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THE VERTEX DETOUR MONOPHONIC NUMBER OF A GRAPH*

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Abstract. In this paper we determine bounds for x-detour monophonic number and characterize graphs which realize these bounds. A connected graph of order p with vertex detour monophonic numbers either $p-1$ or $p-2$ for every vertex is characterized. It is shown that for each triple a, b and p of integers with $1 \leq a \leq b \leq p-4$, there is a connected graph G of order p such that x-monophonic number is a and x-detour monophonic number is b for some vertex x in G. Also, for integers a, b and p with $1 \leq a \leq p-b$ and $b \geq 2$, there is a connected graph G of order p such that x-detour monophonic number is a and monophonic eccentricity of x is b for some vertex x in G.

1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [4]. For vertices x and y in a connected graph G, the *distance*

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$d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an *x-y geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(u) \cup \{v\}$. A vertex v is a *simplicial vertex* of G if the subgraph induced by its neighbors is complete. A *nonseparable graph* is connected, nontrivial, and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. A *caterpillar* is a tree for which the removal of all the end vertices gives a path. The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3].

The concept of vertex geodomination number was introduced in [6] and further studied in [7]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an x-y geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a *g_x-set* of G.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A longest x-y monophonic path P is called an *x-y detour monophonic path*. The *closed interval* $I_m[x, y]$ consists of all vertices lying on some x-y monophonic path of G. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest u-v monophonic path in G. The *monophonic* *eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, rad_mG of G is $rad_mG = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_mG$ of G is $diam_mG = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [8].

The concept of vertex monophonic number was introduced and studied in [9]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-monophonic set* of G if each vertex v of G lies on an x-y monophonic path in G for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G and is denoted by $m_x(G)$. An x-monophonic set of cardinality $m_x(G)$ is called a *m_x-set* of G. The following theorems will be used in the sequel.

Theorem 1.1. [4] Let v be a vertex of a connected graph G. The following statements are equivalent :

(1) v is a cut vertex of G.

(2) There exist u and w distinct from v such that v is on every u-w path.

(3) There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every u-w path.

Theorem 1.2. [4] Every non-trivial connected graph has at least two vertices which are not cut vertices.

Theorem 1.3. [4] Let G be a connected graph with at least three vertices. The following statements are equivalent :

(1) G is a block.

(2) Every two vertices of G lie on a common cycle.

Theorem 1.4. [9] Let x be any vertex of a connected graph G. Then every simplicial vertex of G other than the vertex x (whether x is simplicial vertex or not) belongs to every m_x-set.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Vertex detour monophonic number

Definition 2.1. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-detour monophonic set* if each vertex u of G lies on an x-y detour monoponic path in G for some y in S. The minimum cardinality of an x-detour monophonic set of G is defined as the *x-detour monophonic number* of G and is denoted by $dm_x(G)$. An x-detour monophonic set of cardinality $dm_x(G)$ is called a *dm_x-set* of G.

We observe that for any vertex x in G, x does not belong to any dm_x-set of G.

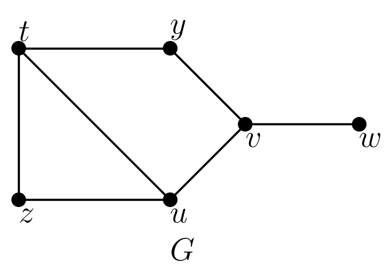

Figure 2.1

Example 2.2. For the graph G given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimum vertex detour monophonic sets and the vertex detour monophonic numbers are given in Table 2.1.

Table 2.1

vertex	minimum vertex monophonic sets	vertex monophonic number	minimum vertex detour mono phonic sets	vertex detour monophonic number
t	$\{z, w\}$	2	$\{z, w\}$	2
y	$\{z, w\}$	2	$\{w, z, t\}$, $\{w, z, u\}$	3
z	$\{w\}$	1	$\{u, w\}$, $\{w, y\}$	2
u	$\{z, w, y\}$	3	$\{w, z, y\}$	3
v	$\{z, w\}$	2	$\{w, t, z\}$, $\{w, u, z\}$	3
w	$\{z\}$	1	$\{t, z\}$, $\{z, u\}$	2

Theorem 2.3. Let x be a vertex of a connected graph G.
(1) Every simplicial vertex of G other than the vertex x (whether x is simplicial vertex or not) belongs to every dm_x-set of G.
(2) No cut vertex of G belongs to any dm_x-set of G.

Proof. (1) Let x be a vertex of G. Then x does not belong to any dm_x-set of G. Let $u \neq x$ be a simplicial vertex and S_x a dm_x-set of G. Suppose that $u \notin S_x$. Then u is an internal vertex of an x-y detour monophonic path, say P, for some $y \in S_x$. Let v and w be the neighbors of u on P. Then v and w are not adjacent and so u is not a simplicial vertex, which is a contradiction.

(2) Let y be a cut vertex of G. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{y\}$ into two subsets U and W such that for any pair of vertices $u \in U$ and $w \in W$, the vertex y is on every u-w path. Hence, if $x \in U$, then for any vertex w in W, y lies on every x-w path so that y is an internal vertex of an x-w detour monophonic path. Let S_x be any dm_x-set of G. Suppose that $S_x \cap W = \emptyset$. Then for any $w_1 \in W$, there exists an element z in S_x such that w_1 lies in some x-z detour monophonic path $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in G.

Now, the x-w_1 subpath of P and w_1-z subpath of P both contain y so that P is not a path in G, which is a contradiction. Hence $S_x \cap W \neq \emptyset$. Let $w_2 \in S_x \cap W$. Then y is an internal vertex of an x-w_2 detour monophonic path. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an x-y detour monophonic path also lies on an x-w_2 detour monophonic path. Hence it follows that S is an x-detour monophonic set of G, which contradicts the fact that S_x is a minimum x-detour monophonic set of G. Thus y does not belong to any dm_x-set. Similarly, if $x \in W$, y does not belong to any dm_x-set. If $x = y$, then obviously y does not belong to any dm_x-set. □

Note 2.4. Even if x is a simplicial vertex of G, x does not belong to any dm_x-set.

Theorem 2.5. For any non-trivial tree T with k end vertices, $dm_x(T) = k - 1$ or k according as x is an end-vertex or not. In fact, if W is the set of all end-vertices of T, then $W - \{x\}$ is the unique dm_x-set of T.

Proof. Let W be the set of all end-vertices of T. It follows from Theorem 2.3 and Note 2.4 that $W - \{x\}$ is the unique dm_x-set of T for any end-vertex x in T and W is the unique dm_x-set of T for any cut vertex x in T. Thus $W - \{x\}$ is the unique dm_x-set of T. □

Theorem 2.6. For any vertex x in a connected graph G of order p, $1 \leq dm_x(G) \leq p - 1$.

Proof. It is clear from the definition of dm_x-set that $dm_x(G) \geq 1$. Also, since the vertex x does not belong to any dm_x-set, it follows that $dm_x(G) \leq p - 1$. □

Remark 2.7. The bounds for $dm_x(G)$ in Theorem 2.6 are sharp, for example $dm_x(C_{2n}) = 1$ for any vertex x in C_{2n}, and $dm_x(K_p) = p - 1$ for any vertex x in K_p.

Now we proceed to characterize graphs for which the bounds in Theorem 2.6 are attained.

Definition 2.8. Let x be any vertex in a connected graph G. A vertex y in G is said to be an x-detour monophonic superior vertex if for any vertex z with $d_m(x, y) < d_m(x, z)$, z lies on an $x-y$ detour monophonic path.

Table 2.2

vertex	t	y	z	u	v	w
vertex detour monophonic superior vertices	$\{w\}$	$\{z\}$	$\{w\}$	$\{y, w\}$	$\{z\}$	$\{z\}$

Example 2.9. For the graph G given in Figure 2.1, the vertex detour monophonic superior vertices are given in Table 2.2.

We give below a property related with monophonic eccentric vertex of x and x-detour monophonic superior vertex in a graph G.

Theorem 2.10. Let x be any vertex in G. Then every monophonic eccentric vertex of x is an x-detour monophonic superior vertex.

Proof. Let y be a monophonic eccentric vertex of x so that $e_m(x) = d_m(x, y)$. If y is not an x-detour monophonic superior vertex, then there exists a vertex z in G such that $d_m(x, y) < d_m(x, z)$ and z does not lie on any $x - y$ detour monophonic path and hence $e_m(x) < d_m(x, z)$, which is a contradiction. □

Note 2.11. The converse of Theorem 2.10 is not true. For the even cycle $C_{2n}(n \geq 3)$, the eccentric vertex of x is an x-detour monophonic superior vertex but it is not a monophonic eccentric vertex of x.

Theorem 2.12. Let G be a connected graph. For a vertex x in G, $dm_x(G) = 1$ if and only if there exists an x-detour monophonic superior vertex y in G such that every vertex of G is on an $x - y$ detour monophomic path.

Proof. Let $dm_x(G) = 1$ and $S = \{y\}$ be a dm_x-set of G. If y is not an x-detour monophonic superior vertex, then there is a vertex z in G with $d_m(x, y) < d_m(x, z)$ and z does not lie on any $x-y$ detour monophonic path. Thus S is not a dm_x-set of G, which is a contradiction. The converse is clear from the definition. □

Theorem 2.13. For any vertex x in a connected graph G of order p, $dm_x(G) = p - 1$ if and only if $deg\ x = p - 1$.

Proof. Let x be any vertex in a connected graph G of order p. Let $dm_x(G) = p - 1$. Suppose that $deg\ x < p - 1$. Then there exists a vertex u in G which is not adjacent to x. Since G is connected, there is a detour monophonic path from x to u, say P, with length greater than or equal to 2. It is clear that $(V(G) - V(P)) \cup \{u\}$ is an x-detour monophonic set of G and hence $dm_x(G) \leq p - 2$, which is a contradiction. Conversely, if $deg\ x = p-1$, then all other vertices of G are adjacent to x and hence all these vertices form the dm_x-set. Thus $dm_x(G) = p - 1$. □

Corollary 2.14. A graph G is complete if and only if $dm_x(G) = p-1$ for every vertex x in G.

Theorem 2.15. Let G be a connected graph. Then $G = K_1 + \cup m_j K_j$ if and only if $dm_x(G) = p - 1$ or $p - 2$ for any vertex x in G.

Proof. Let $G = K_1 + \cup m_j K_j$. Then G has at most one cut vertex. If G has no cut vertex, then $G = K_p$ and so by Corollary 2.14, $dm_x(G) = p - 1$ for every vertex x in G. Suppose that G has exactly one cut vertex. Then all the remaining vertices are simplicial and hence by Theorem 2.3, $dm_x(G) = p - 1$ or $p - 2$ for any vertex x in G.

Conversely, suppose that $dm_x(G) = p - 1$ or $p - 2$ for any vertex x in G. If $p = 2$, then $G = K_2 = K_1 + K_1$. If $p \geq 3$, then by Theorem 1.2, there exists a vertex x, which is not a cut vertex of G. If G has two or more cut vertices, then by Theorem 2.3, $dm_x(G) \leq p-3$, which is a contradiction. Thus, the number of cut vertices k of G is at most one.

Case 1: $k = 0$. Then the graph G is a block. If $p = 3$, $G = K_3 = K_1 + K_2$. For $p \geq 4$, we claim that G is complete. If G is not complete, then there exist two vertices x and y in G such that $d(x, y) \geq 2$. By Theorem 1.3, x and y lie on a common cycle and hence x and y lie on a smallest cycle $C : x, x_1, \ldots, y, \ldots, x_n, x$ of length at least 4. If $d_m(x, y) = 2$, then $V(G) - \{x, x_1, x_n\}$ is an x-detour monophonic set of G and so $dm_x(G) \leq p-3$, which is a contradiction to the assumption. If $d_m(x, y) > 2$, then let P be an $x-y$ detour monophonic path of order at least 4. Clearly $(V(G) - V(P)) \cup \{y\}$ is an x-detour monophonic set of G and so $dm_x(G) \leq p - 3$, which is a contradiction. Hence G is the complete graph K_p and so $G = K_1 + K_{p-1}$.

Case 2: $k = 1$. Let x be the cut vertex of G. If $p = 3$, then $G = P_3 = K_1 + m_j K_1$, where $\sum m_j = 2$. If $p \geq 4$, we claim that $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. It is enough to prove that every block of G is complete. Suppose there exists a block B, which is not complete. Let u and v be two vertices in B such that $d(u,v) \geq 2$. Then by Theorem 1.3, both u and v lie on a common cycle so that u and v lie on a smallest cycle of length at least 4. Then as in Case 1, $dm_u(G) \leq p-3$, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \cup m_j K_j$, where K_1 is the vertex x and $\sum m_j \geq 2$. □

Theorem 2.16. Let G be a connected graph of order $p \geq 3$ with exactly one cut vertex. Then $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$ if and only if $dm_x(G) = p-1$ or $p-2$ for any vertex x in G.

Proof. The proof is contained in Theorem 2.15. □

Now, Corollary 2.14 and Theorem 2.15 lead to the natural question whether there exists a graph G for which $dm_x(G) = p-2$ for every vertex x in G. This is answered in the next theorem.

Theorem 2.17. There is no graph G of order p with $dm_x(G) = p-2$ for every vertex x in G.

Proof. If $dm_x(G) = p-2$ for every vertex x in G, then use Theorem 2.15 to get $G = K_1 + \cup m_j K_j$. If x is K_1, then use Theorem 2.13 to get $dm_x(G) = p-1$. But this contradicts the assumption. Thus there is no graph G with $dm_x(G) = p-2$ for every vertex x in G. □

Theorem 2.18. For any non-trivial tree T with monophonic diameter d_m,
$dm_x(T) = p - d_m$ or $p - d_m + 1$ for any vertex x in T if and only if T is a caterpiller.

Proof. Let T be any non-trivial tree. Let $P : v_0, v_1, \ldots, v_{d_m}$ be a monophonic path of length d_m. Let k be the number of end vertices of T

and l be the number of internal vertices of T other than $v_1, \ldots, v_{d_{m-1}}$. Then $d_m - 1 + l + k = p$. By Theorem 2.5, $dm_x(T) = k$ or $k-1$ for any vertex x in T. Hence $dm_x(T) = p - d_m - l + 1$ or $p - d_m - l$ for any vertex x in T. Hence $dm_x(T) = p - d_m + 1$ or $p - d_m$ for any vertex x in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P, if and only if T is a caterpillar. □

Theorem 2.19. For any vertex x in the cycle C_n $(n \geq 3)$, $dm_x(C_n) = 1$ or 2 according as n is even or odd.

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be the cycle of order n. Let x be any vertex in C_n, say $x = u_1$. If n is even, then $S_x = \{u_{\frac{n}{2}+1}\}$ is an x-detour monophonic set and so $dm_x(C_n) = 1$. If n is odd, then $S_x = \{u_2, u_3\}$ is a minimum x-detour monophonic set and so $dm_x(C_n) = 2$. □

Theorem 2.20. Let $W_n = K_1 + C_{n-1}$ $(n \geq 5)$ be the wheel.
(1) If $n = 5$, then $dm_x(W_n) = n - 1$ or 1 according as x is K_1 or x is in C_{n-1}.
(2) If n is odd and $n \geq 7$, then $dm_x(W_n) = n - 1$ or 2 according as x is K_1 or
x is in C_{n-1}.
(3) If n is even, then $dm_x(W_n) = n - 1$ or 3 according as x is K_1 or
x is in C_{n-1}.

Proof. Let $C_{n-1} : u_1, u_2, \ldots, u_{n-1}, u_1$ be a cycle of order $n-1 \geq 4$ and u be the vertex of K_1. If $x = u$, then by Theorem 2.13, $dm_x(W_n) = n-1$. Let x be any vertex in C_{n-1}, say $x = u_1$. If $n = 5$, then every vertex of W_n lies on an x-u_3 detour monophonic path and so $\{u_3\}$ is an x-detour monophonic set of W_n. Hence it follows that $dm_x(W_n) = 1$. If

$n \geq 7$ and n is odd, then no 1-element subset of $V(W_n)$ is an x-detour monophonic set of W_n. Since $\{u_{\frac{n+1}{2}}, u\}$ is an x-detour monophonic set of W_n, it follows that $dm_x(W_n) = 2$. If n is even, then neither 1-element nor 2-element subset of $V(W_n)$ will form an x-detour monophonic set of G. It is clear that $\{u, u_2, u_3\}$ is an x-detour monophonic set of G and so $dm_x(G) = 3$. □

Theorem 2.21. Let $K_{n_1,n_2,\dots,n_k}$ $(n_i \geq 2)$ be a complete k-partite graph with partition $(V_1, V_2, \dots, V_k)$. Then $dm_x(K_{n_1,n_2,\dots,n_k})$ is $n_i - 1$ according as $x \in V_i$. Moreover, if $n_1 = n_2 = \dots = n_k = r + 1$, then $dm_x(K_{n_1,n_2,\dots,n_k}) = r$ for every vertex x in $K_{n_1,n_2,\dots,n_k}$.

Proof. Let $x \in V_i$. Then it is clear that $V_i - \{x\}$ is a minimum x-detour monophonic set of G and so $dm_x(K_{n_1,n_2,\dots,n_k}) = n_i - 1$. □

For a vertex v in a graph G, the *link* $L(v)$ of v is the subgraph induced by the neighbors of v.

Theorem 2.22. For every integer $k \geq 1$ and every k graphs $G_1, G_2, \dots, G_k$, there exists a connected graph G with a unique dm_x-set $\{v_1, v_2, \dots, v_k\}$ for some x in G such that $L(v_i) = G_i$ for $1 \leq i \leq k$.

Proof. We construct a graph G with the desired property. For each integer i $(1 \leq i \leq k)$, let $F_i = \overline{K_2} + G_i$, where $V(\overline{K_2}) = \{u_i, v_i\}$. Then the graph G is constructed from the graph F_i by adding a new vertex x and the k edges xu_i $(1 \leq i \leq k)$. Thus in G, $L(v_i) = G_i$ for $1 \leq i \leq k$.

If $k = 1$, then every vertex of G lies on an $x - v_1$ detour monophonic path and hence $S = \{v_1\}$ is the unique minimum x-detour monophonic set of G. If $k \geq 2$, then x is a cut-vertex of G. Now, we show that every x-detour monophonic set of G contains an element of every component of $G - \{x\}$. Suppose that there is an x-detour monophonic set S of

G such that S contains no vertex of a component, say B, of $G - \{x\}$. Let $v \in V(B)$. Since S is an x-detour monophonic set, there exists an element $y \in S$ such that v lies in some x-y detour monophonic path $P : x, y_1, \ldots, v, \ldots, y_n = y$ in G. Now the x-v subpath of P and v-y subpath of P both contain x and it follows that P is not a path, which is a contradiction. Thus every x-detour monophonic set of G contains an element of every component of $G - \{x\}$ and so $dm_x(G) \geq k$.

Let $S = \{v_1, v_2, \ldots, v_k\}$. Clearly S is an x-detour monophonic set of G and so $dm_x(G) = k$.

Next, we show that S is a unique dm_x-set of G. Assume, to the contrary, that S' is a dm_x-set of G distinct from S. Clearly S' must contain exactly one vertex from each subgraph F_i $(1 \leq i \leq k)$. Since $S \neq S'$, we may assume that $v_1 \notin S'$. Since u_1 is a cut vertex of G, it follows from Theorem 2.3 that $y \in S$ for some vertex y in G_1. Since $d_m(z, u_1) = 1$ for any z in G_1, v_1 is not an internal vertex of any detour monophonic path from x, which is a contradiction. □

The graph G constructed in the proof of Theorem 2.22 has a cut-vertex and so is not 2-connected. However, we can extend Theorem 2.22 by modifying the structure of the graph G in the proof of Theorem 2.22 to construct a 2-connected graph with the properties described in Theorem 2.22.

Theorem 2.23. For every integer $k \geq 1$ and every k graphs $G_1, G_2, \ldots, G_k$ of order at least two, there exists a 2-connected graph G with a unique dm_x-set $\{v_1, v_2, \ldots, v_k\}$ for some x in G such that $L(v_i) = G_i$ for $1 \leq i \leq k$.

Proof. For each integer i $(1 \leq i \leq k)$, let $F_i = \overline{K_3} + G_i$, where $V(\overline{K_3}) = \{u_i, v_i, w_i\}$. Then a 2-connected graph G is constructed from the graph

F_i by adding a new vertex x and the $4k$ new edges xu_i, xw_i, u_iw_i and u_iw_{i+1} for $1 \leq i \leq k$, where the subscripts are expressed modulo k. For $k = 3$, the graph G is shown in Figure 2.2. Thus in G, $L(v_i) = G_i$ for $1 \leq i \leq k$. Now claim that every x-detour monophonic set of G contains an element of every $F_i (1 \leq i \leq k)$. Suppose that there is an x-detour monophonic set S of G such that S contains no vertex of some F_i, say F_1. Let $y \in V(G_1)$. Since S is an x-detour monophonic set, there exists an element $w \in S$ such that y lies in some x - w detour monophonic path P in G. Since $\{u_1, w_1\}$ is a cut set of G and it separate F_1 from G, the path P contains both the vertices u_1 and w_1 and so P is not a detour monophonic path, which is a contradiction. Thus every x-detour monophonic set of G contains an element of every $F_i(1 \leq i \leq k)$ and hence $dm_x(G) \geq k$. Let $S = \{v_1, v_2, \dots, v_k\}$. Since every vertex of F_i lies on an x - v_i detour monophonic path, S is an x-detour monophonic set of G and so $dm_x(G) = k$.

Next, we show that S is the unique dm_x-set of G. Assume, to the contrary, that S' is a dm_x-set of G distinct from S. Clearly S' must contain exactly one vertex from each subgraph $F_i(1 \leq i \leq k)$. Since $S \neq S'$, we may assume that $v_1 \notin S'$. Since $d_m(x, y) \leq 2$ for any $y \in V(F_1) - \{v_1\}$ and $d_m(x, v_1) = 3$, we have v_1 is not an internal vertex of any detour monophonic path from x, which is a contradiction. □

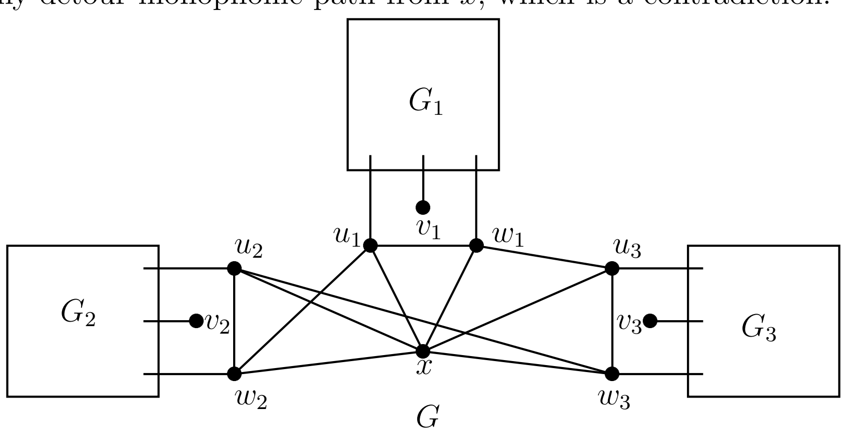

Figure 2.2

Next we present a theorem, which gives the relation between $m_x(G)$ and $dm_x(G)$ of a graph G.

Theorem 2.24. Let x be any vertex in a connected graph G. Then $1 \leq m_x(G) \leq dm_x(G) \leq p-1$.

Proof. It is clear from the definition of m_x-set that $m_x(G) \geq 1$. Since every x-detour monophonic set is an x-monophonic set, we have $m_x(G) \leq dm_x(G)$. Also, since the vertex set x does not belong to any dm_x-set, it follows that $dm_x(G) \leq p-1$. □

The bounds of Theorem 2.24 are sharp. The cycle $C_{2n}(n \geq 2)$ has $m_x(C_{2n}) = dm_x(C_{2n}) = 1$ for every vertex x in C_{2n}. Also, the non-trivial path P_n has $m_x(P_n) = dm_x(P_n) = 1$ for an end-vertex x in P_n. For any vertex x in the complete graph $K_p(p \geq 2)$, $m_x(K_p) = dm_x(K_p) = p-1$. Also, all the inequalities in Theorem 2.24 are strict. For the graph G given in Figure 2.1, $m_y(G) = 2$, $dm_y(G) = 3$ and $p = 6$ so that $1 < m_x(G) < dm_x(G) < p$.

Corollary 2.25. Let x be any vertex in a connected graph G. If $dm_x(G) = 1$, then $m_x(G) = 1$.

Proof. This follows from Theorem 2.24. □

Theorem 2.26. Let x be any vertex in a connected graph G of order p. Then $dm_x(G) \leq p - e_m(x)$.

Proof. Let x be any vertex in G and y a monophonic eccentric vertex of x. Then $d_m(x,y) = e_m(x)$. Let $P : x = x_0, x_1, x_2, \ldots, x_n = y$ be an x-y detour monophonic path in G. Let $S = V(G) - \{x_0, x_1, \ldots, x_{n-1}\}$. Since each $x_i(0 \leq i \leq n-1)$ lies on an x-y detour monophonic path, S is an x-detour monophonic set of G so that $dm_x(G) \leq p - e_m(x)$. □

Remark 2.27. The bound in Theorem 2.26 is sharp. For any vertex x in the odd cycle C_{2n+1}, $e_m(x) = 2n - 1$ and $dm_x(C_{2n+1}) = 2$. Thus $dm_x(C_{2n+1}) = p - e_m(x)$.

Theorem 2.28. For each triple a, b and p of integers with $1 \leq a \leq b \leq p - 4$, there is a connected graph G of order p such that $m_x(G) = a$ and $dm_x(G) = b$ for some vertex x in G.

Proof. Case 1. $1 \leq a = b \leq p - 4$. Let G be a tree of order p with $a+1$ end-vertices. Let x be an end-vertex of G. Then G has the desired properties.

Case 2. $1 \leq a < b \leq p - 4$. Let G be a graph obtained from the cycle $C_{p-b+1} : u_1, u_2, \ldots, u_{p-b+1}, u_1$ of order $p - b + 1$ by (i) adding a new vertices $v_1, v_2, \ldots, v_a$ and joining each vertex $v_i (1 \leq i \leq a)$ to u_{p-b}; and (ii) adding $b - a - 1$ new vertices $w_1, w_2, \ldots, w_{b-a-1}$ and joining each $w_i (1 \leq i \leq b - a - 1)$ to every vertex $y \in \{u_1, u_2, \ldots, u_{p-b}\}$. The graph G has order p and is shown in Figure 2.3. Let $S = \{v_1, v_2, \ldots, v_a\}$ be the set of all simplicial vertices of G. Then by Theorems 1.4 and 2.3, every x-monophonic set and every x-detour monophonic set of G contains S for the vertex $x = u_1$. It is clear that every vertex of G lies on an x-v_i $(1 \leq i \leq a)$ monophonic path so that $m_x(G) = a$. It is clear that S is not an x-detour monophomic set of G. Also, it is easily seen that $w_i (1 \leq i \leq b - a - 1)$ is not an internal vertex of any detour monophonic path starting from x. Thus every x-detour monophonic set of G contains $S_1 = S \cup \{w_1, w_2, \ldots, w_{b-a-1}\}$. Since every vertex $u_i (1 \leq i \leq p - b)$ lies on an x-v_1 detour monophonic path and the vertex u_{p-b+1} lies on an x-u_3 detour monophonic path, $S_1 \cup \{u_3\}$ is an x-detour monophonic set of G. Hence $dm_x(G) = b$. □

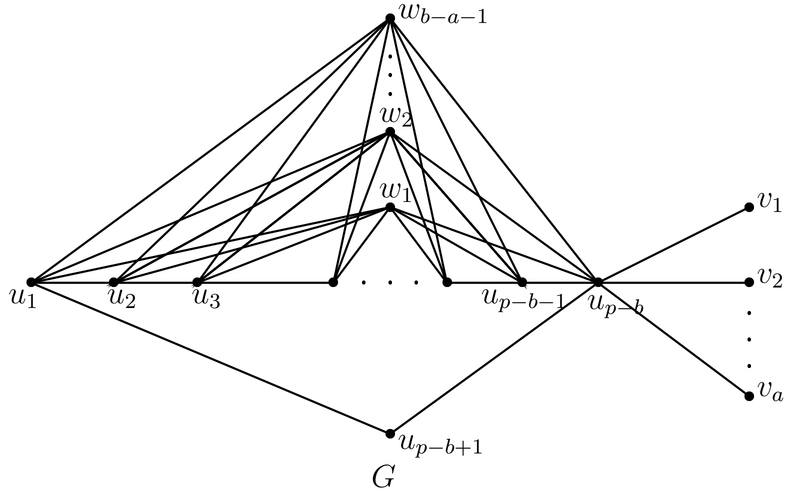

Figure 2.3

Theorem 2.29. For integers a, b and p with $1 \leq a \leq p - b$ and $b \geq 2$, there is a connected graph G of order p such that $dm_x(G) = a$ and $e_m(x) = b$ for some vertex x in G.

Proof. Let $P_{b+1} : u_1, u_2, \ldots, u_{b+1}$ be a path of order $b + 1$. Let G be the graph obtained from P_{b+1} by adding $p - b - 1$ new vertices $v_1, v_2, \ldots, v_{a-1}, w_1, w_2, \ldots,$
w_{p-b-a} and joining each $v_i (1 \leq i \leq a - 1)$ to u_b; also joining each $w_i (1 \leq i \leq p - b - a)$ to both u_1 and u_3. The graph G has order p and is shown in Figure 2.4. Let $x = u_1$ be a vertex in G. Then, clearly $e_m(x) = b$. If $b \geq 3$, then $S = \{v_1, v_2, \ldots, v_{a-1}, u_{b+1}\}$ is the set of all end vertices of G. Since every vertex of G lies on an x - y detour monophonic path for some $y \in S$ and by Theorem 2.3, S is the unique minimum x-detour monophonic set of G. Hence $dm_x(G) = |S| = a$. If $b = 2$, then $S' = S - \{u_3\}$ is the set of all end-vertices of G. By Theorem 2.3, every x-detour monophonic set of G contains S'. Also, it is easily seen that S' is not an x-detour monophonic set of G and so

$dm_x(G) > a - 1$. Then $S' \cup \{u_3\}$ is an x-detour monophonic set of G and so $dm_x(G) = a$. □

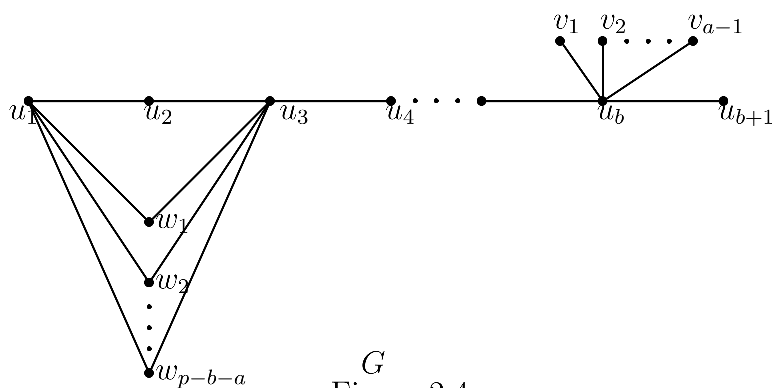

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Figure 2.4

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