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k- KERNELS IN DIGRAPHS FORMED BY SOME OPERATIONS FROM OTHER DIGRAPHS

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ABSTRACT. Let $k \geq 2$ be a positive integer. For a digraph D, a set $J \subseteq V(D)$ is said to be a k- kernel of D if for every $x, y \in J$, $d_D(x, y) \geq k$ and for every $z \in V(D) \backslash J$, there exists $w \in J$ such that $d_D(z, w) \leq k-1$. Given a digraph D, as a generalisation of the operations defined in [9], we define operations on D, each of which results in a digraph with a k- kernel.

1. Introduction

For notation and terminology, in general, we follow [1] and [2]. Let D denote a finite digraph with vertex set $V(D)$ and arc set $A(D)$.

For $x, y \in V(D)$, *distance* from x to y in D, denoted by $d_D(x, y)$, is the number of arcs in a shortest directed path from x to y in D.

For $X, Y \subseteq V(D)$, $d_D(X, Y) = \min\{d_D(x, y) : x \in X,\ y \in Y\}$. When $X = \{x\}$, $d_D(\{x\}, Y) = d_D(x, Y)$. Similar notation holds for $Y = \{y\}$.

For $x \in V(D)$, $N_D^+(x) = \{y \in V(D) \ : \ (x, y) \in A(D)\}$ and $N_D^-(x) = \{y \in V(D) \ : \ (y, x) \in A(D)\}$.

For $x \in V(D)$, and for an integer j, let $\Gamma_D^{+j}(x) = \{y \in V(D) \ : \ d_D(x, y) \leq j\}$ and $\Gamma_D^{-j}(x) = \{y \in V(D) \ : \ d_D(y, x) \leq j\}$. For disjoint sets X and

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Y of vertices, let $X \to Y$ denote the digraph with vertex set $X \cup Y$ and arc set $\{(x, y) : x \in X \text{ and } y \in Y\}$ and let $X \mapsto Y$ denote a spanning subdigraph of $X \to Y$ such that for every $y \in Y$ there exists an $x \in X$ such that $x \to y$. Note that the arc set of the digraph $X \mapsto Y$ is not unique. For convenience, we denote any such digraph using the same notation $X \mapsto Y$.

Let $k \geq 2$ and ℓ be positive intergers. For a set $J \subseteq V(D)$, J is k-*independent* if for every $x, y \in J$, $d_D(x, y) \geq k$; J is ℓ-*absorbent* if for every $z \in V(D) \backslash J$, there exists $w \in J$ such that $d_D(z, w) \leq \ell$; J is a (k, ℓ)-*kernel* of D if J is both k-independent and ℓ-absorbent in D. A k-*kernel* is a $(k, k-1)$-kernel.

It follows from the definition of (k, ℓ)-kernel that, for $2 \leq k_0 \leq k$ and $\ell \leq \ell_0$, every (k, ℓ)-kernel of D is a (k_0, ℓ_0)-kernel of D.

The concept of (k, ℓ)-kernel was introduced by Borowiecki and Kwaśnik (see [11]). A 2-kernel is the classical kernel which was first introduced by Von Neumann and Morgenstern in the context of Game Theory as a solution for co-operative n-player games.

Kernels, on its own, is an interesting research area in digraphs with applications (see [13]) in diverse fields such as logic, decision theory, game theory, artificial intelligence and coding theory. They are directly used to model real-life situations. For example, they are used to find an optimal set of locations to build service centers such as hospitals, schools, etc. From the independence, we will avoid constructing two centers next to one another. Because of the want to spot on, between any two centers maximum distance may be allowed and from any place outside the center, at least one of the service center is reachable at a minimum distance. Here arises the inevitability to study (k, ℓ)-kernels.

Several contributions were made on the study of (k, ℓ)- kernels in digraphs. For a comprehensing view, see the survey article [7]. Specifically, presence of k- kernels in several classes of digraphs has been concentrated widely (see [4], [5], [6] and [10]). The problem of determining whether a digraph has a k- kernel is proved to be NP- complete (see [5]). In addition to the investigation of the existence of k- kernels in a given digraph, constructing digraphs with k- kernels is also an aspect of study (see [3], [8], [9] and [12]).

In [12], the author investigated some unary operations on digraph D (namely, $S(D)$, $R(D)$, $Q(D)$ and $T(D)$) and obtained some necessary or sufficient conditions for the existence or uniqueness of kernels in digraphs formed by these operations from another digraph. In [9], given a digraph D and any integer $k \geq 2$, Galeana-Sánchez and Pastrana have defined four operations $S^k(D)$, $R^k(D)$, $Q^k(D)$ and $T^k(D)$. It is proved that, for $k \geq 2$, $S^k(D)$, $R^k(D)$, $Q^k(D)$ have a k- kernel and for $k \geq 3$, $T^k(D)$ has a k- kernel. Generalising these four operations on D, we introduce four operations that result in digraphs containing a k- kernel.

2. Operations on digraphs that results in digraphs with k-kernels

Let $k \geq 2$ be an integer. As we are dealing with k- kernels of more than one digraph, we use $J(D)$ to denote a k- kernel of the digraph D.

2.1. A generalisation of $S^k(D)$ and $Q^k(D)$.

Let $\mathscr{D}_k$ be the set of all connected digraphs H such that H has a unique k- kernel $J(H)$ with at least 2 vertices.

Given a digraph D with arc set $A(D) = \{e_1, e_2, \ldots, e_m\}$, say $e_i = (x_i, y_i)$, $i \in \{1, 2, \ldots, m\}$, and m digraphs H_i, $i \in \{1, 2, \ldots, m\}$, in

$\mathscr{D}_k$, with $x_i, y_i \in J(H_i)$, $V(H_i) \cap V(D) = \{x_i, y_i\}$ and for $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$, $V(H_i) \cap V(H_j) \subseteq V(D)$, we define two digraphs $S^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ and $Q^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ as follows:

I. $S^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ be the digraph obtained from D by replacing the arcs e_1, e_2, $\ldots, e_m$ by the digraphs $H_1, H_2, \ldots, H_m$, respectively.

II. $Q^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m) = S^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$
$$\cup \Big(\bigcup \big(N^-_{H_i}(y_i) \backslash \Gamma^{+(k-3)}_{H_i}(y_i)\big) \to \big(N^-_{H_j}(y_j) \backslash \Gamma^{+(k-3)}_{H_j}(y_j)\big) \Big),$$
where the union $\bigcup$ runs over all paths $x_i e_i y_i (= x_j) e_j y_j$ of length 2 in D. Here, the path $x_i e_i y_i (= x_j) e_j y_j$ represents the path $x_i e_i y_i e_j y_j$ with $y_i = x_j$.

For example, see Figures 1, 2, 3 and 4.

In I and II, when we take each H_i as a directed path P_{ℓ_i+1} of length $\ell_i \equiv 0 \pmod k$ with origin x_i and terminus y_i, we have $S^k(D; P_{\ell_1+1}, x_1, y_1; \ldots; P_{\ell_m+1}, x_m, y_m)$ and $Q^k(D; P_{\ell_1+1}, x_1, y_1; \ldots; P_{\ell_m+1}, x_m, y_m)$.

In [9], the digraphs $S^k(D; P_{\ell_1+1}, x_1, y_1; \ldots; P_{\ell_m+1}, x_m, y_m)$ and $Q^k(D; P_{\ell_1+1}, x_1, y_1; \ldots; P_{\ell_m+1}, x_m, y_m)$ are denoted, respectively, using the notation $S^k(D)$ and $Q^k(D)$, both of which, in the sense of [9], are not unique. Also, Galeana-Sánchez and Pastrana have proved that both $S^k(D)$ and $Q^k(D)$ contains a k- kernel. We now generalise this result to $S^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ and $Q^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$.

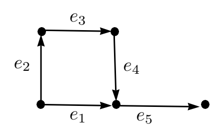

FIGURE 1. D

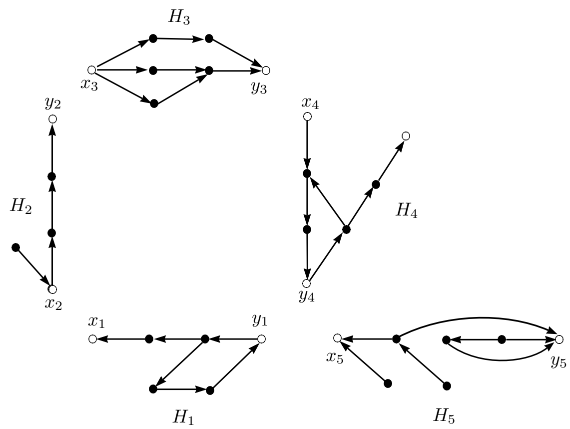

FIGURE 2. H_1, H_2, H_3, H_4 and H_5.
Vertices marked by ∘ constitute the unique 3 - kernel of the respective H_i ' s .

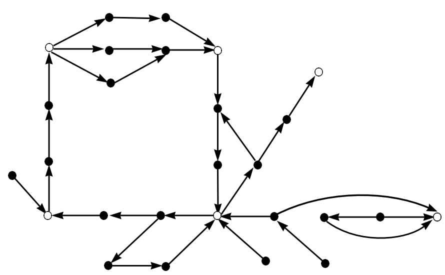

FIGURE 3.
$S^3(D; H_1, x_1, y_1; H_2, x_2, y_2; H_3, x_3, y_3; H_4, x_4, y_4; H_5, x_5, y_5)$

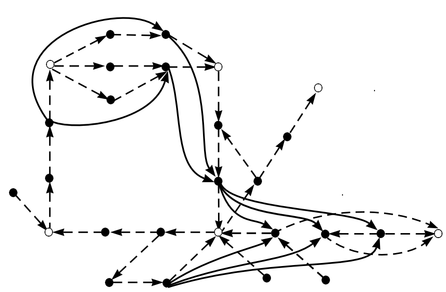

FIGURE 4.
$Q^3(D; H_1, x_1, y_1; H_2, x_2, y_2; H_3, x_3, y_3; H_4, x_4, y_4; H_5, x_5, y_5)$
(Dotted arcs are the arcs of $S^3(D; H_1, x_1, y_1; H_2, x_2, y_2; H_3, x_3, y_3; H_4, x_4, y_4; H_5, x_5, y_5)$.)

2.2. k- kernels in the generalisations.

Theorem 2.1. *The digraphs* $S^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ *and* $Q^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ *contains a* k- *kernel.*

Proof. For the sake of convenience, we use $S^k_{H_1,\ldots,H_m}(D)$ and $Q^k_{H_1,\ldots,H_m}(D)$, respectively, for $S^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ and $Q^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$.

Let $J = \bigcup_{i=1}^{m} J(H_i)$. By the definition of $S^k_{H_1,\ldots,H_m}(D)$ and $Q^k_{H_1,\ldots,H_m}(D)$, $V(D) \subseteq J$.

Claim 1. J is k-independent in $Q^k_{H_1,\ldots,H_m}(D)$.

Let $x, y \in J$. If $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) = \infty$, then there is nothing to prove. So, assume $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) < \infty$. Let P be a shortest directed path from x to y in $Q^k_{H_1,\ldots,H_m}(D)$.

Observe that the first and the last arcs of P are in $S^k_{H_1,\ldots,H_m}(D)$. If the first arc of P is in H_i for some $i \in \{1, 2, \ldots, m\}$, then $x \in J(H_i)$. Similarly, if the last arc of P is in H_j for some j with $j \in \{1, 2, \ldots, m\}$, then $y \in J(H_j)$.

If $P \subseteq H_i$, then $i = j$ and $x, y \in J(H_i)$. Therefore, $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) = d_{H_i}(x, y) \geq k$.

If $P \not\subseteq H_i$, then, let P_0 be the maximal section of P contained in H_i with origin x. If x_i or y_i is the terminus of P_0, then $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) \geq d_{H_i}(x, x_i) \geq k$ or $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) \geq d_{H_i}(x, y_i) \geq k$, respectively. So, assume that neither x_i nor y_i is the terminus of P_0. But then, the terminus of P_0 is in $N^-_{H_i}(y_i) \backslash \Gamma^{+(k-3)}_{H_i}(y_i)$, say, z. Note that $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) = d_{Q^k_{H_1,\ldots,H_m}(D)}(x, z) + d_{Q^k_{H_1,\ldots,H_m}(D)}(z, y) = d_{H_i}(x, z) + d_{Q^k_{H_1,\ldots,H_m}(D)}(z, y)$. Clearly, $d_{Q^k_{H_1,\ldots,H_m}(D)}(z, y) \geq 1$. If $x \neq y_i$, then, as $d_{H_i}(x, y_i) \geq k$, $d_{H_i}(x, z) \geq k - 1$. If $x = y_i$, then, as $z \notin \Gamma^{+(k-3)}_{H_i}(y_i)$, $d_{H_i}(x, z) = d_{H_i}(y_i, z) \geq k - 2$; and as the arc of P with tail z is in $(N^-_{H_i}(y_i) \backslash \Gamma^{+(k-3)}_{H_i}(y_i)) \rightarrow (N^-_{H_t}(y_t) \setminus \Gamma^{+(k-3)}_{H_t}(y_t))$ for some $t \neq i$, where $N^-_{H_t}(y_t) \setminus \Gamma^{+(k-3)}_{H_t}(y_t) \subseteq V(Q^k_{H_1,\ldots,H_m}(D)) \setminus J$, we have $d_{Q^k_{H_1,\ldots,H_m}(D)}(z, y) \geq 2$. Thus, in any case, $d_{Q^k_{H_1,\ldots,H_m}(D)}(x, y) \geq k$.

This proves Claim 1.

Claim 2. J is $(k-1)$-absorbent in $S^k_{H_1,\dots,H_m}(D)$.

Let $z \in V(S^k_{H_1,\dots,H_m}(D)) \setminus J$. We prove $d_{S^k_{H_1,\dots,H_m}(D)}(z,J) \le k-1$. By the definition of $S^k_{H_1,\dots,H_m}(D)$ and J, there exists i such that $z \in V(H_i)\backslash J(H_i)$. As $J(H_i)$ is a k-kernel of H_i, there exists $x \in J(H_i)$ such that $d_{H_i}(z,x) \le k-1$. By the definition of $S^k_{H_1,\dots,H_m}(D)$ and J, $d_{S^k_{H_1,\dots,H_m}(D)}(z,J) \le d_{S^k_{H_1,\dots,H_m}(D)}(z,x) = d_{H_i}(z,x) \le k-1$ and $x \in J$, respectively.

This proves Claim 2.

As $S^k_{H_1,\dots,H_m}(D)$ is a spanning subdigraph of $Q^k_{H_1,\dots,H_m}(D)$, Claims 1 and 2 imply that J is a k-kernel of both $S^k_{H_1,\dots,H_m}(D)$ and $Q^k_{H_1,\dots,H_m}(D)$. □

2.3. **A generalisation of $R^k(D)$ and $T^k(D)$.**

Let $\mathscr{D}^0_k$ be the set of all connected digraphs H such that H has a unique k-kernel $J(H)$. (Let $H \in \mathscr{D}^0_k$ with the unique k-kernel $J(H)$. If $|J(H)| \ge 2$, then $H \in \mathscr{D}_k$; otherwise, $|J(H)| = 1$, then $J(H) = \{v\}$ and $\Gamma^{-(k-1)}_H(v) = V(H)$.) Given a digraph $H_r \in \mathscr{D}^0_k$, we construct two digraphs H'_r and H''_r as follows:

For $k \ge 3$, let $H'_r = H_r \cup \{x^{(r)} \to y^{(r)},\ x^{(r)} \to u^{(r)},\ v^{(r)} \to W^{(r)}_1 \mapsto W^{(r)}_2 \mapsto \cdots \mapsto W^{(r)}_{k-2} \to y^{(r)}\}$, where $u^{(r)}$, $v^{(r)} \in J(H_r)$ (not necessarily distinct); $\{x^{(r)}\}$, $W^{(r)}_1$, $W^{(r)}_2$, $\dots$, $W^{(r)}_{k-2}$, $\{y^{(r)}\}$ are nonempty sets of vertices which are pairwise disjoint and $(\{x^{(r)}\} \cup W^{(r)}_1 \cup W^{(r)}_2 \cup \cdots \cup W^{(r)}_{k-2} \cup \{y^{(r)}\}) \cap V(H_r) = \emptyset$.

For $k = 2$, let $H'_r = H_r \cup \{x^{(r)} \to y^{(r)}, x^{(r)} \to u^{(r)}, v^{(r)} \to y^{(r)}\}$, where $u^{(r)}, v^{(r)} \in J(H_r)$ (not necessarily distinct); $x^{(r)} \ne y^{(r)}$ and $\{x^{(r)}, y^{(r)}\} \cap V(H_r) = \emptyset$.

For $1 \le t \le k-1$, let $H''_{r,t} = H_r \cup \{x^{(r)} \to y^{(r)}\} \cup \{x^{(r)} \to W^{(r)}_1 \mapsto W^{(r)}_2 \mapsto \cdots \mapsto W^{(r)}_t \to u^{(r)}\}$, where $u^{(r)}$, $y^{(r)} \in J(H_r)$ (not necessarily distinct); $\{x^{(r)}\}$, $W^{(r)}_1, W^{(r)}_2, \dots, W^{(r)}_t$ are nonempty sets of vertices which are pairwise disjoint and $(\{x^{(r)}\} \cup W_1^{(r)} \cup W_2^{(r)} \cup \cdots \cup W_t^{(r)}) \cap V(H_r) = \emptyset$.

For $t = 0$, let $H''_{r,0} = H_r \cup \{x^{(r)} \to y^{(r)}\} \cup \{x^{(r)} \to u^{(r)}\}$, where $u^{(r)}$, $y^{(r)} \in J(H_r)$ (not necessarily distinct); and $x^{(r)} \notin V(H_r)$.

Let $H''_r = H''_{r,t}$ for some t with $0 \le t \le k-1$.

Given a digraph D with arc set $A(D) = \{e_1, e_2, \ldots, e_m\}$, say $e_i = (x_i, y_i)$, $i \in \{1, 2, \ldots, m\}$, and m digraphs H_i, $i \in \{1, 2, \ldots, m\}$, in $\mathscr{D}_k^0$, and so, $2m$ digraphs H'_i, H''_i, $i \in \{1, 2, \ldots, m\}$, with

(i) $x_i = x^{(i)}$ and $y_i = y^{(i)}$,

(ii) $V(H'_i) \cap V(D) = V(H''_i) \cap V(D) = \{x_i, y_i\}$ and

(iii) for $i, j \in \{1, 2, \ldots, m\}$ with $i \ne j$,

$V(H'_i) \cap V(H'_j) \subseteq V(D)$, $V(H'_i) \cap V(H''_j) \subseteq V(D)$, $V(H''_i) \cap V(H''_j) \subseteq V(D)$, we define two digraphs $R^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ and $T^k(D; H_1, x_1, y_1; \ldots;\ H_m, x_m, y_m)$ as follows:

I. $R^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ be the digraph obtained from D by replacing every arc of D such that the below conditions are satisfied:

if $d_D^+(y_i) \ge 1$, then replace e_i by H'_i such that $x_i = x^{(i)}$ and $y_i = y^{(i)}$;

if $d_D^+(y_i) = 0$, then replace e_i by H''_i such that $x_i = x^{(i)}$ and $y_i = y^{(i)}$.

II. $T^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m) = R^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m) \cup \big(\bigcup (W_{k-2}^{(i)} \to A_j)\big)$, where the union $\bigcup$ runs over all paths $x_i e_i y_i (= x_j) e_j y_j$ of length 2 in D, and $A_j = W_{k-2}^{(j)}$ if $d_D^+(y_j) \ge 1$; $A_j = N^-_{H''_j}(y^{(j)}) \setminus \{x^{(j)}\}$ if $d_D^+(y_j) = 0$.

For example, see Figures 1, 5, 6 and 7.

For every $i \in \{1, 2, \ldots, m\}$, when we take

(i) $H_i = P_{\ell_i+1}$, where $\ell_i \equiv 0 \pmod k$,

(ii) H'_i with $|W_1^{(i)}| = |W_2^{(i)}| = \cdots = |W_{k-2}^{(i)}| = 1$, $u^{(i)}$ and $v^{(i)}$, respectively, as the origin and terminus of P_{ℓ_i+1}, and

(iii) $H''_i = H''_{i,k-1}$ with $|W_1^{(i)}| = |W_2^{(i)}| = \ldots = |W_{k-1}^{(i)}| = 1$, $u^{(i)}$ and $y^{(i)}$, respectively, as the origin and terminus of P_{ℓ_i+1},

we have the digraphs $R^k(D; P_{\ell_1+1}, x_1, y_1; \ldots; P_{\ell_m+1}, x_m, y_m)$ and $T^k(D; P_{\ell_1+1}, x_1, y_1; \ldots; P_{\ell_m+1}, x_m, y_m)$ as in [9], which are denoted by $R^k(D)$ and $T^k(D)$, respectively. Note that, in the sense of the authors of [9], $R^k(D)$ and $T^k(D)$ are not unique. In [9], they have proved that $R^k(D)$ contains a k- kernel for $k \geq 2$ and $T^k(D)$ contains a k- kernel for $k \geq 3$. We now generalise these results to $R^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ and $T^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$.

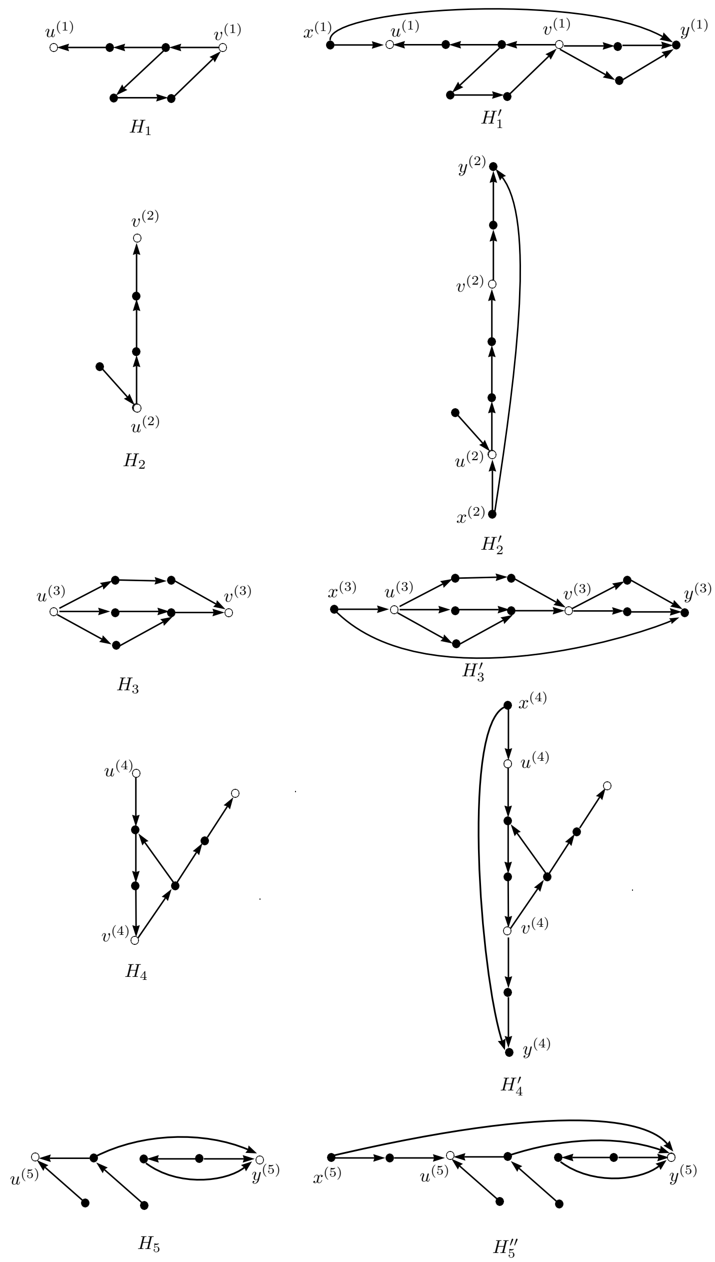

FIGURE 5. H_1, H_2, H_3, H_4 and H_5, corresponding H_1', H_2', H_3', H_4' and H_5''. Vertices marked by $\circ$ constitute the unique 3 - kernel of the respective H_i's.

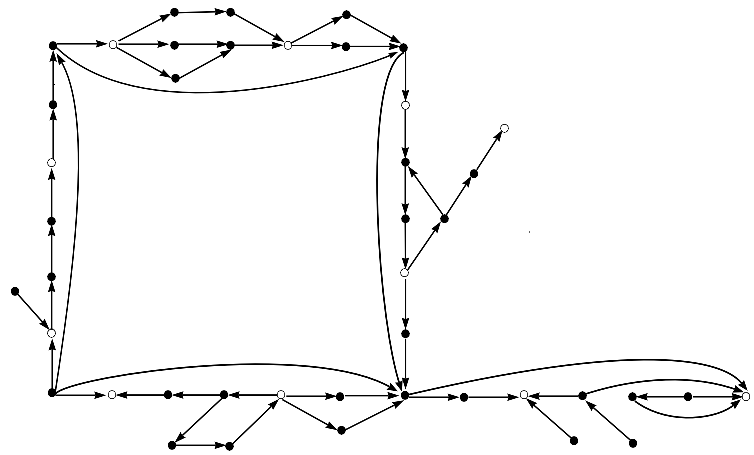

FIGURE 6.
$R^3(D; H_1, x_1, y_1; H_2, x_2, y_2; H_3, x_3, y_3; H_4, x_4, y_4; H_5, x_5, y_5)$

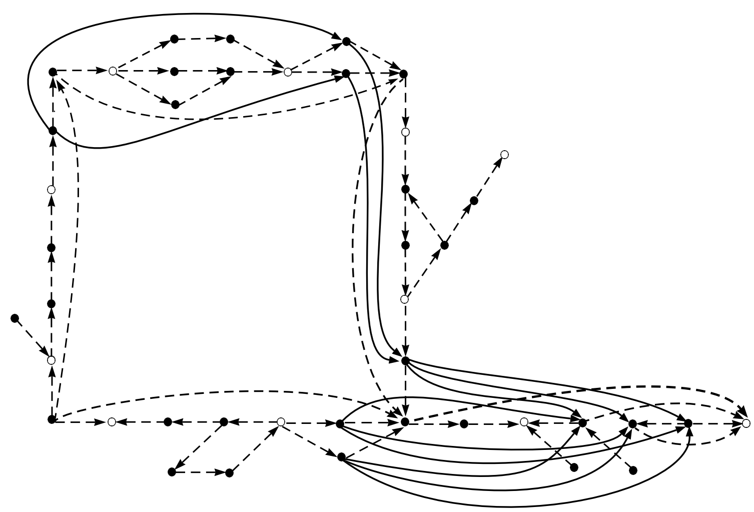

FIGURE 7.
$T^3(D; H_1, x_1, y_1; H_2, x_2, y_2; H_3, x_3, y_3; H_4, x_4, y_4; H_5, x_5, y_5)$
(Dotted arcs are the arcs of
$R^3(D; H_1, x_1, y_1; H_2, x_2, y_2; H_3, x_3, y_3; H_4, x_4, y_4; H_5, x_5, y_5)$.)

2.4. k- kernels in the generalisations.

Theorem 2.2. *For $k \geq 2$, the digraph $R^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ contains a k- kernel and for $k \geq 3$, the digraph $T^k(D; H_1, x_1, y_1; \ldots; H_m, x_m, y_m)$ contains a k- kernel.*

Proof. For the sake of convenience, we use $R^k_{H_1,\ldots,H_m}(D)$ and $T^k_{H_1,\ldots,H_m}(D)$, respectively, for $R^k(D;H_1,x_1,y_1;\ldots;H_m,x_m,y_m)$ and $T^k(D;H_1,x_1,y_1;\ldots;H_m,x_m,y_m)$.

Let $J=\bigcup\limits_{i=1}^{m} J(H_i)$.

Claim 1. For $k\geq 3$, J is k- independent in $T^k_{H_1,\ldots,H_m}(D)$.

Let $x,y\in J$. If $d_{T^k_{H_1,\ldots,H_m}(D)}(x,y)=\infty$, then there is nothing to prove. So, assume $d_{T^k_{H_1,\ldots,H_m}(D)}(x,y)<\infty$. Let P be a shortest directed path from x to y in $T^k_{H_1,\ldots,H_m}(D)$.

By the definition of $T^k_{H_1,\ldots,H_m}(D)$ and J, there exists $e_i=(x_i,y_i)$, $e_j=(x_j,y_j)\in A(D)$ such that $x\in J(H_i)$ and $y\in J(H_j)$, where $i,j\in\{1,2,\ldots,m\}$. Note that $x\neq x_i$ and $y\neq x_j$.

If $P\subseteq H_i$, then $i=j$. Therefore, $d_{T^k_{H_1,\ldots,H_m}(D)}(x,y)=d_{H_i}(x,y)\geq k$.

If $P\nsubseteq H_i$, then $d^+_D(y_i)\geq 1$ and $i\neq j$. Let P_0 be the maximal section of P contained in H'_i with origin x. If y_i is the terminus of P_0, then $d_{T^k_{H_1,\ldots,H_m}(D)}(x,y)=d_{T^k_{H_1,\ldots,H_m}(D)}(x,y_i)+d_{T^k_{H_1,\ldots,H_m}(D)}(y_i,y)=d_{H'_i}(x,y_i)+d_{T^k_{H_1,\ldots,H_m}(D)}(y_i,y)\geq k-1+1=k$. So, assume that y_i is not the terminus of P_0. Then, by the definition of $T^k_{H_1,\ldots,H_m}(D)$, the terminus of P_0 is in $W^{(i)}_{k-2}$, say, z. Let z' be the successor of z in P. Then, there exists a path $x_ie_iy_i(=x_p)e_py_p$ of length 2 in D such that: $z'\in W^{(p)}_{k-2}$ if $d^+_D(y_p)\geq 1$; $z'\in N^-_{H''_p}(y^{(p)})\backslash\{x^{(p)}\}$ if $d^+_D(y_p)=0$; for some $p\in\{1,2,\ldots,m\}$ with $p\neq i$. In either case, $z'\in V(T^k_{H_1,\ldots,H_m}(D))\backslash J$. Therefore, $d_{T^k_{H_1,\ldots,H_m}(D)}(x,y)=d_{T^k_{H_1,\ldots,H_m}(D)}(x,z)+d_{T^k_{H_1,\ldots,H_m}(D)}(z,z')+d_{T^k_{H_1,\ldots,H_m}(D)}(z',y)=d_{H'_i}(x,z)+1+d_{T^k_{H_1,\ldots,H_m}(D)}(z',y)\geq k-2+1+1=k$.

This proves Claim 1.

Claim 2. For $k=2$, J is independent in $R^k_{H_1,\ldots,H_m}(D)$.

Let $x,y\in J$. If $d_{R^2_{H_1,\ldots,H_m}(D)}(x,y)=\infty$, then there is nothing to prove. So, assume $d_{R^2_{H_1,\ldots,H_m}(D)}(x,y)<\infty$. Let P be a shortest directed path from x to y in $R^2_{H_1,\ldots,H_m}(D)$.

By the definition of $R^2_{H_1,\ldots,H_m}(D)$ and J, there exists $e_i = (x_i, y_i)$, $e_j = (x_j, y_j) \in A(D)$ such that $x \in J(H_i)$ and $y \in J(H_j)$. Note that $x \neq x_i$ and $y \neq x_j$.

If $P \subseteq H_i$, then $i = j$. In the case, $d_{R^2_{H_1,\ldots,H_m}(D)}(x, y) = d_{H_i}(x, y) \geq 2$.

If $P \nsubseteq H_i$, then $d^+_D(y_i) \geq 1$ and $i \neq j$. Let P_0 be the maximal section of P contained in H'_i with origin x. By the definition of $R^2_{H_1,\ldots,H_m}(D)$, y_i is the terminus of P_0 and thus $d_{R^2_{H_1,\ldots,H_m}(D)}(x, y) = d_{R^2_{H_1,\ldots,H_m}(D)}(x, y_i) + d_{R^2_{H_1,\ldots,H_m}(D)}(y_i, y) \geq 1 + 1 = 2$.

This proves Claim 2.

Claim 3. J is $(k-1)$- absorbent in $R^k_{H_1,\ldots,H_m}(D)$.

Let $z \in V(R^k_{H_1,\ldots,H_m}(D)) \backslash J$. We prove $d_{R^k_{H_1,\ldots,H_m}(D)}(z, J) \leq k - 1$.

Suppose $z = x_i$ for some $x_i \in V(D)$. By the definition of $R^k_{H_1,\ldots,H_m}(D)$ and J, $z \to u^{(i)} \in J$ or $z \to y_i \in J$, respectively, if $d^+_D(y_i) \geq 1$ or $d^+_D(y_i) = 0$. So, assume, for any $i \in \{1, 2, \ldots, m\}$, $z \neq x_i$.

If $z = y_j$ for some $j \in \{1, 2, \ldots, m\}$, then, as $z \neq x_i$, $d^+_D(y_j) = 0$. But, $y_j \in J$. So, assume, for any $j \in \{1, 2, \ldots, m\}$, $z \neq y_j$.

Now, by the definition of $R^k_{H_1,\ldots,H_m}(D)$ and J, there exists $e_i = (x_i, y_i) \in A(D)$ such that $z \in V(H'_i) \backslash (J(H_i) \cup \{x_i, y_i\})$ or $z \in V(H''_i) \backslash (J(H_i) \cup \{x_i\})$, respectively, if $d^+_D(y_i) \geq 1$ or $d^+_D(y_i) = 0$. If $z \in V(H_i) \backslash J(H_i)$, then, by the definition of $J(H_i)$, there exists $x \in J(H_i)$ such that $d_{R^k_{H_1,\ldots,H_m}(D)}(z, J) \leq d_{R^k_{H_1,\ldots,H_m}(D)}(z, x) = d_{H_i}(z, x) \leq k - 1$. So, assume $z \notin V(H_i)$. We consider two cases.

Case 1. $d^+_D(y_i) \geq 1$.

Then $z \in W^{(i)}_1 \cup W^{(i)}_2 \cup \cdots \cup W^{(i)}_{k-2}$. As $d^+_D(y_i) \geq 1$, there exists a path $x_i e_i y_i (= x_p) e_p y_p$ of length 2 in D. Note that $d_{R^k_{H_1,\ldots,H_m}(D)}(z, J) = d_{R^k_{H_1,\ldots,H_m}(D)}(z, y_i) + d_{R^k_{H_1,\ldots,H_m}(D)}(y_i, J) \leq k-2 + d_{R^k_{H_1,\ldots,H_m}(D)}(x_p, J)$. If $d^+_D(y_p) \geq 1$, then $d_{R^k_{H_1,\ldots,H_m}(D)}(x_p, J) = d_{H'_p}(x_p, u^{(p)}) = 1$. If $d^+_D(y_p) = 0$, then $d_{R^k_{H_1,\ldots,H_m}(D)}(x_p, J) = d_{R^k_{H_1,\ldots,H_m}(D)}(x_p, y_p) = 1$. Hence, in both the cases, $d_{R^k_{H_1,\ldots,H_m}(D)}(z, J) \leq k-1$.

Case 2. $d^+_D(y_i) = 0$.

Then $z \in W_1^{(i)} \cup W_2^{(i)} \cup \ldots \cup W_t^{(i)}$ and therefore, $d_{R^k_{H_1,\ldots,H_m}(D)}\ (z, J)$ $= d_{R^k_{H_1,\ldots,H_m}(D)}(z, u^{(i)}) \leq k-1$.

This proves Claim 3.

As $R^k_{H_1,\ldots,H_m}(D)$ is a spanning subdigraph of $T^k_{H_1,\ldots,H_m}(D)$, by Claims 1, 2 and 3, for $k \geq 2$, $R^k_{H_1,\ldots,H_m}(D)$ contains a k- kernel and by Claims 1 and 3, for $k \geq 3$, $T^k_{H_1,\ldots,H_m}(D)$ contains a k- kernel. □

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