Jordan Journal of Mathematics and Statistics (JJMS) 14(1), 2021, pp 177 - 196

GEOMETRY OF CONTACT CR-WARPED PRODUCT SUBMANIFOLDS ON NEARLY COSYMPLECTIC MANIFOLDS

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Abstract. In this paper, we study contact CR-warped products submanifolds on nearly cosymplectic manifolds. We work out the characterizations in terms of tensor fields under which a contact CR-submanifold of a nearly cosymplectic manifold reduces to a warped product submanifold. In the beginning, we obtain some theorems and lemmas and then develop the general sharp inequalities for squared norm of the second fundamental form on nearly cosymplectic manifolds.

1. Introduction

B. Y. Chen ([10, 11]) introduced by the notion of CR-warped product submanifolds as a natural generalization of CR-products. Chen [9] studied the notion of a CR-warped product submanifold in a Kaehler manifold. Also he derived inequalities for the second fundamental form in terms of warping functions. Recently, Sahin[21] set up a general inequality for warped product pseudo-slant (also named hemi-slant) isometrically immersed in a Kaehler manifold for mixed totally geodesic. Later, I. Hesigawa and I. Mihai proved an inequality for contact CR-warped product submanifolds of Sasakian manifolds [12]. Moreover, I. Mihai in [16] improved same inequality

2020 *Mathematics Subject Classification.* 53C40, 53C42, 53C15.

Key words and phrases. Inequalities, mean curvature, submanifolds, warped product, nearly cosymplectic manifolds, mixed totally geodesic.

Received: July 14 , 2019 Accepted: Nov. 17, 2020 .

for contact CR-warped product submanifolds of Sasakian space form. Later on, Uddin, et al.[14] obtained some inequalities of warped product submanifolds in different structures.

The almost contact manifolds with Killing structures tensors were defined in [4] as nearly cosymplectic manifolds. Later, these manifolds were studied by Blair and Showers from the topological point of view [5]. A totally geodesic hypersurface S^5 of a 6-dimensional sphere S^6 is a nearly cosymplectic manifold. A normal nearly cosymplectic manifold is cosymplectic (see [6]). In [22], [23] Uddin et al. worked on warped product semi-invariant and semi-slant submanifolds of nearly cosymplectic manifolds. Warped product manifolds take an important role in differential geometry specially in general relativity. Several authors studied different types of warped product slant submanifolds of different structures in [1], [3], [15], [17], [18], [19], [20]. In the present paper, we define a contact CR-submanifold in a nearly cosymplectic manifolds and derive the integrability conditions and totally geodesic foliation of involving distributions. Here, we also consider contact CR-warped product submanifolds of a nearly cosymplectic generalized cosymplectic space form and obtained a geometric inequality for existence of contact CR-warped product submanifolds. Lastly, we obtain some results on characterizations of warped product submanifolds in terms of endomorphisms T and F.

2. Preliminaries

Let $\tilde{M}$ be $(2n+1)$-dimensional smooth manifold with almost contact structure (φ, ξ, η) i.e., a (1, 1) tensor field φ, a vector field ξ and a 1-form η on $\tilde{M}$ such that

$$\varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X). \tag{2.4}$$

A nearly cosymplectic manifold [5] is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0 \tag{2.5}$$

for every vector field X, Y. i.e.,

$$(\nabla_X \varphi)X = 0. \tag{2.6}$$

Given an almost contact metric manifold $\tilde{M}$, it is said to cosymplectic space form [2], if there exist a function c on $\tilde{M}$ such that

$$\begin{aligned} \tilde{R}(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \frac{c}{4}\{g(X,\varphi Z)\varphi Y \\ &\quad - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z + \frac{c}{4}\{\eta(X)\eta(Z)Y \\ &\quad - \eta(X)\eta(Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned} \tag{2.7}$$

for any vector fields X, Y, Z on $\tilde{M}$, where $\tilde{R}$ denotes the curvature tensor of $\tilde{M}$.
Let M be a submanifold of an almost contact manifold $\tilde{M}$ with induced metric g and ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and normal bundle $T^{\perp}M$ of M respectively. $\mathcal{F}(M)$ denote the algebra of smooth function on M and $\Gamma(TM)$ denotes the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.9}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\tilde{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.10}$$

where g denotes the Riemannian metric on $\tilde{M}$ as well as the one induced on M. The mean curvature vector H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, where n is the dimension of M and $\{e_1, e_2......, e_m\}$ is a local orthonormal frame of vector fields on M. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *totally umbilical* if the second fundamental form satisfies $h(X,Y) = g(X,Y)H$, for all $X, Y \in \Gamma(TM)$. The submanifold M is said to *totaly geodesic* if $h(X,Y) = 0$, for all $X, Y \in \Gamma(TM)$ and minimal if $H = 0$.

Let $\{e_1, e_2......, e_m\}$ is a local orthonormal basis of tangent space TM and e_r belong to the orthonormal basis $\{e_{m+1}, e_{m+2}......, e_{2n+1}\}$ of the normal bundle $T^{\perp}M$, we put

$$h^r_{ij} = g(h(e_i, e_j), e_r) \text{ and } \|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.11}$$

For a diferentiable function φ on M, the gradient $\vec{\nabla}\varphi$ is defined by

$$g(\vec{\nabla}\varphi, X) = X\varphi \tag{2.12}$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\vec{\nabla}\varphi\|^2 = \sum_{i=1}^{m} (e_i(\varphi))^2. \tag{2.13}$$

For any $X \in \Gamma(TM)$,

$$\varphi X = TX + FX, \tag{2.14}$$

where PX is the tangential component and FX is the normal component of φX. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be invariant if F is identically zero, that is $\varphi X \in \Gamma(TM)$ and anti-invariant if P is identically zero, that is $\varphi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

Similarly, for $N \in T^{\perp}M$, we can write

$$\varphi N = tN + fN, \tag{2.15}$$

where tN and fN are the tangential and normal components of φN respectively. The covariant differentiation of the tensors φ, T, F, t and f are defined as respectively

$$(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y, \tag{2.16}$$

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.17}$$

$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{2.18}$$

$$(\tilde{\nabla}_X t)Y = \nabla_X tN - t\nabla^{\perp}_X N \tag{2.19}$$

$$(\tilde{\nabla}_X f)Y = \nabla_X fN - f\nabla^{\perp}_X N \tag{2.20}$$

Furthermore, for any $X, Y \in TM$; the tangential and normal parts of $(\tilde{\nabla}_X \varphi)Y$ are denoted by $P_X Y$ and $Q_X Y$ i.e.,

$$(\tilde{\nabla}_X \varphi)Y = P_X Y + Q_X Y \tag{2.21}$$

Using (2.1) and (2.16), we have

$$(\tilde{\nabla}_X \varphi)\varphi Y = -\varphi(\tilde{\nabla}_X \varphi Y) - \eta(\nabla_X Y)\xi, \tag{2.22}$$

using (2.8)-(2.18) and (2.21), we have

$$P_X Y = (\tilde{\nabla}_X T)Y - A_{FY}X - th(X, Y), \tag{2.23}$$

$$Q_X Y = (\tilde{\nabla}_X F)Y + h(X, tY) - fh(X, Y), \tag{2.24}$$

Basically $P_X N$ and $Q_X N$ are the tangential and normal parts of $(\tilde{\nabla}_X \varphi)N$ for $N \in T^{\perp}M$. Now we have

$$P_X N = (\tilde{\nabla}_X t)N + TA_N X - A_f NX, \tag{2.25}$$

$$Q_X N = (\tilde{\nabla}_X f)N + h(tN, X) + FA_N X, \tag{2.26}$$

From the equation (2.6) and (2.21), we have

$$P_X Y + P_Y X = 0 \tag{2.27}$$

and

$$Q_X Y + Q_Y X = 0 \tag{2.28}$$

for any $X, Y \in TM$. It is straightforward to verify the following properties of P and Q, which will be further used:

$$\begin{cases} i) P_{X+Y} W = P_X W + P_Y W, \\ ii) Q_{X+Y} W = Q_X W + Q_Y W, \\ iii) P_X W + Z = P_X W + P_X Z, \\ iv) Q_X W + Z = Q_X W + Q_X Z, \\ v) g(P_X Y, W) = -g(Y, P_X W), \\ vi) g(Q_X Y, N) = -g(Y, Q_X N), \\ vii) P_X \varphi Y + Q_X \varphi Y = \varphi(P_X Y + Q_X Y) \end{cases} \tag{2.29}$$

3. Contact CR- Submanifolds

Let M be a submanifold tangent to the structure vector field ξ isometrically immersed into an almost contact metric manifold $\tilde{M}$. Then M is said to be a contact CR-submanifold if there exists a pair of orthogonal distributions $D : p \longrightarrow D_p$ and $D^{\perp} : p \longrightarrow D_p^{\perp}$, for all $p \in M$ such that:

$(i) TM = D \oplus D^{\perp} \oplus < \xi >$, where $< \xi >$ is the 1-dimensional distribution spanned by the structure vector field ξ.
$(ii) D$ is invariant, i.e., $\varphi D \subseteq D$,
$(iii) D^{\perp}$ is anti-invariant, i.e., $\varphi D^{\perp} \subseteq T^{\perp}M$.

Invariant and anti-invariant submanifolds are the special cases of a contact CR-submanifold. If we denote the dimensions of the distributions D and $D^{\perp}$ by d_1 and d_2, respectively. Then M is *invariant* (resp. *anti-invariant*) if $d_2 = 0$(resp. $d_1 = 0$). Moreover, if ν is an invariant subspace under φ of the normal bundle $T^{\perp}M$, then in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = FD^{\perp} \oplus \nu$. Let us denote the orthogonal projections on D and $D^{\perp}$ by P_1 and P_2, respectively. Then, for any $X \in \Gamma(TM)$, we have

$$X = P_1X + P_2X + \eta(X)\xi, \tag{3.1}$$

where $P_1X \in \Gamma(D)$ and $P_2X \in \Gamma(D)^{\perp}$. From (2.10), (2.14) and (3.1), we have

$$TX = \varphi P_1X, FX = \varphi P_2X, \tag{3.2}$$

is straightforward to observe that we obtain:

$$i)TP_2 = 0,\ ii)FP_1 = 0,\ iii)t(T^{\perp}M) \subseteq D^{\perp},\ iv)f(T^{\perp}M) \subset \nu.$$

Now we have following lemma:

Lemma 3.1. *Let M be a contact CR-submanifold of a nearly Cosymplectic manifold $\tilde{M}$. Then $\boldsymbol{D} \oplus \xi$ is integrable if and only if,*

$$2g(\nabla_X Y, Z) = g(A_{\varphi Z}X, \varphi Y) + g(A_{\varphi Z}y, \varphi X).$$

for any $X, Y \in \Gamma(\mathbf{D} \oplus \langle \xi \rangle)$ and $Z \in \Gamma(D^{\perp})$.

Proof. From the equation (2.3) and (2.8), we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi \tilde{\nabla}_X Y, \varphi Z) + \eta(\tilde{\nabla}_X Y)\eta(Z).$$

Since $\eta(Z) = 0$, so, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi \tilde{\nabla}_X Y, \varphi Z).$$

From (2.16), we get

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z).$$

From the Gauss formula and the structure equations and (2.5), we have

$$g(\nabla_X Y, Z) = g(h(X, \varphi Y), \varphi Z) + g((\tilde{\nabla}_Y \varphi)X, \varphi Z).$$

Now from (2.1), (2.2), (2.3), (2.4), (2.8) and (2.16), we have

$$g(\nabla_X Y, Z) = g(h(X, \varphi Y), \varphi Z) + g(h(Y, \varphi X), \varphi Z) - g(\tilde{\nabla}_Y X, Z).$$

Then, by the relation between the second fundamental form and the shape operator, we get our result. □

4. Warped Product Submanifolds

In this section, we discuss the warped product submanifolds of a nearly cosymplectic manifold. These manifolds were studied by Bishop and O'Neil [7],[17]. They defined these manifolds as follows: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f a positive differentiable function on N_1. Then their warped product $M = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_{1*}(X)\|^2 + (f \circ \pi_1)^2 \|\pi_{2*}(X)\|^2,$$

for any vector field X tangent to M, where $*$ is the symbol for the tangent maps. Thus we have,

$$g = g_1 + f^2 g_2.$$

The function f is called the warping function on M. It was proved in [7] that for any $X \in \Gamma(TN_1)$ and $Y \in \Gamma(TN_2)$, the following holds:

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{4.1}$$

where ∇ denote the Levi-Civita connection M. ∇f is the gradient of f and is defined as

$$g(\nabla f, X) = Xf, \tag{4.2}$$

for all $X \in TM$.

A warped product manifold $M = N_1 \times_f N_2$ is said to be trivial if the warping function f is constant. If $M = N_1 \times_f N_2$ is a warped product manifold then the base manifold N_1 is totally geodesic and the fiber N_2 is a totally umbilical submanifold of M, respectively [7].

We recall below several results for warped product manifolds.

Lemma 4.1. [22] *Let a CR-warped product submanifold $M_T \times_f M_\perp$ of a nearly cosymplectic manifold $\tilde{M}$ be, such that M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\tilde{M}$, respectively. Then we have*
$(i)\xi lnf = 0$,
$(ii)g(h(X,Y),\varphi Z) = 0$,
$(iii)g(h(X,Z),\varphi Z) = -(\varphi X lnf)\|Z\|^2$.

Let M be a m-dimensional Riemannian manifold with Riemannian metric g and let $\{e_1, ...e_m\}$ be an orthogonal basis of TM: As a consequence of (4.2), we have

$$\|\nabla f\|^2 = \sum_{i=1}^{m}(e_i(f))^2. \tag{4.3}$$

the laplacian of f is defined by

$$\Delta f = \sum_{i=1}^{m}\{\nabla_{e_i}e_i)f - e_ie_if\}. \tag{4.4}$$

Now Hopf's lemma [8] state:

Hopf's lemma: Let M be a n-dimensional compact Riemannian manifold. If ψ is differentiable function on M such that $\Delta\psi \geqslant 0$ everywhere on M(or $\Delta\psi \leqslant 0$ everywhere on M), then ψ is a constant function.

5. Contact CR-Warped Product Submanifolds

In this section we consider contact CR-warped product of the type $M_T \times_f M_\perp$of the nearly cosymplectic manifolds $\tilde{M}$, where M_T and $M_\perp$ are the invariant and anti-invariant submanifolds of $\tilde{M}$, respectively. Throughout, this section, we consider ξ tangent to M_T. We first discuss several properties of contact CR-warped product submanifolds of a nearly cosymplectic manifold.

For any $X, Y \in \Gamma(TM_T)$, from the properties (2.23) and (2.29)i, we have

$$(\tilde{\nabla}_X T)Y + (\tilde{\nabla}_Y T)X = 2th(X, Y). \tag{5.1}$$

Since, M_T is totaly geodesic in M. Thus, by comparing the component which is tangent to M_T in formula (5.1), we obtain $th(X, Y) = 0$, which implies that $h(X, Y) \in \upsilon$ and

$$(\tilde{\nabla}_X T)Y + (\tilde{\nabla}_Y T)X = 0. \tag{5.2}$$

If we put $Y = \xi$ in the above equation, we have

$$(\tilde{\nabla}_\xi T)X = 0. \tag{5.3}$$

Now we have the following lemma:

Lemma 5.1. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\tilde{M}$. We have:*

$$(\tilde{\nabla}_Z T)X = (TXlnf)Z,$$

$$(\tilde{\nabla}_U T)Z = g(\mathrm{P}_2 U, Z)T\nabla lnf,$$

for each $U \in \Gamma(TM)$, $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$.

Proof. Using (2.17), (3.1) and (4.1), we completes the proof of lemma. □

Using the above lemma we have the following theorem:

Theorem 5.1. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$, with both the distributions integrable. Then M is locally isometric to a CR-warped product if and only if*

$$(\tilde{\nabla}_U T)U = (T\mathrm{P}_1 U\mu)\mathrm{P}_2 U + \|\mathrm{P}_2 U\|^2 T\nabla\mu \tag{5.4}$$

or,

$$(\tilde{\nabla}_U T)V + (\tilde{\nabla}_V T)U = (T\mathrm{P}_1 V lnf)\mathrm{P}_2 U + (T\mathrm{P}_1 U lnf)\mathrm{P}_2 V + 2g(\mathrm{P}_2 U, \mathrm{P}_2 V)T\nabla lnf \tag{5.5}$$

for each $U, V \in \Gamma(TM)$ *and for any* C^∞*-function* μ *on* M*, with* $Z\mu = 0$ *for each* $Z \in \Gamma(D^\perp)$.

Proof. Let M be a contact CR-warped product submanifold of $\tilde{M}$. Now using the property (3.1), we have,

$$(\tilde{\nabla}_U T)U = (\tilde{\nabla}_{P_1} UT)P_1 U + (\tilde{\nabla}_{P_2} UT)P_2 U + \eta(U)(\tilde{\nabla}_\xi T)P_1 U + (\tilde{\nabla}_U T)P_2 U + \eta(U)(\tilde{\nabla}_U T)\xi \tag{5.6}$$

From (5.2) and (5.3) and using the lemma 5.1, we have

$$(\tilde{\nabla}_U T)U = (TP_1 U lnf)P_2 U + \|P_2 U\|^2 T\nabla lnf.$$

Since $lnf = \mu$, we get

$$(\tilde{\nabla}_U T)U = (TP_1 U\mu)P_2 U + \|P_2 U\|^2 T\nabla \mu.$$

□

Which gives the result (5.4) by replacing U by $U + V$ in the above equation, we get (5.5).

Now conversely, suppose that M is a contact CR-submanifold of $\tilde{M}$, with both distributions integrable on M such that (5.5) holds for any C^∞-function μ on M, with $Z\mu = 0$ for each $Z \in \Gamma(D^\perp)$. Now for any $X, Y \in \Gamma(D\oplus <\xi>$ and $P_2 X = 0$, using the equation (5.5), we have

$$(\tilde{\nabla}_X T)Y + (\tilde{\nabla}_Y T)X = 0. \tag{5.7}$$

Since, $\tilde{M}$ is a nearly Cosymplectic manifold. Now using the equation (2.23) and (2.27), we have

$$(\tilde{\nabla}_X T)Y + (\tilde{\nabla}_Y T)X = 2th(X,Y). \tag{5.8}$$

Now from (5.7) and (5.8), we have $th(X,Y) = 0$. From the lemma 5.17, we have $g(\nabla_X Y, Z) = 0$ for each $Z \in \Gamma(D^{\perp})$. Now using the equation (5.5), we obtain

$$(\tilde{\nabla}_Z T)W + (\tilde{\nabla}_W T)Z = 2g(Z,W)T\nabla\mu \tag{5.9}$$

Now using the equation (2.23) and (2.27), we get

$$(\tilde{\nabla}_Z T)W + (\tilde{\nabla}_W T)Z = A_{FW}Z + A_{FZ}W + 2th(Z,W). \tag{5.10}$$

From the equation (5.9) and (5.10), we get

$$A_{FW}Z + A_{FZ}W = 2g(Z,W)T\nabla\mu - 2th(X,Y) \tag{5.11}$$

Using (2.10) and (2.14), we have

$$g(h(Z,\varphi X),\varphi W) + g(h(W,\varphi X),\varphi Z) = 2g(Z,W)g(T\nabla\mu,\varphi X) \tag{5.12}$$

From (2.18), we have

$$g(\tilde{\nabla}_Z \varphi X, \varphi W) + g(\tilde{\nabla}_W \varphi X, \varphi Z) = 2g(Z,W)g(T\nabla\mu,\varphi X).$$

Using the orthogonality of vector fields, we have

$$g(\tilde{\nabla}_Z \varphi W, \varphi X) + g(\tilde{\nabla}_W \varphi Z, \varphi X) = -2g(Z,W)g(T\nabla\mu,\varphi X).$$

Now using the property of (2.16), we have

$$\begin{aligned} -2g(Z,W)g(\varphi\nabla\mu,\varphi X) &= g((\tilde{\nabla}_Z\varphi)W + (\tilde{\nabla}_W\varphi)Z, \varphi X) \\ &\quad + g(\varphi\tilde{\nabla}_Z W, \varphi X) + g(\varphi\tilde{\nabla}_W Z, \varphi X) \end{aligned} \tag{5.13}$$

Now using (2.1), (2.2), (2.3) and (2.4) in (5.13), we have,

$$g(\tilde{\nabla}_Z W, X) + g(\tilde{\nabla}_W Z, X) = -2g(Z, W)g(\nabla\mu, X) + 2g(Z, W)\eta(\nabla\mu)\eta(X),$$

which implies

$$g(\nabla_Z W, X) + g(\nabla_W Z, X) = -2g(Z, W)g(\nabla\mu, X) + 2g(Z, W)(\xi lnf)\eta(X).$$

Since $D^\perp$ is integrable and $\xi lnf = 0$, we have

$$g(\nabla_Z W, X) = -g(Z, W)g(\nabla\mu, X). \tag{5.14}$$

Let $M_\perp$ be a leaf of $D^\perp$ and let $h^\perp$ be the second fundamental form of the immersion of $M_\perp$ into M. Then we have,

$$g(h^\perp(Z, W), X) = -g(Z, W)g(\nabla\mu, X), \tag{5.15}$$

i.e.,

$$h^\perp(Z, W) = -g(Z, W)\nabla\mu.$$

From the above equation, we conclude that $M_\perp$ is totally umbilical in M with the mean curvature vector satisfies $H = -\nabla\mu$. Now we prove that H is parallel corresponding to the normal connection D of $M_\perp$ in M. So,

$$g(D_Z\nabla\lambda, Y) = 0. \tag{5.16}$$

Since $Z(\lambda) = 0$ for all $Z \in \Gamma(D^\perp)$, we obtain $\nabla_Y\nabla\lambda \in \Gamma(D\oplus < \xi >)$. Hence from [13], we conclude that M is a warped product submanifold. So we complete the proof of the theorem.

Now we have a following lemma:

Lemma 5.2. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then*

$$g(h(\varphi X, Z), \varphi h(X, Z)) = \|h(X, Z)\|^2 - g(\phi h(X, Z), \mathrm{Q}_X Z),$$

for any $X \in TN_T$ and $Z \in TN_\perp$.

Proof. Using (2.8) and (2.16), we have

$$h(\varphi X, Z) = (\tilde{\nabla}_Z \varphi)X + \varphi \nabla_Z X + \varphi h(X, Z) - \nabla_Z \varphi X. \tag{5.17}$$

Now from (2.21) and (4.1), we get

$$h(\varphi X, Z) = P_Z X + Q_Z X + X lnf \varphi Z + \varphi h(X, Z) - \varphi X lnf Z, \tag{5.18}$$

If we take the normal part of the above equation, we have

$$h(\varphi X, Z) = Q_Z X + X lnf \varphi Z + \varphi h(X, Z) \tag{5.19}$$

By using (2.28), we get

$$g(h(\varphi X, Z), \varphi h(X, Z)) = \|h(X, Z)\|^2 - g(\varphi h(X, Z), Q_X Z). \tag{5.20}$$

□

Now we prove the following theorem with the help of Hopes's lemma:

Theorem 5.2. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of nearly cosymplectic space form $\tilde{M}(f_1, f_2, f_3)$ such that M_T is compact. Then M is contact CR-product submanifold if either one of the following inequality holds*

i)

$$\sum_{i=1}^{2u} \sum_{j=1}^{v} \|h_\mu(e_i, e^j\|^2 \geqslant 2u.v.\frac{c}{4},$$

where h_μ denotes the component of h in μ, $2u+1$ and v are the dimensions of M_T and $M_\perp$.

Proof. Any unit vector field X tangent to M_T and orthogonal to ξ, and Z tangent to $M_\perp$, then from (2.7), we have

$$\tilde{R}(X, \varphi X, Z, \varphi Z) = -2.\frac{c}{4}.\|X\|^2\|Z\|^2. \tag{5.21}$$

Now from Coddazi equation, we have

$$\begin{aligned} \tilde{R}(X, \varphi X, Z, \varphi Z) &= g(\nabla_X^\perp h(\varphi X, Z), \varphi Z) - g(h(\nabla_X \varphi X, Z), \varphi Z) \\ &\quad - g(h(\varphi X, \nabla_X Z), \varphi Z) - g(\nabla_\varphi X^\perp h(X, Z), \varphi Z) \\ &\quad + g(h(\nabla_{\varphi X} X, Z), \varphi Z) + g(h(\nabla_{\varphi X} Z, X), \varphi Z). \end{aligned} \tag{5.22}$$

Now using (2.8), (2.16), (2.21), we have

$$\begin{aligned} g(\nabla_X^\perp h(\varphi X, Z), \varphi Z) &= Xg(h(\varphi X, Z), \varphi Z) - g(h(\varphi X, Z), \tilde{\nabla}_X \varphi Z) \\ &\quad + X(Xlnf g(Z, Z)) - g(h(\varphi X, Z), (\tilde{\nabla}_X \varphi)Z \\ &\quad + \varphi \tilde{\nabla}_X Z). \end{aligned} \tag{5.23}$$

After some calculations and using lemma 5.2, we have

$$\begin{aligned} g(\nabla_X^\perp h(\varphi X, Z), \varphi Z) &= X^2 lnf g(Z, Z) + (Xlnf)^2 g(Z, Z) - \|h_\mu(X, Z)\|^2 \\ &\quad - g(\varphi h(X, Z) - h(\varphi X, Z), Q_X Z). \end{aligned} \tag{5.24}$$

Using (2.8), (2.16), (2.28) and (4.1) in (5.24), we have,

$$\begin{aligned} g(\nabla_X^\perp h(\varphi X, Z), \varphi Z) &= X^2 lnf g(Z, Z) + (Xlnf)^2 g(Z, Z) \\ &\quad - \|h_\mu(X, Z)\|^2 \end{aligned} \tag{5.25}$$

Similarly, we have

$$\begin{aligned} -g(\nabla_\varphi^\perp h(\phi X, Z), \phi Z) &= (\phi X)^2 lnf g(Z, Z) + (\phi Xlnf)^2 g(Z, Z) \\ &\quad - \|h_\mu(\phi X, Z)\|^2 \end{aligned} \tag{5.26}$$

Also, we have

$$g(A_{\varphi Z}Z, \varphi X) = Xlnf, \tag{5.27}$$

replacing X by $\nabla_X X$,

$$g(A_{\varphi Z}Z, \varphi \nabla_X X) = \nabla_X Xlnf, \tag{5.28}$$

By using the Gauss formula in last equation, we get

$$g(A_{\varphi Z}Z, \varphi(\tilde{\nabla}_X X - h(X, X)) = \nabla_X Xlnf, \tag{5.29}$$

Now using (2.6), (2.8),(2.16) and (4.1), we have $h(X, X) \in \mu$, applying this in (5.29),

$$g(h(\nabla_X \varphi X, Z), \varphi Z) = (\nabla_X X)lnfg(Z, Z), \tag{5.30}$$

Similarly,

$$g(h(\nabla_{\varphi X} X, Z), \varphi Z) = (\nabla_{\varphi X}\varphi X)lnfg(Z, Z), \tag{5.31}$$

Using (2.16), we have

$$g(h(\varphi X, \nabla_X Z), \varphi Z) = (Xlnf)^2 g(Z, Z) \tag{5.32}$$

and

$$g(h(X, \nabla_{\varphi X} Z), \varphi Z) = -(\varphi Xlnf)^2 g(Z, Z). \tag{5.33}$$

Using (5.26), (5.27), (5.31), (5.32), (5.33), (5.34) in (5.23), we have

$$\begin{aligned} \tilde{R}(X, \varphi X, Z, \varphi Z) &= X^2 lnfg(Z, Z) + (\varphi X)^2 lnfg(Z, Z) - \nabla_X Xlnfg(Z, Z) \\ &\quad -\nabla_{\varphi X}\varphi Xlnfg(Z, Z) - \|h_\mu(X, Z)\|^2 \\ &\quad -\|h_(\varphi X, Z)\|^2 \end{aligned} \tag{5.34}$$

Let $\{e_0 = \xi, e_1, e_2, ...e_u, e_{u+1} = \varphi e_1, e_{u+2} = \phi e_2, ..., e_{2u} = \phi e_u, e^1, e^2, ...e^v\}$ be an orthonormal frame of TM such that $\{e_1, e_2, ...e_u, \varphi e_1, \varphi e_2, ..., \varphi e_u\}$ are tangent to

TN_T and $\{e^1, e^2;, ..., e^v\}$ are tangent to $TN_\perp$. Using (5.21) and (4.4) in (5.34) and summing over $i = 1, 2, ..., u$ and $j = 1, 2, ..., v$, we get, we have

$$v\Delta lnf = 2.u.v.\frac{c}{4} - \sum_{i=1}^{2u}\sum_{j=1}^{v} \|h_\mu(e_i, e^j\|^2. \tag{5.35}$$

Using Hope's lemma, we have

$$\sum_{i=1}^{2u}\sum_{j=1}^{v} \|h_\mu(e_i, e^j\|^2 \geq 2.u.v.\frac{c}{4}. \tag{5.36}$$

then M is simply contact CR-product submanifold, which proves the theorem completely. □

Acknowledgement

The authors wish to express their sincere thanks and gratitude to the editor and the referees for the valuable suggestions towards the improvement of the paper.

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