrm{l}(1), t) = \mathcal{Q}(x_\lambda, t)$$

$$(G \circ \varepsilon_4)(t) = \beta(t)$$

$$(\mathrm{G} \circ \varepsilon_4)(t) = \mathscr{G}(\varepsilon_4(t))$$

$$=\mathrm{G}(1,\ \mathrm{t})$$

$$=\mathrm{Q}(l(1),\ \mathrm{t}) = \mathrm{Q}(\mathrm{x}_\lambda, t)$$

$$(\mathrm{G} \circ \varepsilon_4)(t) = \beta(t)$$

Our aim is to prove that there exist a fuzzy $\mathscr{F}^*$-structure homotopy $\mathscr{H} : (I, \Im(\zeta)) \times (I, \Im(\zeta)) \to (X_2, \Im_2)$ which starts with the fuzzy $\mathscr{F}^*$-loop $\overline{\beta} * (\psi \circ l) * \beta$ and terminates with the fuzzy $\mathscr{F}^*$-loop $\varphi \circ l$ and also the fuzzy end points are fixed.

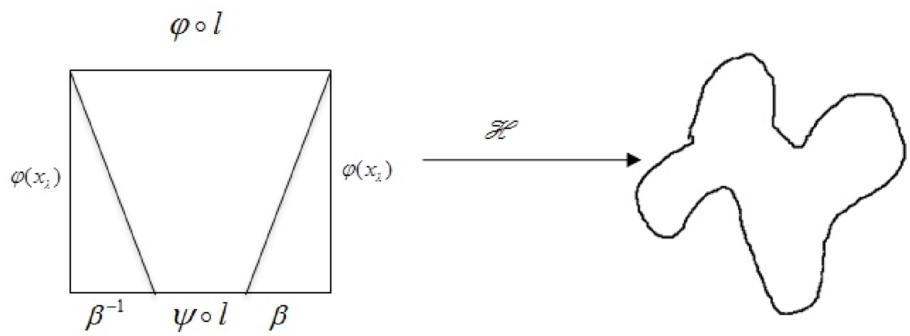

FIGURE 3

Hence the required fuzzy $\mathscr{F}^*$-structure homotopy is defined by

$$\mathscr{H}(t_{\varpi}, s) = \begin{cases} \overline{\beta}(2t) & \text{if } 0 \leq t \leq \frac{1-s}{2} \text{ and } 0 \leq s \leq 1 \\ \mathscr{G}\left(\left(\dfrac{4t+2s-2}{3s+1}\right), s\right) & \text{if } \frac{1-s}{2} \leq t \leq \frac{s+3}{4} \text{ and } 0 \leq s \leq 1 \\ \beta(4t-3) & \text{if } \frac{s+3}{4} \leq t \leq 1 \text{ and } 0 \leq s \leq 1. \end{cases}$$

$$= \begin{cases} \overline{\beta}(0) = y_\mu, & \text{if t} = 0 \\ \overline{\beta}(t) = z_\delta, & \text{if } 0 < t < \frac{1-s}{2} \text{ and } 0 \leq s \leq 1 \\ \overline{\beta}(1-s) = x_\lambda, & \text{if } t = \frac{1-s}{2} \text{ and } 0 \leq s \leq 1 \\ \mathscr{G}(0,s) = \beta(s), & \text{if } t = \frac{1-s}{2} \text{ and } 0 \leq s \leq 1 \\ \mathscr{G}(1,s) = \beta(s), & \text{if } t = \frac{s+3}{4} \text{ and } 0 \leq s \leq 1 \\ \beta(s), & \text{if } t = \frac{s+3}{4} \text{ and } 0 \leq s \leq 1 \\ \beta(1) = y_\mu, & \text{if t} = 1. \end{cases}$$

Therefore $\overline{\beta} * (\psi \circ l) * \beta \cong \varphi \circ l$ and

$$\begin{aligned} \varphi_\sharp[l\,] &= [\varphi \circ l] \\ &= [\overline{\beta} * (\psi \circ l) * \beta] \\ &= \widehat{\gamma}_\beta[\psi \circ l] \\ &= \widehat{\gamma}_\beta(\psi_\sharp[l\,]) \\ &= (\widehat{\gamma}_\beta \circ \psi_\sharp)[l\,]. \end{aligned}$$

Since l is arbitrary, it is verified that $\varphi_\sharp = \widehat{\gamma}_\beta \circ \psi_\sharp$. □

Proposition 3.9. *Let $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ be any two fuzzy $\mathscr{F}^*$-path connected spaces which are of same fuzzy $\mathscr{F}^*$-homotopy type. Then their corresponding $\mathscr{F}^*$-fundamental groups are fuzzy $\mathscr{F}^*$-structure isomorphic.*

Proof. Let $\pi_1((X_1, \mathfrak{I}_1), x_\lambda)$ and $\pi_1((X_2, \mathfrak{I}_2), y_\mu)$ be any two $\mathscr{F}^*$-fundamental groups of $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ respectively. By Remark 5, $\mathscr{F}^*$-fundamental groups $\pi_1((X_1, \mathfrak{I}_1), x_\lambda)$ and $\pi_1((X_2, \mathfrak{I}_2), y_\mu)$ of $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ respectively, are independent of the fuzzy base points x_λ, y_μ because each of them are fuzzy $\mathscr{F}^*$-path connected spaces. Since $(X_1, \mathfrak{I}_1)$ and $(X_2, \mathfrak{I}_2)$ are of same fuzzy $\mathscr{F}^*$-structure homotopy type, there exist fuzzy $\mathscr{F}^*$-structure continuous functions $\psi : (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$ and $\varphi : (X_2, \mathfrak{I}_2) \to$

$(X_1, \mathfrak{I}_1)$ such that $\varphi \circ \psi \cong \mathcal{I}_{X_1}$ by a fuzzy $\mathscr{F}^*$-homotopy, say $\mathscr{F}$ and $\psi \circ \varphi \cong \mathcal{I}_{X_2}$ by some other fuzzy $\mathscr{F}^*$-homotopy, say $\mathscr{G}$. Let $x_\lambda \in \mathcal{FP}(X_1)$ be a fuzzy base point in $(X_1, \mathfrak{I}_1)$. Let

$$\psi_\sharp : \pi_1((X_1, \mathfrak{I}_1), x_\lambda) \to \pi_1((X_2, \mathfrak{I}_2), \psi(x_\lambda))$$

and

$$\varphi_\sharp : \pi_1((X_2, \mathfrak{I}_2), \psi(x_\lambda)) \to \pi_1((X_1, \mathfrak{I}_1), \varphi(\psi(x_\lambda)))$$

be the induced fuzzy $\mathscr{F}^*$-structure homomorphisms. It is enough to prove that $\psi_\sharp$ is a bijective function. Let β be a fuzzy $\mathscr{F}^*$-path joining x_λ to $\varphi(\psi(x_\lambda))$ defined by the fuzzy $\mathscr{F}^*$-homotopy $\mathscr{F}$. Then by Proposition 3.8, the triangle is commutative.

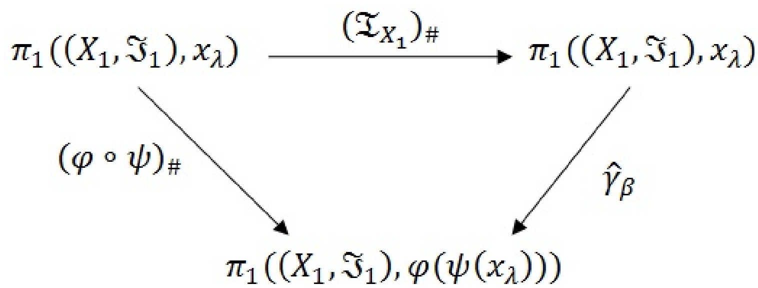

FIGURE 4

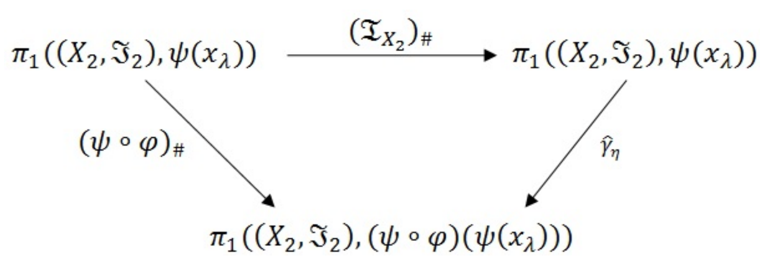

FIGURE 5

That is, $\widehat{\gamma}_\beta : \pi_1((X_2, \mathfrak{I}_2), \psi(x_\lambda)) \to \pi_1((X_2, \mathfrak{I}_2), \widehat{\gamma}_\beta(x_\lambda))$ be such that $\widehat{\gamma}_\beta([l]) = [\overline{\beta} * l * \beta]$. Thus $(\varphi \circ \psi)_\sharp = \widehat{\gamma}_\beta \circ (\mathcal{I}_{X_1})_\sharp$ and so $\varphi_\sharp \circ \psi_\sharp = \widehat{\gamma}_\beta$. By Proposition 3.3, $\widehat{\gamma}_\beta$ is fuzzy $\mathscr{F}^*$-structure isomorphism, it is found that $\psi_\sharp$ is a one-to-one function.

Let η be the fuzzy $\mathscr{F}^*$-path joining $\psi(x_\lambda)$ to $(\psi\circ\varphi)(\psi(x_\lambda))$ defined by the fuzzy $\mathscr{F}^*$-homotopy $\mathscr{G}$. Then by Proposition 3.8, the triangle is commutative.
That is,

$$(\psi\circ\varphi)_\sharp = \widehat{\gamma}_\eta \circ (\mathcal{I}_{X_2})_\sharp$$

$$(\psi\circ\varphi)_\sharp = \widehat{\gamma}_\eta$$

$$\varphi_\sharp \circ \psi_\sharp = \widehat{\gamma}_\eta.$$

By Proposition 3.3, $\widehat{\gamma}_\eta$ is fuzzy $\mathscr{F}^*$-structure isomorphism, $\psi_\sharp$ is also onto. Hence $\mathscr{F}^*$-fundamental groups are fuzzy $\mathscr{F}^*$-structure isomorphic. □

Conclusion

The concept of fuzzy $\mathscr{F}^*$-structure isomorphisms between $\mathscr{F}^*$-fundamental groups are studied. Also it is discussed that for every fuzzy $\mathscr{F}^*$-structure continuous function, there is a induced fuzzy $\mathscr{F}^*$-structure group homomorphism $\psi_\sharp$ between their $\mathscr{F}^*$-fundamental groups. Some properties related with $\mathscr{F}^*$-fundamental group are also discussed. Finally some characterization in fuzzy $\mathscr{F}^*$-path connected spaces are also discussed.

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