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QUASI-$\mathcal{N}$-OPEN SETS IN (a)TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduced the notion of $\mathcal{N}$-open sets in (a)topological spaces which is a set equipped with countable number of topologies. We investigated the three types of $\mathcal{N}$-open sets in (a)topological spaces and via them several types of compactness are introduced. Also, we introduced the notion of quasi-$\mathcal{N}$-open sets in (a)topological spaces and related compactness.

1. INTRODUCTION

J. C. Kelly [8] introduced the notion of bitopological spaces (X, τ_1, τ_2) (a non empty set X endowed with two topologies τ_1 and τ_2) which is a wider structure than the classical topological spaces. Several authors worked on bitopology, three topologies and countable number of topologies in [4, 5, 6, 7, 14, 15, 16, 20, 21]. Tyagi et. al. [23] studies new types of sets via γ-open sets in (a)topological spaces. Datta [9] introduced the notion of quasi-open sets in bitopological spaces. In [1, 10, 11, 17, 18, 19, 24, 25], several modifications of concept of quasi-open sets are introduced. In this paper, we introduced the notion of $\mathcal{N}$-open sets in (a)topological spaces and investigated the three types of $\mathcal{N}$-open sets and via them several types of compactness are introduced. We define and investigate quasi-$\mathcal{N}$-open sets in (a)topological spaces and use them to define new type of compactness.

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Let (X, τ) be a topological space and let G be a subset of X. A point $x \in X$ is said to be an infinite point [2, 3] (resp. condensation point [12]) of G if for each $U \in \tau$ with $x \in U$, $U \cap G$ is infinite (resp. uncountable). In case, the set G contains all its infinite points (resp. condensation points), G is called $\mathcal{N}$-closed [2, 3] (resp. ω-closed [12]). A set is said to be $\mathcal{N}$-open [2, 3] (resp. ω-open [13]) if its complement is $\mathcal{N}$-closed (resp. ω-closed).

Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers. $\tau_{n\mathcal{N}}, \tau_{n\omega}$ denotes the topology on X consisting of all $\mathcal{N}$-open, ω-open sets in (X, τ_n), respectively. τ_{lr}, τ_{rr}, τ_I denotes the left ray topology, the right ray topology and the indiscrete topology on the set of all real numbers, respectively. If there is no scope of confusion, we denote the (a)topological space $(X, \{\tau_n\}_{n\in\mathbb{N}})$ by $(X, \{\tau_n\})$.

2. (a)TOPOLOGICAL SPACES

Definition 2.1. [22] *If $\{\tau_n\}$ is a sequence of topologies on a set X, then the pair $(X, \{\tau_n\}_{n\in\mathbb{N}})$ is called an (a)topological space.*

Definition 2.2. *Let $(X, \{\tau_n\})$ be an (a)topological space. The smallest topology on X containing $\cup_{n\in\mathbb{N}} \tau_n$ is called the least upper bound topology on X.*

The least upper bound topology on (a)topological space $(X, \{\tau_n\})$ will be denoted by $(< X, \{\tau_n\} >)$. If there is no scope of confusion, we denotes $(< X, \{\tau_n\} >)$ by $< \tau_n >$.

Definition 2.3. *A set $A \subseteq X$ is said to be s-open if it is open in the least upper bound topology on $(X, \{\tau_n\})$.*

Proposition 2.4. *A set $A \subseteq X$ is s-open if and only if for each $x \in A$ there exist $U_n \in \tau_n$(for each $n \in \mathbb{N}$) such that $x \in \cap_{n\in\mathbb{N}} U_n \subseteq A$, where $U_n = X$ for all except finitely many n.*

Proof. Let $A \subseteq X$ be s-open and $x \in A$ be arbitrary. Then A is open in the least upper bound topology of $(X, \{\tau_n\})$. This implies that there exist $U_{j_i} \in \tau_{j_i}$, $i = 1, 2, \ldots, n$ such that $x \in \cap_{i=1}^{n} U_{j_i} \subseteq A$. Let $U_k = X$ for all $k \neq j_1, j_2, \ldots, j_n$. Thus, for each $x \in A$ there exist $U_n \in \tau_n$ such that $x \in \cap_{n\in\mathbb{N}} U_n \subseteq A$, where $U_n = X$ for all except finitely many n.

Conversely, let for each $x \in A$ there exist $U_n \in \tau_n$ such that $x \in \cap_{n\in\mathbb{N}} U_n \subseteq A$, where $U_n = X$ for all except finitely many n, say $i_1, i_2, \ldots, i_k$. Then $x \in \cap_{j=1}^{k} U_{i_j} \subseteq A$. Then A is open in the least upper bound topology of $(X, \{\tau_n\})$ and hence, A is s-open. □

Definition 2.5. *A set $A \subseteq X$ is said to be μ-open if $A \in \cup_{n\in\mathbb{N}} \tau_n$.*

The family of all μ-open sets in $(X, \{\tau_n\})$ will be denoted by $\mu(\tau_n)$.

Definition 2.6. *A set $A \subseteq X$ is said to be quasi-open if for every $x \in A$ there exists $U_n \in \tau_n$ such that $x \in U_n \subseteq A$ for some $n \in \mathbb{N}$.*

Definition 2.7. *A set $A \subseteq X$ is said to be quasi-closed if $X \backslash A$ is quasi-open.*

The family of all quasi-open sets in $(X, \{\tau_n\})$ is denoted by $q(\tau_n)$. It is observe that $\tau_n \subseteq q(\tau_n)$ for all $n \in \mathbb{N}$.

Proposition 2.8. *A set $A \subseteq X$ is quasi-open if and only if $A = \cup_{n\in\mathbb{N}} G_n$, where $G_n \in \tau_n$ for all $n \in \mathbb{N}$.*

Proof. Let A be a quasi-open set in $(X, \{\tau_n\})$ and let $x \in A$ be arbitrary. Then there exists $U_n \in \tau_n$(depending upon x) such that $x \in U_n \subseteq A$ for some $n \in \mathbb{N}$. This implies that $A = \cup_{x\in A} U_n$. If required, take $U_n = \emptyset$ for some naturals n. Then $A = \cup_{n\in\mathbb{N}} U_n$, where $U_n \in \tau_n$ for all $n \in \mathbb{N}$. Converse follows by Definition 2.6. □

Proposition 2.9. *Let $(X, \{\tau_n\})$ be an (a)topological space. Then we have the following:*

(a). $\mu(\tau_n) \subseteq q(\tau_n) \subseteq < \tau_n >$.

(*b*). *$q(\tau_n)$ is closed under arbitrary union.*

Proof. The proof follows by definitions. □

Following example shows that $q(\tau_n)$ is not closed under finite intersection.

Example 2.10. *Consider the* (a)*topological space* $(\mathbb{R}, \{\tau_n\})$ *where* $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ *for all* $n \neq 1$. *Then the set* $A = (-\infty, 2) \in \tau_1 \subset q(\tau_n)$ *and* $B = (0, \infty) \in \tau_2 \subset q(\tau_n)$. *But* $A \cap B = (0, 2) \notin q(\tau_n)$.

In general, $< \tau_n > \neq q(\tau_n)$ and $\cup_{n\in\mathbb{N}} \tau_n \neq q(\tau_n)$. For example, Consider the (a)topological space $(\mathbb{R}, \{\tau_n\})$ where $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ for all $n \neq 1$. Then the set $(0, 2) = (-\infty, 2) \cap (0, \infty)$ and thus $(0, 2) \in < \tau_n >$ But $(0, 2) \notin q(\tau_n)$. Hence, $< \tau_n > \neq q(\tau_n)$.
It is observe that the set $A = (-\infty, 0) \cup (2, \infty) \in q(\tau_n)$ but $A \notin \tau_n$ for any $n \in \mathbb{N}$. Hence, $\cup_{n\in\mathbb{N}} \tau_n \neq q(\tau_n)$.

Definition 2.11. *Let* $(X, \{\tau_n\})$ *be an* (a)*topological space and let* $A \subseteq X$. *Then*

(*a*). *A is said to be μ-$\mathcal{N}$ (resp. μ-ω) open in $(X, \{\tau_n\})$ if $A \in \cup_{n\in\mathbb{N}} \tau_{n\mathcal{N}}$ (resp. $A \in \cup_{n\in\mathbb{N}} \tau_{n\omega}$).*

(*b*). *A is said to be μ-$\mathcal{N}$ (resp. μ-ω) closed in $(X, \{\tau_n\})$ if $X \backslash A$ is μ-$\mathcal{N}$ (resp. μ-ω) open in $(X, \{\tau_n\})$.*

(*c*). *A is said to be s-$\mathcal{N}$ open in $(X, \{\tau_n\})$ if $A \in < \tau_{n\mathcal{N}} >$. That is, A is s-$\mathcal{N}$ open in $(X, \{\tau_n\})$ if A is open in the least upper bound topology on X via $\{\tau_{n\mathcal{N}}\}$.*

The family of all μ-$\mathcal{N}$ (resp. μ-ω) open sets in $(X, \{\tau_n\})$ is denoted by μ-$\mathcal{N}(\tau_n)$ (resp. μ-$\omega(\tau_n)$).

Theorem 2.12. (*a*). *Every μ-open set in (a)topological spaces is μ-$\mathcal{N}$-open.*
(*b*). *Every μ-$\mathcal{N}$-open set in (a)topological spaces is μ-ω-open.*

Proof. (a). Let A be a μ-open set in an (a)topological space $(X, \{\tau_n\})$. Then $A \in \mu(\tau_n)$. It is obvious that every open set is $\mathcal{N}$-open so $\tau_n \subseteq \tau_{n\mathcal{N}}$ for all $n \in \mathbb{N}$. Thus, $\mu(\tau_n) \subseteq \mu(\tau_{n\mathcal{N}})$ and hence, $A \in \mu(\tau_{n\mathcal{N}})$. So A is μ-$\mathcal{N}$-open.
(b). Let A be a μ-$\mathcal{N}$-open set in an (a)topological space $(X, \{\tau_n\})$. Then $A \in \mu(\tau_{n\mathcal{N}})$. It is obvious that every $\mathcal{N}$-open set is ω-open so $\tau_{n\mathcal{N}} \subseteq \tau_{n\omega}$ for all $n \in \mathbb{N}$. Thus, $\mu(\tau_{n\mathcal{N}}) \subseteq \mu(\tau_{n\omega})$ and hence, $A \in \mu(\tau_{n\omega})$. So A is μ-ω-open. □

The following example shows that the converse of Theorem 2.12 is not true.

Example 2.13. *Consider the (a)topological space $(\mathbb{R}, \{\tau_n\})$ where $\tau_1 = \tau_I$, $\tau_n = \tau_{lr}$ for all $n \neq 1$. Then clearly $\mathbb{R}\backslash\mathbb{Q}$ is μ-ω-open but not μ-$\mathcal{N}$-open. It is also observe that complement of any finite set is μ-$\mathcal{N}$-open but not μ-open.*

Theorem 2.14. *Let $(X, \{\tau_n\})$ be an (a)topological space. Then $< \tau_n >_{\mathcal{N}} = < \tau_{n\mathcal{N}} >$, where $< \tau_n >_{\mathcal{N}}$ denotes the family of all $\mathcal{N}$-open sets in $< \tau_n >$.*

Proof. Let $A \in < \tau_n >_{\mathcal{N}}$ and let $x \in A$. Then there exist $U \in < \tau_n >$ and finite sets $F \subseteq X$ such that $x \in U\backslash F \subseteq A$. Since $x \in U$ and $U \in < \tau_n >$, there exist $U_n \in \tau_n$ such that $x \in \cap_{n\in\mathbb{N}} U_n \subseteq U$, where $U_n = X$ for all n except finitely many n, say $n = i_1, i_2, \ldots, i_k$. For all $n \neq i_1, i_2, \ldots, i_k$, take $F = \emptyset$. It is observe that for all $n \in \mathbb{N}$, $U_n\backslash F \in \tau_{n\mathcal{N}}$, $U_n\backslash F = X$ for all $n \neq i_1, i_2, \ldots, i_k$ and $x \in \cap_{n\in\mathbb{N}} (U_n\backslash F) \subseteq (\cap_{n\in\mathbb{N}} U_n)\backslash F \subseteq U\backslash F \subseteq A$. It follows that $A \in < \tau_{n\mathcal{N}} >$.
Conversely, let $A \in < \tau_{n\mathcal{N}} >$ and let $x \in A$. Then there exist $U_n \in \tau_{n\mathcal{N}}$ such that $x \in \cap_{n\in\mathbb{N}} U_n \subseteq A$, where $U_n = X$ for all n except finitely many n, say $n = i_1, i_2, \ldots, i_k$. For each $n = i_1, i_2, \ldots, i_k$, there exist $G_n \in \tau_n$ and finite sets F_n such that $x \in G_n\backslash F_n \subseteq U_n$. For each $n \neq i_1, i_2, \ldots, i_k$, take $G_n = X$ and $F_n = \emptyset$. We observe that $x \in \cap_{n\in\mathbb{N}} G_n \backslash \cup_{n\in\mathbb{N}} F_n \subseteq \cap_{n\in\mathbb{N}} U_n \subseteq A$. Thus, $A \in < \tau_n >_{\mathcal{N}}$. □

Theorem 2.15. *Let $(X, \{\tau_n\})$ be an (a)topological space. Then μ-$\mathcal{N}(\tau_n) \subseteq < \tau_n >_{\mathcal{N}}$.*

Proof. By Definition 2.11, $\mu\text{-}\mathcal{N}(\tau_n) = \cup_{n\in\mathbb{N}} \tau_{n\mathcal{N}} \subseteq <\tau_{n\mathcal{N}}>$. By Theorem 2.14, $<\tau_{n\mathcal{N}}> = <\tau_n>_{\mathcal{N}}$. Hence, $\mu\text{-}\mathcal{N}(\tau_n) \subseteq <\tau_n>_{\mathcal{N}}$. □

In general, inclusion in Theorem 2.15 cannot be replaced by equality. This is shown by following example.

Example 2.16. *Consider the (a)topological space $(\mathbb{R}, \{\tau_n\})$ where $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ for all $n \neq 1$. Then any finite interval $(a,b) \in <\{\tau_n\}>_{\mathcal{N}}$ because $(a,b) \in <\{\tau_n\}>$. But $(a,b) \notin \mu\text{-}\mathcal{N}(\tau_n)$.*

Definition 2.17. *A set $A \subseteq X$ is said to be quasi-ω-open (q-ω-open) if for each $x \in A$ there exists $G \in \tau_{n\omega}$ for at least one natural n such that $x \in G \subseteq A$. Equivalently, a set $A \subseteq X$ is q-ω-open if and only if $A \in q(\tau_{n\omega})$.*

The family of all quasi-ω-open (q-ω-open) sets in $(X, \{\tau_n\})$ is denoted by $q\text{-}\omega(\tau_n)$ and a set in $(X, \{\tau_n\})$ is q-ω-closed if its complement is q-ω-open.

Definition 2.18. *A set $A \subseteq X$ is said to be quasi-$\mathcal{N}$-open (q-$\mathcal{N}$-open) if for each $x \in A$ there exists $G \in \tau_{n\mathcal{N}}$ for at least one natural n such that $x \in G \subseteq A$. Equivalently, a set $A \subseteq X$ is q-$\mathcal{N}$-open if and only if $A \in q(\tau_{n\mathcal{N}})$.*

The family of all quasi-$\mathcal{N}$-open (q-$\mathcal{N}$-open) sets in $(X, \{\tau_n\})$ is denoted by $q\text{-}\mathcal{N}(\tau_n)$ and a set in $(X, \{\tau_n\})$ is q-$\mathcal{N}$-closed if its complement is q-$\mathcal{N}$-open.

Theorem 2.19. *A set $A \subseteq X$ is q-$\mathcal{N}$-open if and only if for each $x \in A$, there exists $U \in \mu(\tau_n)$ and a finite set $F \subseteq X$ such that $x \in U \backslash F \subseteq A$.*

Proof. Let A be a q-$\mathcal{N}$-open set and let $x \in A$. Then for each $n \in \mathbb{N}$ there exists $U_n \in \tau_{n\mathcal{N}}$ such that $A = \cup_{n\in\mathbb{N}} U_n$. Without loss of generality assume that $x \in U_1$. Since $U_1 \in \tau_{1\mathcal{N}}$, there exists $U \in \tau_1$ and finite set $F \subseteq X$ such that $x \in U\backslash F \subseteq U_1$. Also $U \in \tau_1 \subseteq \mu(\tau_n)$ and $U_1 \subseteq A$. Thus, there exist $U \in \mu(\tau_n)$ and a finite set $F \subseteq X$

such that $x \in U \backslash F \subseteq A$.

Conversely, suppose that for each $x \in A$ there exists $U_x \in \mu(\tau_n)$ and a finite set $F_x \subseteq X$ such that $x \in U_x \backslash F_x \subseteq A$. Let $V_n = \cup\{U_x \backslash F_x \colon U_x \in \tau_n\}$. It is observe that $V_n \in \tau_{n\mathcal{N}}$ for all $n \in \mathbb{N}$ and $A = \cup_{n \in \mathbb{N}} V_n$. Thus, A is q-$\mathcal{N}$-open. □

Theorem 2.20. *For an (a)topological space $(X, \{\tau_n\})$, following results hold:*

(a). μ-$\mathcal{N}(\tau_n) \subseteq q$-$\mathcal{N}(\tau_n)$.

(b). q-$(\tau_n) \subseteq q$-$\mathcal{N}(\tau_n)$.

(c). q-$\mathcal{N}(\tau_n) \subseteq < \tau_n >_{\mathcal{N}}$.

(d). The family of all q-$\mathcal{N}$-open sets is closed under arbitrary union.

(e). The family of all q-$\mathcal{N}$-closed sets is closed under arbitrary intersection.

(f). q-$\mathcal{N}(\tau_n) \subseteq q$-$\omega(\tau_n)$.

Proof. (a). Since $\mu(\tau_n) \subseteq q(\tau_n)$, $\mu(\tau_{n\mathcal{N}}) \subseteq q(\tau_{n\mathcal{N}})$. Also $\mu(\tau_{n\mathcal{N}}) = \mu$-$\mathcal{N}(\tau_n)$ and $q(\tau_{n\mathcal{N}}) = q$-$\mathcal{N}(\tau_n)$. So μ-$\mathcal{N}(\tau_n) \subseteq q$-$\mathcal{N}(\tau_n)$.

(b). Since $\tau_n \subseteq \tau_{n\mathcal{N}}$ for all n, q-$(\tau_n) \subseteq q$-$(\tau_{n\mathcal{N}})$. But q-$(\tau_{n\mathcal{N}}) = q$-$\mathcal{N}(\tau_n)$, so q-$(\tau_n) \subseteq q$-$\mathcal{N}(\tau_n)$.

(c). since q-$(\tau_n) \subseteq < \tau_n >$, q-$(\tau_{n\mathcal{N}}) \subseteq < \tau_{n\mathcal{N}} >$. Also q-$(\tau_{n\mathcal{N}}) = q$-$\mathcal{N}(\tau_n)$ and $< \tau_{n\mathcal{N}} > = < \tau_n >_{\mathcal{N}}$. So q-$\mathcal{N}(\tau_n) \subseteq < \tau_n >_{\mathcal{N}}$.

(d). Since q-$\mathcal{N}(\tau_n) = q$-$(\tau_{n\mathcal{N}})$ and q-$(\tau_{n\mathcal{N}})$ is closed under arbitrary union, so the family of all q-$\mathcal{N}$-open sets is closed under arbitrary union.

(e). Follows by part (d).

(f). In (a)topological spaces $(X, \{\tau_n\})$, every $\mathcal{N}$-open set is ω-open. Thus, q-$\mathcal{N}(\tau_n) \subseteq q$-$\omega(\tau_n)$. □

Following example shows that inclusion in part (a). of Theorem 2.20 cannot be replaced by equality, in general.

Example 2.21. *Consider the* (a)*topological space* $(\mathbb{R}, \{\tau_n\})$ *where* $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ *for all* $n \neq 1$. *Then the set* $A = (-\infty, 1) \cup (2, \infty)$ *is a* q*-*$\mathcal{N}$*-open set but not a* μ*-*$\mathcal{N}$*-open set.*

Following example shows that inclusion in part (b). of Theorem 2.20 cannot be replaced by equality, in general.

Example 2.22. *Consider the* (a)*topological space* $(\mathbb{R}, \{\tau_n\})$ *where* $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ *for all* $n \neq 1$. *Then the set* $A = (-\infty, 0) \backslash \{-1\}$ *is a* q*-*$\mathcal{N}$*-open set but not* q*-open.*

Following example shows that inclusion in part (c). of Theorem 2.20 cannot be replaced by equality, in general.

Example 2.23. *Consider the* (a)*topological space* $(\mathbb{R}, \{\tau_n\})$ *where* $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ *for all* $n \neq 1$. *Let* $A = (a, b)$ *be any finite interval. Then* $A \in <\tau_n> \subseteq <\tau_n>_{\mathcal{N}}$, *but* $A \neq q$*-*$\mathcal{N}(\tau_n)$.

Following example shows that the family of all q-$\mathcal{N}$-open sets in an (a)topological space does not form a topological space, in general.

Example 2.24. *Consider the* (a)*topological space* $(\mathbb{R}, \{\tau_n\})$ *where* $\tau_1 = \tau_{lr}$, $\tau_n = \tau_{rr}$ *for all* $n \neq 1$. *Let* $A = (-\infty, 1)$, $B = (0, \infty)$. *Both* A *and* B *are* q*-*$\mathcal{N}$*-open sets in* $(\mathbb{R}, \{\tau_n\})$ *but* $A \cap B$ *is not a* q*-*$\mathcal{N}$*-open set in* $(\mathbb{R}, \{\tau_n\})$.

Theorem 2.25. *In an* (a)*topological space,* $q(\tau_n)$ *forms a topology if and only if* $q(\tau_n) =< \{\tau_n\} >$.

Proof. Let us suppose that $q(\tau_n) =< \tau_n >$. Since $< \tau_n >$ is a topology on X, $q(\tau_n)$ forms a topology on X.
Conversely, suppose that $q(\tau_n)$ forms a topology on X. Since $q(\tau_n) \subseteq < \tau_n >$, it is sufficient to show that $< \tau_n > \subseteq q(\tau_n)$. Let $A \in < \tau_n >$ and $x \in A$ be arbitrary.

Then there exist $U_n^x \in \tau_n$ such that $x \in \cap_{n\in\mathbb{N}} U_n^x \subseteq A$, where $U_n^x = X$ for all except finitely many $n \in \mathbb{N}$. Since $U_n^x \in \tau_n \subset q(\tau_n)$ and $q(\tau_n)$ is a topology on X, $\cap_{n\in\mathbb{N}} U_n^x \in q(\tau_n)$ as $\cap_{n\in\mathbb{N}} U_n^x$ is just a finite intersection of member of $q(\tau_n)$. This implies that $A = \cup_{x\in A} \cap_{n\in\mathbb{N}} U_n^x$, where $\cap_{n\in\mathbb{N}} U_n^x \in q(\tau_n)$. Since $q(\tau_n)$ is a topology on X, $A \in q(\tau_n)$. Thus, $< \tau_n > \subseteq q(\tau_n)$ and hence proved. □

Theorem 2.26. *In an (a)topological space, q-$\mathcal{N}(\tau_n)$ forms a topology if and only if q-$\mathcal{N}(\tau_n) = < \tau_n >_{\mathcal{N}}$.*

Proof. Let us suppose that q-$\mathcal{N}(\tau_n) = < \tau_n >_{\mathcal{N}}$. Since $< \tau_n >_{\mathcal{N}} = < \tau_{n\mathcal{N}} >$ and $< \tau_{n\mathcal{N}} >$ is a topology on X, q-$\mathcal{N}(\tau_n)$ forms a topology on X.
Conversely, suppose that q-$\mathcal{N}(\tau_n)$ forms a topology on X. Since q-$\mathcal{N}(\tau_n) \subseteq < \tau_{n\mathcal{N}} >$, it is sufficient to show that $< \tau_{n\mathcal{N}} > \subseteq q$-$\mathcal{N}(\tau_n)$. Let $A \in < \tau_{n\mathcal{N}} >$ and $x \in A$ be arbitrary. Then there exist $U_n^x \in \tau_{n\mathcal{N}}$ such that $x \in \cap_{n\in\mathbb{N}} U_n^x \subseteq A$, where $U_n^x = X$ for all except finitely many $n \in \mathbb{N}$. Since $U_n^x \in \tau_{n\mathcal{N}} \subset q$-$\mathcal{N}(\tau_n)$ and q-$\mathcal{N}(\tau_n)$ is a topology on X, $\cap_{n\in\mathbb{N}} U_n^x \in q$-$\mathcal{N}(\tau_n)$ as $\cap_{n\in\mathbb{N}} U_n^x$ is just a finite intersection of member of q-$\mathcal{N}(\tau_n)$. This implies that $A = \cup_{x\in A} \cap_{n\in\mathbb{N}} U_n^x$, where $\cap_{n\in\mathbb{N}} U_n^x \in q$-$\mathcal{N}(\tau_n)$. Since q-$\mathcal{N}(\tau_n)$ is a topology on X, $A \in q$-$\mathcal{N}(\tau_n)$. Thus, $< \tau_{n\mathcal{N}} > \subseteq q$-$\mathcal{N}(\tau_n)$ and hence proved. □

Definition 2.27. *A cover α of an (a)topological space $(X, \{\tau_n\})$ is said to be:*

- (a). *$\{\tau_n\}$-open if $\alpha \subseteq \mu(\tau_n)$.*
- (b). *p-open if it is $\{\tau_n\}$-open and α contains at least one non empty member of each τ_n.*

Definition 2.28. *An (a)topological space $(X, \{\tau_n\})$ is said to be:*

- (a). *s-compact if every $\{\tau_n\}$-open cover of $(X, \{\tau_n\})$ has a finite subcover.*
- (b). *p-compact if every p-open cover of $(X, \{\tau_n\})$ has a finite subcover.*

Theorem 2.29. *Let* $(X, \{\tau_n\})$ *be an* (a)*topological space and let* $\mathcal{A} = \{W \backslash F \colon W \in \mu(\tau_n)$ *and* $F \subseteq X$ *is a finite set* $\}$. *Then* $(X, \{\tau_{n\mathcal{N}}\})$ *is* s*-compact if and only if every cover of* X *consists of elements of* $\mathcal{A}$ *has a finite subcover.*

Proof. Suppose that $(X, \{\tau_{n\mathcal{N}}\})$ is s-compact and let $\mathcal{F}$ be a cover of X with $\mathcal{F} \subseteq \mathcal{A}$. Since $\mathcal{F} \subseteq \mathcal{A} \subseteq \cup_{n\in\mathbb{N}}\, \tau_{n\mathcal{N}} = \mu(\tau_{n\mathcal{N}})$. This implies that $\mathcal{F}$ is a $\{\tau_{n\mathcal{N}}\}$-open cover of $(X, \{\tau_{n\mathcal{N}}\})$. Since $(X, \{\tau_{n\mathcal{N}}\})$ is s-compact, there exists a finite family of members of $\mathcal{F}$ covers X.

Conversely, let $\mathcal{F} = \{F_\alpha \colon \alpha \in \Delta\}$ be a $\{\tau_{n\mathcal{N}}\}$-open cover of $(X, \{\tau_{n\mathcal{N}}\})$. Then for each $\alpha \in \Delta$, $F_\alpha \in \cup_{n\in\mathbb{N}}\, \tau_{n\mathcal{N}}$. That is, $F_\alpha \in \tau_{n\mathcal{N}}$ for some $n \in \mathbb{N}$. Therefore, there exists $A_\alpha \in \mu(\tau_n)$ and finite sets B_α such that $F_\alpha = A_\alpha \backslash B_\alpha$. Thus $F_\alpha \in \mathcal{A}$ and hence $F \subseteq \mathcal{A}$. By hypothesis, F has a finite subcover. Hence, $(X, \{\tau_{n\mathcal{N}}\})$ is s-compact. □

Theorem 2.30. *For an* (a)*topological space* $(X, \{\tau_n\})$, *the following statements are equivalent*:

(*a*). $(X, \{\tau_n\})$ *is* s*-compact.*

(*b*). $(X, \{\tau_{n\mathcal{N}}\})$ *is* s*-compact.*

(*c*). *Each cover of* X *of members of* q-$\mathcal{N}(\tau_n)$ *has a finite subcover.*

(*d*). *Each cover of* X *of members of* $q(\tau_n)$ *has a finite subcover.*

Proof. $(a) \Rightarrow (b)$: Let $(X, \{\tau_n\})$ be s-compact. Let $\mathcal{A} = \{W \backslash F \colon W \in \mu(\tau_n)$ and $F \subseteq X$ is a finite set$\}$ and let $\mathcal{F} \subseteq \mathcal{A}$ be a cover of X. Let $\mathcal{F} = \{W_\alpha \backslash F_\alpha \colon \alpha \in \Delta,$ $W_\alpha \in \mu(\tau_n)$ and $F_\alpha \subseteq X$ is a finite set$\}$. Then $X = \cup_{\alpha\in\Delta}\, W_\alpha \backslash F_\alpha$. This implies that $X = \cup_{\alpha\in\Delta}\, W_\alpha$. By part $(a.)$, there exists a finite set $\Delta_1 \subseteq \Delta$ such that $\{W_\alpha \colon \alpha \in \Delta_1\}$ covers X. Let $H = \cup_{\alpha\in\Delta_1}\, F_\alpha$. For each $x \in H$, choose $\alpha_x \in \Delta$ such that $x \in W_{\alpha x} \backslash F_{\alpha x}$. Thus, $\{W_{\alpha x} \backslash F_{\alpha x} \colon x \in H\} \cup \{W_\alpha \backslash F_\alpha \colon \alpha \in \Delta_1\}$ is a finite subcover of $\mathcal{F}$.

$(b) \Rightarrow (c)$: Suppose $(X, \{\tau_{n\mathcal{N}}\})$ is s-compact and let $\mathcal{F} = \{F_\alpha \colon \alpha \in \Delta\}$ be a cover of X consists of elements of q-$\mathcal{N}(\tau_n)$. For each $\alpha \in \Delta$, there exist $A_{n\alpha} \in \tau_{n\mathcal{N}}$ such that

$F_\alpha = \cup_{n\in\mathbb{N}} A_{n\alpha}$. Since $\{F_\alpha \colon \alpha \in \Delta\}$ covers X and $\{A_{n\alpha} \colon n \in \mathbb{N}, \alpha \in \Delta\} \subseteq \mu(\tau_{n\mathcal{N}})$. By (b), there exists a finite set $\Delta_1 \subseteq \Delta$ such that $\{A_{n\alpha} \colon n \in \mathbb{N}, \alpha \in \Delta_1\}$ covers X. It follows that $\{F_\alpha \colon \alpha \in \Delta_1\}$ is a finite subcover of $\mathcal{F}$.

$(c) \Rightarrow (d)$: Let F be a cover of X with $F \subseteq q(\tau_n)$. Since $q(\tau_n) \subseteq q\text{-}\mathcal{N}(\tau_n)$, then $F \subseteq q\text{-}\mathcal{N}(\tau_n)$. Therefore, by part (c), $\mathcal{F}$ has a finite subcover.

$(d) \Rightarrow (a)$: Let F be a cover of X with $F \subseteq \mu(\tau_n)$. Since $\mu(\tau_n) \subseteq q(\tau_n)$, by part (d) every cover of X with members of $\mu(\tau_n)$ has a finite subcover. It follows that $(X, \{\tau_n\})$ is s-compact. □

Theorem 2.31. *Let $(X, \{\tau_n\})$ be an (a)topological space. Then $(X, \{\tau_n\})$ is p-compact if and only if $(X, \{\tau_{n\mathcal{N}}\})$ is p-compact.*

Proof. Let $(X, \{\tau_n\})$ be p-compact. Let $\mathcal{F} = \{F_\alpha \colon \alpha \in \Delta\}$ be a p-open cover of $(X, \{\tau_{n\mathcal{N}}\})$. For each $n \in \mathbb{N}$, there exist $\alpha_n \in \Delta$ such that $F_{\alpha_n} \in \tau_{n\mathcal{N}} \backslash \{\emptyset\}$. For each $\alpha \in \Delta$, there exists an indexed set ω_α such that $F_{\alpha_n} = \cup_{\beta\in\omega_\alpha} A_\beta \backslash B_\beta$ where $\{A_\beta \colon \beta \in \omega_\alpha\} \subseteq \tau_n$ for some $n \in \mathbb{N}$ and $\{B_\beta \colon \beta \in \omega_\alpha\}$ is a family of finite subsets of X. For every $n \in \mathbb{N}$, choose $\beta_n \in \omega_{\alpha_n}$ such that $A_\beta \in \tau_n \backslash \{\emptyset\}$. Thus, $\{A_\beta \colon \beta \in \cup_{\alpha\in\Delta} \omega_\alpha\}$ is a p-open cover of $(X, \{\tau_n\})$. Since $(X, \{\tau_n\})$ is p-compact, then there exists a finite set $\Delta_1 \subseteq \Delta$ such that for every $\alpha \in \Delta_1$, there exists a finite set $\gamma_\alpha \subseteq \omega_\alpha$ such that $\{A_\beta \colon \beta \in \cup_{\alpha\in\Delta_1} \gamma_\alpha\}$ covers X. Take $G = \{B_\beta \colon \beta \in \cup_{\alpha\in\Delta_1} \gamma_\alpha\}$. For each $x \in G$, choose $\alpha_x \in \Delta$ such that $x \in F_{\alpha_x}$. Then we have $\{F_\alpha \colon \alpha \in \Delta_1\} \cup \{F_{\alpha_x} \colon x \in G\}$ is a finite subcover of $\mathcal{F}$.

Conversely, $(X, \{\tau_n\}) \subseteq (X, \{\tau_{n\mathcal{N}}\})$, every p-open cover of $(X, \{\tau_n\})$ is also a p-open cover of $(X, \{\tau_{n\mathcal{N}}\})$ and thus every p-open cover of $(X, \{\tau_n\})$ has a finite subcover. Hence, $(X, \{\tau_n\})$ is p-compact. □

Theorem 2.32. *Let $(X, \{\tau_n\})$ be an s-compact (a)topological space. Then every q-$\mathcal{N}$-closed subset in $(X, \{\tau_n\})$ is s-compact in $(X, \{\tau_n\})$.*

Proof. Let $(X, \{\tau_n\})$ be an s-compact (a)topological space and A is a q-$\mathcal{N}$-closed subset in $(X, \{\tau_n\})$. Let $\mathcal{F}$ be a $\{\tau_n\}$-open cover of A. Then $\mathcal{F} \subseteq q(\tau_n) \subseteq q$-$\mathcal{N}(\tau_n)$. since $X \backslash A$ is q-$\mathcal{N}$-open, $\mathcal{F} \cup \{X \backslash A\} \subseteq q$-$\mathcal{N}(\tau_n)$ is a cover of X. Since $(X, \{\tau_n\})$ is a s-compact, $(X, \{\tau_{n\mathcal{N}}\})$ is also s-compact. So $\mathcal{F} \cup \{X \backslash A\}$ has a finite subcover say $\mathcal{H}$. Thus $\mathcal{H} \backslash \{X \backslash A\}$ is a finite subcover of A and hence A is s-compact in $(X, \{\tau_n\})$. □

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References

[1] T. A. Al-Hawary, A. Al-Omari, Quasi b-open sets in bitopological spaces, *Abhath AL-Yarmouk Journal*, **21** (2012) 1-14.

[2] A. Al-Omari, MS. Noorani, New characterization of compact spaces, Proceedings of the 5th Asian Mathematical Conference, Malaysia, 53-60, 2009.

[3] A. Al-Omari, MS. Noorani, Characterization of strongly compact spaces, *Int. J. Math. Math. Sci.*, Art. ID 573038, 9 pages (2009).

[4] M. k. Bose, A. Mukharjee, On countable families of topologies on a set, *Novi Sad J. Math.*, **40** (2) (2010) 7-16.

[5] M. K. Bose, R. Tiwari, (ω)topological connectedness and hyperconnectedness, *Note Mat.*, **31** (2011), 93-101.

[6] M. K. Bose, R. Tiwari, On increasing sequences of topologies on a set, *Riv. Mat. Univ. Parma.*, (7) **7** (2007), 173-183.

[7] M. K. Bose, R. Tiwari, On (ω)topological spaces, *Riv. Mat. Univ. Parma.*, (7) **9** (2008), 125-132.

[8] J. C. Kelly, Bitopological spaces, *J. Proc. London Math. Soc.*, **13** (1963), 71-89.

[9] M. Datta, Projective bitopological spaces, *J. Austral. Math. Soc.*, **13** (1962), 65-78.

[10] S. Al. Ghour, S. Issa, On μ-ω-open and q-ω-open sets in bitopological spaces, *Missouri J. Math. Sci.*, **24** (2012), 37-53.

[11] S. Al. Ghour, H. Saleh, Quasi-$\mathcal{N}$-Open sets and related compactness concepts in bitopological spaces, *Proyecciones* (*Antofagasta*), **37**(3) (2018), 491-501.

[12] H. Hdeib, ω-closed mappings, *Revista Colomb. De Matem.*, **16** (1982), 65-78.

[13] H. Hdeib, ω-continuous functions, *Dirasat Journal*, **16** (1989), 136-153.

[14] M. M. Kovár, On 3-topological version of θ-regularity, *Int. J. Math. Math. Sci.*, **23** (2000), 393-398.

[15] M. M. Kovár, A note on the comparison of topologies, *Int. J. Math. Math. Sci.*, **26** (2001), 233-237.

[16] E. P. Lane, Bitopological spaces and quasi-uniform spaces, *Proc. Lond. Math. Soc.*, (3) **17** (1967), 241-256.

[17] J. Y. Lee, J. J. Lee, Quasi-semi-open sets and quasi-semi-continuity, *Ulsan Inst. Tach. Rep.*, **13** (1982), 171-173.

[18] S. N Maheshwari, P. C. Jain, G. I. Chae, On quasiopen sets, *Ulsan Inst. Tech. Rep.*, **11** (1980), 291-292.

[19] S. N. Maheshwari, G. I. Chae, S. S. Thakur, Quasi semiopen sets, *Univ. Ulsan Rep.*, **17** (1986), 133-137.

[20] C. W. Patty, Bitopological spaces, *Duke Mathn Journ.*, **34** (1967), 387-391.

[21] W. J. Pervin, Connectedness in bitopological spaces, *Ind. math.*, **29** (1967), 369-372.

[22] A. Roy Choudhuri, A. Mukharjee, M. K. Bose, Hyperconnectedness and extremal disconnectedness in (a)topological spaces, *Hacettepe Journal of mathematics and Statistics*, **44** (2) (2015), 289-294.

[23] B. K. Tyagi, S. Luthra, Harsh V. S. Chauhan, On new types of sets via γ-open in (a)topological spaces, *CUBO*, **20** (2) (2018), 41-52.

[24] S. S. Thakur, P. Palik, Quasi alpha-sets, *J. Indian. Acad. Math.*, **7** (1985), 91-95.

[25] S. S. Thakur, P. Verma, Quasi semi preopen sets, *Vikram Math. J.*, **11** (1991), 57-61.

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