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A RADIUS PROBLEM FOR A CERTAIN CLASS OF SCHLICHT FUNCTIONS

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Abstract. In this work we apply a lemma of Tuan and Anh [5] to solve a radius problem for certain class of schlicht functions defined by a product of expressions having geometric meaning. Many interesting consequences of the result are derived for some well known classes of functions, most especially the novel radius of convexity of order $\frac{1}{2}$ for functions of bounded turning in the unit disk $|z| < 1$.

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots$$

which are holomorphic in the unit disk $E = |z| < 1$. Denote by $P(\beta)$, the class of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$$

holomorphic in E and satisfying Re $p(z) > \beta$ for some real number $0 \leq \beta < 1$.

In [2], the authors introduced and gave some characterizations of a class of functions, $\mathcal{J}_n^\alpha(\beta)$, consisting of $f \in A$ satisfying

$$\text{Re}\, \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} > \beta$$

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where D^n is the Salagean derivative [4] defined by

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]', \ n \in N_0 = \{0, 1, 2, ...\}$$

with $D^0 f(z) = f(z)$ and, α and β are real numbers restricted by $0 \leq \beta < 1$ and $\alpha \geq 0$. Furthermore powers mean principal determinations only. They proved, among others, that for $n \geq 1$, $\mathcal{J}_n^\alpha(\beta)$ consists of schlicht functions in E. They also remarked that for $n = 0$, the cases $\alpha = 0, 1/2$ and 1 coincide respectively with well-known classes of schlicht maps, namely, (i) starlike functions of order β

$$Re \, \frac{zf'(z)}{f(z)} > \beta,$$

(ii) bounded turning functions of order β

$$Re \, f'(z) > \beta,$$

and (iii) Bazilevic functions of type 2, order β

$$Re \, \frac{zf'(z)f(z)^{\alpha-1}}{z^\alpha} > \beta$$

(see Remark 1 [2]).

Oversightedly, the authors missed the fact that the case $n = 0$ generally consists of schlicht Bazilevic maps of type 2α, order β defined by

$$\text{Re} \, \frac{f(z)^{2\alpha-1} f'(z)}{z^{2\alpha-1}} > \beta$$

by which we conclude (in passing) that the new class of functions consists of schlicht functions only and so provides a unified treatment for many known classes of schlicht functions in the unit disk. The result of this paper with its many interesting new and existing corollaries for many classes of functions further justifies the introduction of the new class $\mathcal{J}_n^\alpha(\beta)$.

The object of the present work is to employ a lemma of Tuan and Anh [5] to solve a radius problem for the class $\mathcal{J}_0^\alpha(\beta)$. Some of the consequences of our result include the determination of the radius of the subdisk $|z| < r$ for which functions in $\mathcal{J}_0^\alpha(\beta)$ satisfy certain not-linear combinations (see [1]), and in particular, the radius of convexity for some special cases of the new class.

In the next section we state the lemma of Tuan and Anh [5] on which we depend for our main result, which we prove in Section 3 of the paper.

2. The Lemma

Lemma 2.1. [5] *Let $p \in P$. Then for $0 \leq \beta < 1$*

$$Re\ \frac{zp'(z)}{\frac{\beta}{1-\beta} + p(z)} \geq \begin{cases} -\frac{2(1-\beta)r}{(1+r)(1+(2\beta-1)r)}, & \text{for } R_1 \leq R_2, \\ -\frac{\beta}{1-\beta} + \frac{1}{1-\beta}\left(2R_1 - \frac{1-(2\beta-1)r^2}{1-r^2}\right), & \text{for } R_2 \leq R_1. \end{cases}$$

where $R_1 = \left(\frac{\beta-\beta(2\beta-1)r^2}{1-r^2}\right)^{\frac{1}{2}}$ and $R_2 = \frac{1+(2\beta-1)r}{1+r}$. The functions given by

$$p(z) = \begin{cases} \frac{1-z}{1+z}, & \text{for } R_1 \leq R_2, \\ \frac{1}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right), & \text{for } R_2 \leq R_1. \end{cases}$$

show that the inequalities are sharp, where $\cos\theta$ satisfies the equation

$$\begin{aligned}(2R_1 - a - \alpha) &- 2\cos\theta[(2R_1 - a - \alpha)(1+\alpha) \\ &+ (1-\alpha)^2]r + [2\alpha(2R_1 - a - \alpha)(1 + 2\cos^2\theta) + 4(1-\alpha)^2]r^2 \\ &- 2\cos\theta[(2R_1 - a - \alpha)(3\alpha - 1) + (1-\alpha)^2]r^3 + (2\alpha - 1)r^4 = 0\end{aligned}$$

with $a = (1 - (2\alpha - 1)r^2)/(1 - r^2)$.

3. Main Result

The following radius problem is our main result. Some important consequences are indicated after the proof.

Theorem 3.1. *Let $f \in \mathcal{J}_n^\alpha(\beta)$. Then*

$$Re\left(\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2}f(z)^\alpha}{D^{n+1}f(z)^\alpha}\right) > 2\rho,\ \rho \geq -\frac{1}{4}$$

for $|z| < r_0(\alpha, \beta, \rho)$ *where* $r_0(\alpha, \beta, \rho)$ *is given by*

$$r_0(\alpha,\beta,\rho) = \begin{cases} \frac{F}{2(\alpha-\rho)(1-2\beta)}, & \text{if } 0 \le \beta < \beta_0(\alpha,\rho), \\ \left(\frac{G}{(4(\alpha-\rho)^2+1)(1-\beta)+4(\alpha-\rho)(1-3\beta)}\right)^{\frac{1}{2}}, & \text{if } \beta_0(\alpha,\rho) \le \beta < 1, \beta \ne \mu, \\ \left(\frac{2(\alpha-\rho)}{2(\alpha-\rho)+1}\right)^{\frac{1}{2}}, & \text{if } \beta = \mu \end{cases}$$

where also

$$\beta_0(\alpha,\rho) = \min\left\{\frac{1}{2}, \frac{1}{4(\alpha-\rho)+1}\right\}$$

and

$$F = [(4(\alpha-\rho)^2+1)(1-\beta)^2 - 4(\alpha-\rho)\beta(1-\beta)]^{\frac{1}{2}} + 2(\alpha-\rho)\beta + \beta - 1$$

$$G = 4[2\beta(\alpha-\rho)(1-\beta)]^{\frac{1}{2}} - [(1-4(\alpha-\rho)^2)(1-\beta) + 8(\alpha-\rho)\beta]$$

$$\mu = \frac{(2(\alpha-\rho)+1)^2}{1+12(\alpha-\rho)+4(\alpha-\rho)^2}.$$

Proof. Since $f \in \mathcal{J}_n^\alpha(\beta)$, then there exists a $p(z) \in P$ such that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} = \beta + (1-\beta)p(z). \tag{3.1}$$

Differentiation of equation (3.1) and some computation yields

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2} f(z)^\alpha}{D^{n+1} f(z)^\alpha} = 2\alpha + \frac{zp'(z)}{\frac{\beta}{1-\beta} + p(z)}.$$

If $R_1 < R_2$, then by the condition of the theorem and the lemma, we have

$$\text{Re}\left(\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2} f(z)^\alpha}{D^{n+1} f(z)^\alpha}\right) \ge 2\alpha - \frac{2(1-\beta)r}{(1+r)(1+(2\beta-1)r)}$$

where R_1 and R_2 are as given in the lemma. Therefore we deduce that

$$\text{Re}\left(\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2} f(z)^\alpha}{D^{n+1} f(z)^\alpha}\right) > 2\rho \tag{3.2}$$

provided

$$(\alpha-\rho) - \frac{(1-\beta)r}{(1+r)(1+(2\beta-1)r)} > 0$$

which is true if $|z| = r < r_0(\alpha, \beta, \rho)$ where

$$r_0(\alpha, \beta, \rho) = \frac{F}{2(\alpha - \rho)(1 - 2\beta)}$$

is the smallest positive root of the equation

$$(\alpha - \rho)(2\beta - 1)r^2 + (2(\alpha - \rho)\beta + \beta - 1)r + (\alpha - \rho) = 0.$$

Now, this root is real if

$$(2(\alpha - \rho)\beta + \beta - 1)^2 - 4(\alpha - \rho)^2(2\beta - 1) \geq 0$$

and since this root must be less than one, then

$$\beta < \beta_0(\alpha, \rho) = \min\left\{\frac{1}{2}, \frac{1}{4(\alpha - \rho) + 1}\right\}.$$

Next, suppose $R_2 \leq R_1$, then by the lemma again and the condition of the theorem,

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2}f(z)^\alpha}{D^{n+1}f(z)^\alpha}\right)$$
$$\geq 2\alpha - \frac{\beta}{1-\beta} + \frac{1}{1-\beta}\left[2\left(\frac{\beta - \beta(2\beta - 1)r^2}{1 - r^2}\right)^{\frac{1}{2}} - \frac{1 - (2\beta - 1)r^2}{1 - r^2}\right].$$

and the condition (3.2) is satisfied provided

$$2(\alpha - \rho) - \frac{\beta}{1-\beta} + \frac{1}{1-\beta}\left[2\left(\frac{\beta - \beta(2\beta - 1)r^2}{1 - r^2}\right)^{\frac{1}{2}} - \frac{1 - (2\beta - 1)r^2}{1 - r^2}\right] > 0$$

which is the case if $|z| = r < r_0(\alpha, \beta, \rho)$ where $r_0(\alpha, \beta, \rho)$ is the smallest positive root of the equation

$$ar^4 + br^2 + c = 0 \tag{3.3}$$

where

$$a = (4(\alpha - \rho)^2 + 1)(1 - \beta)^2 + 4(\alpha - \rho)(1 - \beta)(1 - 3\beta),$$

$$b = 2(1 - 4(\alpha - \rho)^2)(1 - \beta)^2 + 16(\alpha - \rho)\beta(1 - \beta)$$

and

$$c = (4(\alpha-\rho)^2+1)(1-\beta)^2 - 4(\alpha-\rho)(1-\beta)(1+\beta).$$

This root is given by

$$r_0(\alpha,\beta,\rho) = \left(\frac{4[2\beta(\alpha-\rho)(1-\beta)]^{\frac{1}{2}} - [(1-4(\alpha-\rho)^2)(1-\beta)+8(\alpha-\rho)\beta]}{(4(\alpha-\rho)^2+1)(1-\beta)+4(\alpha-\rho)(1-3\beta)}\right)^{\frac{1}{2}}$$

where $\beta \neq (2(\alpha-\rho)+1)^2/(1+12(\alpha-\rho)+4(\alpha-\rho)^2)$.

If $\beta = (2(\alpha-\rho)+1)^2/(1+12(\alpha-\rho)+4(\alpha-\rho)^2)$ in (3.3), we obtain the third root given in the theorem. □

For $n = 0$, we have the following important corollaries from the main result.

Corollary 3.1. *Let $f \in \mathcal{J}_0^\alpha(\beta)$ (that is, f is Bazilevic of type 2α, order β). Then*

$$Re\left\{1+\frac{zf''(z)}{f(z)}+(2\alpha-1)\frac{zf'(z)}{f(z)}\right\} > 2\rho, \quad \rho \geq -\frac{1}{4} \tag{3.4}$$

for $|z| < r_0(\beta)$ where $r_0(\beta)$ is as given in the theorem.

It is worthy of mention that Babalola in [1] proved that (3.4) is sufficient for schlichtness in the open unit disk. Now take $\rho = \alpha - \frac{1}{4}$ in (3.4), we have the following.

Corollary 3.2. *Let $f \in \mathcal{J}_0^\alpha(\beta)$ (that is, f is Bazilevic of type 2α, order β). Then*

$$Re\left\{1+\frac{zf''(z)}{f(z)}+(2\alpha-1)\frac{zf'(z)}{f(z)}\right\} > 2\alpha-\frac{1}{2}$$

for $|z| < r_0(\beta)$ where $r_0(\beta)$ is given by

$$r_0(\beta) = \begin{cases} \frac{3\beta-2+\sqrt{(1-\beta)(5-9\beta)}}{1-2\beta}, & \text{if } 0 \leq \beta < \frac{1}{2}, \\ \left(\frac{8\sqrt{2\beta(1-\beta)}-5\beta-3}{9-17\beta}\right)^{\frac{1}{2}}, & \text{if } \frac{1}{2} \leq \beta < 1,\ \beta \neq \frac{9}{17}, \\ \frac{\sqrt{3}}{3}, & \text{if } \beta = \frac{9}{17}. \end{cases}$$

In particular, if $\alpha = 0$, we have:

Corollary 3.3. *Let $f \in \mathcal{J}_0^0(\beta)$ (that is, f is starlike of order β). Then*

$$Re\left\{1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{1}{2} \tag{3.5}$$

for $|z| < r_0(\beta)$ where $r_0(\beta)$ is as given in Corollary 3.2 above.

Also if we take $\alpha = \frac{1}{2}$, which consists of functions of bounded turning of order β, then we have the radius of convexity of order $\frac{1}{2}$ for such functions as:

Corollary 3.4. *Let $f \in \mathcal{J}_0^{\frac{1}{2}}(\beta)$ (that is, f is of bounded turning order β). Then*

$$Re\left\{1 + \frac{zf''(z)}{f(z)}\right\} > \frac{1}{2} \tag{3.6}$$

for $|z| < r_0(\beta)$ where $r_0(\beta)$ is also as given in Corollary 3.2 above.

This radius of convexity of order $\frac{1}{2}$ for bounded-turning functions of order β is novel, and advances over similar result of Macgregor [3] for the radius of convexity of order zero. Particular cases for $\beta = 0$ and $\beta = \frac{1}{2}$ are mentioned in Corollaries 3.8 and 3.12 below.

Also if $\alpha = 1$, then we have:

Corollary 3.5. *Let $f \in \mathcal{J}_0^1(\beta)$ (that is, f is starlike of order β). Then*

$$Re\left\{1 + \frac{zf''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > \frac{3}{2} \tag{3.7}$$

for $|z| < r_0(\beta)$ where $r_0(\beta)$ is as given in Corollary 3.2 above.

Finally, we note the plausible limiting case $r_0(\beta) \to 1$ as $\beta \to 1$ in Corollary 3.2 above, and also in particular, for the cases $\beta = 0$ and $\beta = 1/2$ we have the following.

Corollary 3.6. *Let $f \in \mathcal{J}_0^\alpha$. Then f satisfies (3.4) in the subdisk $|z| < \sqrt{5} - 2$.*

Corollary 3.7. *Every starlike function ($f \in \mathcal{J}_0^0$) satisfies (3.5) in the subdisk $|z| < \sqrt{5} - 2$.*

Corollary 3.8. *Every bounded turning function* ($f \in \mathcal{J}_0^{\frac{1}{2}}$) *satisfies* (3.6) *in the subdisk* $|z| < \sqrt{5}-2$. *That is every bounded turning function is convex of order* $\frac{1}{2}$ *in the subdisk subdisk* $|z| < \sqrt{5}-2$.

This radius of convexity of order $\frac{1}{2}$ for bounded-turning functions is new.

Corollary 3.9. *Every type 2 Bazilevic function* ($f \in \mathcal{J}_0^{1}$) *satisfies* (3.7) *in the subdisk* $|z| < \sqrt{5}-2$.

Corollary 3.10. *Let* $f \in \mathcal{J}_0^{\alpha}(\frac{1}{2})$. *Then* f *satisfies* (3.4) *in the subdisk*

$$|z| < \sqrt{8\sqrt{2}-11} = 0.5600968657.$$

Corollary 3.11. *Every starlike function of order* $\frac{1}{2}$ ($f \in \mathcal{J}_0^{0}(\frac{1}{2})$) *satisfies* (3.5) *in the subdisk* $|z| < \sqrt{8\sqrt{2}-11}$.

Corollary 3.12. *Every bounded turning function of order* $\frac{1}{2}$ ($f \in \mathcal{J}_0^{\frac{1}{2}}(\frac{1}{2})$) *satisfies* (3.6) *in the subdisk* $|z| < \sqrt{8\sqrt{2}-11}$.

Also this radius of convexity of order $\frac{1}{2}$ for bounded-turning functions of order $\frac{1}{2}$ is also new.

Corollary 3.13. *Every Type 2 Bazilevic function of order* $\frac{1}{2}$ ($f \in \mathcal{J}_0^{1}(\frac{1}{2})$) *satisfies* (3.7) *in the subdisk* $|z| < \sqrt{8\sqrt{2}-11}$.

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