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A PROXIMAL POINT ALGORITHM CONVERGING STRONGLY TO A MINIMIZER OF A CONVEX FUNCTION

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Abstract. In this paper, we obtain a new modified proximal point algorithm in the setting of CAT(0) spaces and establish some strong convergence results of the proposed algorithm. In process, several relevant results of the existing literature are generalized and improved.

1. Introduction

Monotone operator theory holds an important place in nonlinear analysis. It plays a crucial role in convex analysis, optimization, variational inequalities, semigroup theory and evolution equations. Many nonlinear operator equations are of the form $0 \in A(x)$, where A is a monotone operator in a Hilbert space H. A zero of a maximal monotone operator is a solution of variational inequality associated to the monotone operator also an equilibrium point of an evolution equation governed by the monotone operator as well as a solution of a minimization problem for a convex function when the monotone operator

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is a subdifferential of the convex function. Therefore, existence and approximation of a zero of a maximal monotone operator is the center of consideration of many recent researchers.

The most popular method for approximation of a zero of a maximal monotone operator is the proximal point algorithm popularly known as PPA. Its origin goes back to Martinet [29], Rockafellar [32], and Brézis and Lions [7]. Martinet introduced PPA for variational inequalities whereas Rockafellar showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces. Güler's counterexample [18] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper, and lower semicontinuous function. Following this, Many mathematicians have tried to modify the PPA in such a way that the new iterative methods generate sequences which converges strongly ([22, 36, 48, 6]).

The literature on the subject has become too extensive to be even partially listed here. For some generalization in Hilbert and Banach spaces the reader can consult ([23, 27, 3, 33, 45]). Recently, many convergence results for the PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds ([15, 28, 31, 46]).

Let X be a Hilbert space and $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization theory is to solve $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

We denote by

$$arg\min_{y\in X} f(y),$$

the set of a minimizer of a convex function.

The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds (see [1, 34, 44, 47]).

Owing to the usefulness of PPA, Bačák [5] introduced the proximal point algorithm in CAT(0) space in 2013. Bačák generalized Brézis and Lions [7] on the proximal point algorithm in Hilbert spaces to complete CAT(0) spaces. Inspired by this, numerous results have been obtained for proximal point algorithm in the setting of CAT(0) spaces (see [9, 10, 19, 24]).

Recently, a number of iteration schemes have been constructed in various spaces along with their applications in real world (for e.g. [28]-[32]). Motivated by the research going on in this direction, we propose the modified proximal point algorithm using the Thakur iteration process [38] for three nonexpansive mappings in CAT(0) spaces and prove some convergence theorems of the proposed processes under some mild conditions. Our main results generalize the results of Thakur et al. [38] from one nonexpansive mapping to three nonexpansive mappings involving the convex and lower semi-continuous function in CAT(0) spaces.

2. Preliminaries

In this section, we will mention some basic concepts, definitions, notations and few Lemmas for use in the next section.

A metric space (X, d) is said to be a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane (see more details in [8]). A complete CAT(0) space is then called a Hadamard space. Euclidean spaces, Hilbert spaces, the Hilbert ball [17], hyperbolic spaces [26], R-tress [39] and any complete, simply connected Riemannian manifold having non-positive sectional curvature are some examples of a CAT(0) space. Recently, many fixed point theorem has been proved in the setting of CAT(0) spaces (for example [16, 40, 41, 42, 43]).

Definition 2.1. A subset C of a CAT(0) space X is said to be convex if C includes every geodesic segment joining ant two of its points, that is, for any $x, y \in C$, we have $[x, y] \subset C$, where $[x, y] := \{\alpha x \oplus (1-\alpha)y : 0 \leq \alpha \leq 1\}$ is the unique geodesic joining x and y.

Definition 2.2. A mapping $T : C \to C$ is said to be non-expansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$.

First we state the following Lemma to be used later on.

Lemma 2.1. *([12]) Let (X, d) be a CAT(0) space, then the following assertions hold:*

(i) *For $x, y \in X$ and $t \in [0, 1]$, there exists a unique $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1-t)d(x, y).$$

(ii) *For* $x, y, z \in X$ *and* $t \in [0, 1]$, *we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$$

and

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2.$$

We use the notation $(1-t)x \oplus ty$ for the unique point z of the above Lemma.

Now, we collect some basic geometric properties, which are instrumental throughout the discussions.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X. For $x \in X$ write:

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\})\}.$$

It is well known that, in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point [13]. We now give the definition and some basic properties of the Δ- convergence which will be fruitful for our subsequent discussion.

Definition 2.3. ([25]) A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-convergent to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2. *([25]) Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*

Lemma 2.3. *([14]) If C is a closed convex subset of a complete CAT(0) space X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.4. *([12]) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in C such that $\Delta - \lim_n x_n = x$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, then x is a fixed point of T.*

Lemma 2.5. *([12]) If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Definition 2.4. A function $f : C \to (-\infty, \infty]$ defined on a convex subset C of a CAT(0) space is convex if, for any geodesic $\gamma : [a, b] \to C$, the function $fo\gamma$ is convex, i.e., $f(\alpha x \oplus (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ for all $x, y \in C$.

For some important examples one can refer [4]. Now, a function f defined on C is said to be lower semi-continuous at $x \in C$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

for each sequence $\{x_n\}$ such that $x_n \to x$ as $n \to \infty$. A function f is said to be lower semi-continuous on C if it is lower semi-continuous at any point in C.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in CAT(0) space as follows:

$$J_\lambda(x) = arg \min_{y\in C}[f(y) + \frac{1}{2\lambda} d^2(y, x)]$$

for all $x \in C$.

Now, we list few results which will be frequently used throughout the text.

Lemma 2.6. *([4]) Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function, then the set $F(J_\lambda)$ of the fixed point of the resolvent J_λ associated with f coincides with the set $arg \min_{y \in C} f(y)$ of minimizers of f.*

Lemma 2.7. *([21]) For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Lemma 2.8. *([2]) Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function, then for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y).$$

Lemma 2.9. *([21, 30]) Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Then the following identity holds:*

$$J_\lambda x = J_\mu\left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x\right)$$

for all $x \in X$ and $\lambda > \mu > 0$.

3. Main Results

Lemma 3.1. *Let (X, d) be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function and $T_1, T_2, T_3 : C \to C$ be three nonexpansive mappings with $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$ and*

$$\omega := F(T_1) \cap F(T_2) \cap F(T_3) \cap arg \min_{y \in X} f(y) \neq \phi.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be sequences in* $(0,1)$ *with* $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ *for all* $n \in \mathbb{N}$ *and for some* a, b *constants in* $(0,1)$ *and* $\{\lambda_n\}$ *be a sequence with* $\lambda_n \geq \lambda > 0$ *for all* $n \in \mathbb{N}$ *and for some* λ. *Let* $\{x_n\}$ *be the sequence generated in the following manner:*

$$\begin{aligned} w_n &= arg\min_{y \in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)] \\ z_n &= (1-\gamma_n)w_n \oplus \gamma_n T_1 w_n \\ y_n &= (1-\beta_n)z_n \oplus \beta_n T_2 z_n \\ x_{n+1} &= (1-\alpha_n)T_2 z_n \oplus \alpha_n T_3 y_n \end{aligned} \tag{3.1}$$

for each $n \in \mathbb{N}$. *Then, the following statements hold:*

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in \omega$,
(ii) $\lim_{n\to\infty} d(x_n, w_n) = 0$,
(iii) $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \lim_{n\to\infty} d(x_n, T_3 x_n) = 0.$

Proof. Let $p \in \omega$. Then, $p = T_1 p = T_2 p = T_3 p$ and $f(p) \leq f(y)$ for all $y \in C$. Therefore, we have

$$f(p) + \frac{1}{2\lambda_n} d^2(p,p) \leq f(y) + \frac{1}{2\lambda_n} d^2(y,p)$$

for all $y \in C$ and hence $p = J_{\lambda_n} p$ for each $n \in \mathbb{N}$.

(i) Note that $w_n = J_{\lambda_n} x_n$ and J_{λ_n} is nonexpansive map for each $n \in \mathbb{N}$. So, we have

$$d(w_n, p) = d(J_{\lambda_n} x_n, J_{\lambda_n} p) \leq d(x_n, p). \tag{3.2}$$

Also, by (3.1) and (3.2), we get

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)w_n \oplus \gamma_n T_1 w_n, p) \\ &\leq (1-\gamma_n)d(w_n, p) + \gamma_n d(T_1 w_n, p) \\ &\leq (1-\gamma_n)d(w_n, p) + \gamma_n d(w_n, p) \\ &= d(w_n, p) \\ &\leq d(x_n, p) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n T_2 z_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(T_2 z_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &= d(z_n, p) \\ &\leq d(w_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.4}$$

Therefore, by using (3.3) and (3.4), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)T_2 z_n \oplus \alpha_n T_3 y_n, p) \\ &\leq (1-\alpha_n)d(T_2 z_n, p) + \alpha_n d(T_3 y_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(z_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.5}$$

This shows that $\lim_{n\to\infty} d(x_n, p)$ exists and so we assume that

$$\lim_{n\to\infty} d(x_n, p) = c \geq 0. \tag{3.6}$$

(ii) Next, we show that $\lim_{n\to\infty} d(x_n, w_n) = 0$. By Lemma 2.8, we get

$$\frac{1}{2\lambda_n}\{d^2(w_n, p) - d^2(x_n, p) + d^2(x_n, w_n)\} \leq f(p) - f(w_n).$$

Since $f(p) \leq f(w_n)$ for each $n \in \mathbb{N}$, it follows that

$$d^2(x_n, w_n) \leq d^2(x_n, p) - d^2(w_n, p). \tag{3.7}$$

So, in order to show that $\lim_{n\to\infty} d(x_n, w_n) = 0$, it is sufficient to show that

$$\lim_{n\to\infty} d(w_n, p) = c.$$

Now, from (3.3) and (3.5), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(z_n, p) \\ &\leq d(w_n, p). \end{aligned}$$

Hence, we get $c \leq \liminf_{n\to\infty} d(w_n, p)$.

Also, from (3.2), we see that

$$\limsup_{n\to\infty} d(w_n, p) \leq c.$$

Therefore, we have

$$\lim_{n\to\infty} d(w_n, p) = c. \tag{3.8}$$

This shows that

$$\lim_{n\to\infty} d(x_n, w_n) = 0, \tag{3.9}$$

which proves (ii).

(iii) Now, we show that

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0.$$

On using Lemma 2.1 and nonexpansiveness of T, we obtain

$$\begin{aligned} d^2(z_n,p) &= d^2((1-\gamma_n)w_n \oplus \gamma_n T_1 w_n, p) \\ &\leq (1-\gamma_n)d^2(w_n,p) + \gamma_n d^2(T_1 w_n, p) - \gamma_n(1-\gamma_n)d^2(w_n, T_1 w_n) \\ &\leq (1-\gamma_n)d^2(w_n,p) + \gamma_n d^2(w_n, p) - \gamma_n(1-\gamma_n)d^2(w_n, T_1 w_n) \\ &= d^2(w_n,p) - \gamma_n(1-\gamma_n)d^2(w_n, T_1 w_n) \\ &\leq d^2(w_n,p) - a(1-b)d^2(w_n, T_1 w_n)), \end{aligned}$$

which yields

$$d^2(w_n, T_1 w_n) \leq \frac{1}{a(1-b)}[d^2(w_n,p) - d^2(z_n,p)]. \tag{3.10}$$

Now, from (3.4), we have

$$\limsup_{n\to\infty} d(y_n,p) \leq c$$

and from (3.5), we get

$$\begin{aligned} d(x_{n+1},p) &\leq (1-\alpha_n)d(z_n,p) + \alpha_n d(y_n,p) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p), \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(x_n,p) &\leq \frac{1}{\alpha_n}(d(x_n,p) - d(x_{n+1},p)) + d(y_n,p) \\ &\leq \frac{1}{a}(d(x_n,p) - d(x_{n+1},p)) + d(y_n,p). \end{aligned}$$

Since $d(x_{n+1},p) \leq d(x_n,p)$ and $\alpha_n \geq a > 0$ for all $n \in \mathbb{N}$, we deduce that

$$c \leq \liminf_{n\to\infty} d(y_n,p),$$

which yields

$$\lim_{n\to\infty} d(y_n,p) = c. \tag{3.11}$$

We see that, from (3.3) and (3.6), we have

$$\limsup_{n\to\infty} d(z_n,p) \le \limsup_{n\to\infty} d(x_n,p) = c \tag{3.12}$$

and from (3.4) and (3.10), we get

$$c = \liminf_{n\to\infty} d(y_n,p) \le \liminf_{n\to\infty} d(z_n,p). \tag{3.13}$$

Thus, from (3.11) and (3.12), we have

$$\lim_{n\to\infty} d(z_n,p) = c. \tag{3.14}$$

From (3.8), (3.10) and (3.14), it follows that

$$d^2(w_n, T_1 w_n) \to 0 \ \ as \ \ n \to \infty,$$

i.e.,

$$\lim_{n\to\infty} d(w_n, T_1 w_n) = 0. \tag{3.15}$$

Now, triangle inequality, nonexpansiveness of T, (3.9) and (3.15) yields

$$\begin{aligned} d(x_n, T_1 x_n) &\le d(x_n, w_n) + d(w_n, T_1 w_n) + d(T_1 w_n, T_1 x_n) \\ &\le d(x_n, w_n) + d(w_n, T w_n) + d(w_n, x_n) \\ &\to 0 \end{aligned} \tag{3.16}$$

as $n \to \infty$.

Similarly, we get

$$\begin{aligned} d^2(y_n,p) &= d^2((1-\beta_n)z_n \oplus \beta_n T_2 z_n, p) \\ &\le (1-\beta_n)d^2(z_n,p) + \beta_n d^2(T_2 z_n, p) - \beta_n(1-\beta_n)d^2(z_n, T_2 z_n) \\ &\le (1-\beta_n)d^2(z_n,p) + \beta_n d^2(z_n, p) - \beta_n(1-\beta_n)d^2(z_n, T_2 z_n) \\ &= d^2(z_n,p) - \beta_n(1-\beta_n)d^2(z_n, T_2 z_n) \\ &\le d^2(z_n,p) - a(1-b)d^2(z_n, T_2 z_n)), \end{aligned}$$

which yields

$$d^2(z_n, T_2 z_n) \leq \frac{1}{a(1-b)}[d^2(z_n, p) - d^2(y_n, p)].$$

Using (3.11) and (3.14), we get

$$\lim_{n\to\infty} d(z_n, T_2 z_n) = 0. \tag{3.17}$$

Now, we have

$$\begin{aligned} d(x_n, T_2 x_n) &\leq d(x_n, z_n) + d(z_n, T_2 z_n) + d(T_2 z_n, T_2 x_n) \\ &\leq 2d(x_n, z_n) + d(z_n, T_2 z_n). \end{aligned} \tag{3.18}$$

Also,

$$d(x_n, z_n) \leq d(x_n, w_n) + d(w_n, z_n)$$

and

$$\begin{aligned} d(w_n, z_n) &= d(w_n, (1-\gamma_n)w_n \oplus \gamma_n T_1 w_n) \\ &\leq (1-\gamma_n)d(w_n, w_n) + \gamma_n d(w_n, T_1 w_n) \\ &= \gamma_n d(w_n, T_1 w_n). \end{aligned}$$

Using (3.15), we obtain

$$\lim_{n\to\infty} d(w_n, z_n) = 0, \tag{3.19}$$

which yields

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.20}$$

Thus, using (3.18) we have

$$\lim_{n\to\infty} d(x_n, T_2 x_n) = 0. \tag{3.21}$$

Now, consider

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_n)T_2 z_n \oplus \alpha_n T_3 y_n, p) \\ &\leq (1-\alpha_n)d^2(T_2 z_n, p) + \alpha_n d^2(T_3 y_n, p) - \alpha_n(1-\alpha_n)d^2(T_2 z_n, T_3 y_n) \\ &\leq (1-\alpha_n)d^2(z_n, p) + \alpha_n d^2(y_n, p) - \alpha_n(1-\alpha_n)d^2(T_2 z_n, T_3 y_n) \\ &\leq d^2(x_n, p) - a(1-b)d^2(T_2 z_n, T_3 y_n), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(T_2 z_n, T_3 y_n) = 0. \tag{3.22}$$

Observe that

$$\begin{aligned} d(y_n, x_n) &= d((1-\beta_n)z_n \oplus \beta_n T_2 z_n, x_n) \\ &\leq (1-\beta_n)d(z_n, x_n) + \beta_n d(T_2 z_n, x_n) \end{aligned}$$

and

$$d(T_2 z_n, x_n) \leq d(T_2 z_n, z_n) + d(z_n, x_n).$$

From (3.17) and (3.20), we get

$$\lim_{n\to\infty} d(T_2 z_n, x_n) = 0,$$

which gives

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{3.23}$$

Again by triangle inequality, (3.9), (3.15), (3.16), (3.17), (3.19), (3.22) and (3.23), we obtain

$$\begin{aligned} d(x_n, T_3x_n) &\leq d(x_n, T_1x_n) + d(T_1x_n, T_1w_n) + d(T_1w_n, w_n) + d(w_n, z_n) \\ &+ d(z_n, T_2z_n) + d(T_2z_n, T_3y_n) + d(T_3y_n, T_3x_n) \\ &\leq d(x_n, T_1x_n) + d(x_n, w_n) + d(T_1w_n, w_n) + d(w_n, z_n) \\ &+ d(z_n, T_2z_n) + d(T_2z_n, T_3y_n) + d(y_n, x_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

This completes the proof. Next, we prove the Δ-convergence of our iteration.

Theorem 3.1. *Let (X, d) be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function and $T_1, T_2, T_3 : C \to C$ are three nonexpansive mappings with $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$ and*

$$\omega := F(T_1) \cap F(T_2) \cap F(T_3) \cap arg \min_{y \in X} f(y) \neq \phi.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0, 1)$ with $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b constants in $(0, 1)$ and $\{\lambda_n\}$ be a sequence with with $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. If $\{x_n\}$ is the sequence defined by (3.1), then $\{x_n\}$ Δ-converges to an element of ω.

Proof. In fact, it follows from (3.9) and Lemma 2.13 that

$$\begin{aligned} d(J_\lambda x_n, x_n) &\leq d(J_\lambda x_n, w_n) + d(w_n, x_n) \\ &= d(J_\lambda x_n, J_{\lambda_n} x_n) + d(w_n, x_n) \\ &= d(J_\lambda x_n, J_\lambda(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n)) + d(w_n, x_n) \\ &\leq d(x_n, (1 - \frac{\lambda}{\lambda_n}) J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n) + d(w_n, x_n) \\ &\leq (1 - \frac{\lambda}{\lambda_n}) d(x_n, J_{\lambda_n} x_n) + \frac{\lambda}{\lambda_n} d(x_n, x_n) + d(w_n, x_n) \\ &= (1 - \frac{\lambda}{\lambda_n}) d(x_n, w_n) + d(w_n, x_n) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. By Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \omega$, and $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \lim_{n\to\infty} d(x_n, T_3 x_n) = 0$. Let $W_\omega(\{x_n\}) =: \cup A(\{u_n\})$, where union is taken over all subsequences $\{u_n\}$ over $\{x_n\}$. In order to show the Δ-convergence of $\{x_n\}$ to a point of ω, firstly we will prove $W_\omega(\{x_n\}) \subset \omega$ and thereafter argue that $W_\omega(\{x_n\})$ is a singleton set.

To show $W_\omega(\{x_n\}) \subset F(T)$, let $y \in W_\omega(\{x_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A(\{y_n\}) = y$. By Lemma 2.2, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n z_n = z$ and $z \in C$. By Lemma 2.4, $z \in \omega$. So, $y = z$ by Lemma 2.5. This shows that $W_\omega(\{x_n\}) \subset \omega$. Now it is left to show that $W_\omega(\{x_n\})$ consists of single element only. For this, let $\{y_n\}$ be a subsequence of $\{x_n\}$. Again, by using Lemma 2.2, we can find a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n z_n = z$. Let $A(\{y_n\}) = y$ and $A(\{x_n\}) = x$. It is enough to show that $z = x$. Since $z \in \omega$, by Lemma 3.1, $\{d(x_n, z)\}$ is convergent. Again, by Lemma 2.5, we have $z = x$ which proves that $W_\omega(\{x_n\}) = \{x\}$. Hence the conclusion follows.

If $T_1 = T_2 = T_3 = T$ in Theorem 3.1, then we obtain the following result.

Corollary 3.1. *Let* (X, d) *be a complete CAT(0) space and* C *be a nonempty closed convex subset of* X*. Let* $f : X \to (-\infty, \infty]$ *be a proper convex and lower semi-continuous function and* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \phi$ *and*

$$\omega := F(T) \cap arg \min_{y \in X} f(y) \neq \phi.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be a sequences in* $(0, 1)$ *with* $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ *for all* $n \in \mathbb{N}$ *and for some* a, b *constants in* $(0, 1)$ *and* $\{\lambda_n\}$ *be a sequence with* $\lambda_n \geq \lambda > 0$ *for all* $n \in \mathbb{N}$ *and for some* λ*. If* $\{x_n\}$ *is the sequence generated in the following manner:*

$$\begin{aligned} w_n &= arg \min_{y \in X} [f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)] \\ z_n &= (1 - \gamma_n) w_n \oplus \gamma_n T w_n \\ y_n &= (1 - \beta_n) z_n \oplus \beta_n T z_n \\ x_{n+1} &= (1 - \alpha_n) T z_n \oplus \alpha_n T y_n \end{aligned}$$

for each $n \in \mathbb{N}$*, then* $\{x_n\}$ Δ*-converges to an element of* ω*.*

Since every real Hilbert space H is a complete CAT(0) space. The following result can be obtained from Theorem 3.1.

Corollary 3.2. *Let* C *be a nonempty closed and convex subset of real Hilbert space* H*. Suppose that* $T_1, T_2, T_3, f, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\lambda\}$ *and* ω *satisfy all the hypothesis in Theorem 3.1. If* $\{x_n\}$ *is the sequence* *generated in the following manner:*

$$\begin{aligned} w_n &= arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2] \\ z_n &= (1-\gamma_n)w_n + \gamma_n T_1 w_n \\ y_n &= (1-\beta_n)z_n + \beta_n T_2 z_n \\ x_{n+1} &= (1-\alpha_n)T_2 z_n + \alpha_n T_3 y_n \end{aligned}$$

for each $n \in \mathbb{N}$, then $\{x_n\}$ converges weakly to an element in ω.

Next, we establish the strong convergence theorems of our iteration.

Theorem 3.2. *Under the hypothesis of Theorem 3.1, the sequence $\{x_n\}$ converges to an element of ω if and only if $\liminf_{n\to\infty} d(x_n, \omega) = 0$.*

Proof. If the sequence $\{x_n\}$ converges to a point $x \in \omega$, then it can be easily seen that $\liminf_{n\to\infty} d(x_n, \omega) = 0$.
For the converse part, since

$$d(x_{n+1}, p) \leq d(x_n, p)$$

for all $p \in \omega$, it follows that

$$d(x_{n+1}, \omega) \leq d(x_n, \omega)$$

Therefore, $\lim_{n\to\infty} d(x_n, \omega)$ exists and $\lim_{n\to\infty} d(x_n, \omega) = 0$.
Now, we prove that $\{x_n\}$ is a Cauchy sequence in C. Let ϵ >0 be arbitrarily chosen. Since $\liminf_{n\to\infty} d(x_n, \omega) = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$d(x_n, \omega) < \frac{\epsilon}{4}.$$

In particular,

$$\inf\{d(x_{n_0}, p) : p \in \omega\} < \frac{\epsilon}{4},$$

so there must exist a $p \in \omega$ such that

$$d(x_{n_0}, p) < \frac{\epsilon}{2}.$$

Thus, for $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p) + d(x_n, p) < 2d(x_{n_0}, p) < 2\frac{\epsilon}{2} = \epsilon$$

which shows that $\{x_n\}$ is a cauchy sequence. Thus, $\{x_n\}$ converges to a point x^* in X and so $d(x^*, \omega) = 0$. Since ω is closed, we have $x^* \in \omega$ which completes the proof.

Let C be a nonempty closed convex subset of a CAT(0) space (X, d). A family $\{P, Q, R, S\}$ of mappings is said to satisfy condition (Ω) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Px) \geq f(d(x, F))$$

or

$$d(x, Qx) \geq f(d(x, F))$$

or

$$d(x, Rx) \geq f(d(x, F))$$

or

$$d(x, Sx) \geq f(d(x, F))$$

for all $x \in X$, where $F = F(P) \cap F(Q) \cap F(R) \cap F(S)$.

Theorem 3.3. *Under the hypothesis of Theorem 3.1, suppose that the family* $\{T_1, T_2, T_3, J_\lambda\}$ *satisfy the condition* (Ω). *Then the sequence* $\{x_n\}$ *defined by (3.1) strongly converges to an element of* ω.

Proof. From Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \omega$. This implies that $\lim_{n\to\infty} d(x_n, \omega)$ exists.
Also, by the condition (Ω), we have

$$\lim_{n\to\infty} f(d(x_n, \omega)) \leq \lim_{n\to\infty} d(x_n, T_1 x_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \omega)) \leq \lim_{n\to\infty} d(x_n, T_2 x_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \omega)) \leq \lim_{n\to\infty} d(x_n, T_3 x_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \omega)) \leq \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} f(d(x_n, \omega)) = 0.$$

By using the property of f, we get $\lim_{n\to\infty} d(x_n, \omega) = 0$. Thus, the proof follows from Theorem 3.2.

A mapping $T : C \to C$ is said to be semi-compact if any sequence $\{x_n\}$ in C satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 3.4. *Under the hypothesis of Theorem 3.1, suppose that T_1 or T_2 or T_3 or J_λ is semi-compact. Then the sequence $\{x_n\}$ defined by (3.1) strongly converges to an element of ω.*

Proof. Suppose that T_1 is semi-compact. By Lemma 3.1(iii), we have

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0.$$

Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to p^* \in X$. Since

$$\lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0,$$

we have $d(p^*, T_2p^*) = d(p^*, T_3p^*) = 0$ and $d(p^*, J_\lambda p^*) = 0$, which shows that $p^* \in \omega$. In other cases, we can prove the strong convergence of $\{x_n\}$ to an element of ω. This completes the proof.

4. Numerical Example

In this section, we give the numerical example to show the convergence of our iteration scheme and support our main theorem in a space of real numbers.

Let $X = \mathbb{R}$ with the Euclidean norm and $C = \{x : -3 \leq x \leq 3\}$. For each $x \in C$, we define mappings T_1, T_2 and T_3 on C as follows:

$$T_1(x) = \frac{x}{2},$$

$$T_2(x) = \frac{x}{3}$$

and

$$T_3(x) = \frac{x}{4}.$$

Clearly, T_1, T_2 and T_3 are nonexpansive mappings.

Also, for each $x \in C$, we define $f : C \to (-\infty, \infty]$ by

$$f(x) = x^2.$$

We can easily check that f is a proper convex and lower semi-continuous function.

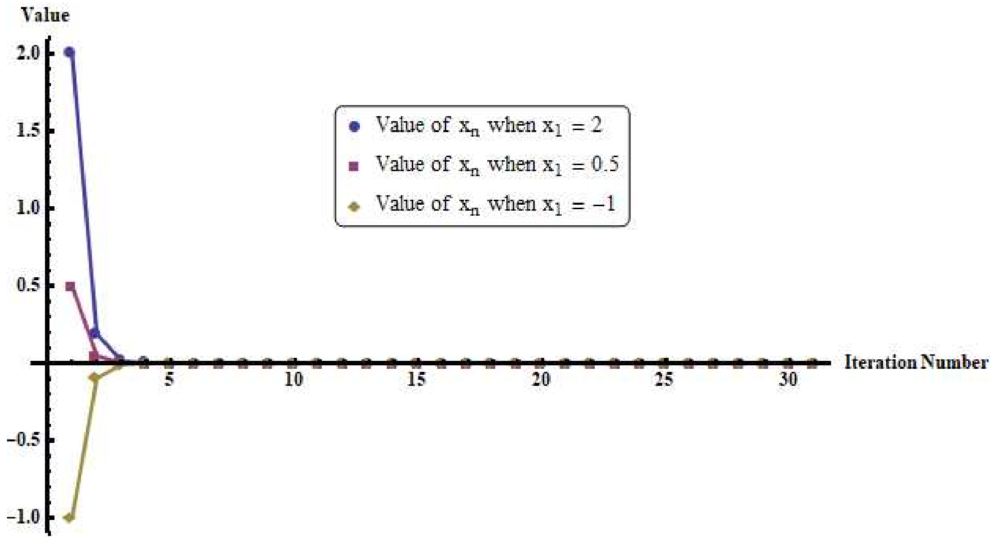
We choose $\alpha_n = \frac{n}{n+5}$, $\beta_n = \frac{2}{n+3}$ and $\gamma_n = \frac{n+2}{n+3}$. Also, we set $\lambda_n = \frac{1}{2}$ $\forall$ n. It can be observed that all the assumptions of Theorem 3.1 are satisfied. Using the algorithm (3.1) with the initial points $x_1 = 2$, $x_1 = 0.5$ and $x_1 = -1$, we obtain the following table of iteration values:

Step	When $x_1 = 2$	When $x_1 = 0.5$	When $x_1 = -1$
1	2	0.5	-1
2	0.190972222222	0.0477430555556	-0.0954861111111
3	0.0166418650794	0.00416046626984	-0.00832093253968
4	0.00136515299479	0.000341288248698	-0.000682576497396
5	0.000107313614103	0.0000268284035258	-0.0000536568070515
6	$8.17427919926 \times 10^{-6}$	$2.04356979981 \times 10^{-6}$	$-4.08713959963 \times 10^{-6}$
7	$6.07795731707 \times 10^{-7}$	$1.51948932927 \times 10^{-7}$	$-3.03897865853 \times 10^{-7}$
8	$4.43395427887 \times 10^{-8}$	$1.10848856972 \times 10^{-8}$	$-2.21697713944 \times 10^{-8}$
9	$3.18523098228 \times 10^{-9}$	$7.9630774557 \times 10^{-10}$	$-1.59261549114 \times 10^{-9}$
10	$2.25936523049 \times 10^{-10}$	$5.64841307622 \times 10^{-11}$	$-1.12968261524 \times 10^{-10}$
11	$1.58571491357 \times 10^{-11}$	$3.96428728393 \times 10^{-12}$	$-7.92857456787 \times 10^{-12}$
12	$1.10294663751 \times 10^{-12}$	$2.75736659378 \times 10^{-13}$	$-5.51473318755 \times 10^{-13}$
13	$7.61249443929 \times 10^{-14}$	$1.90312360982 \times 10^{-14}$	$-3.80624721964 \times 10^{-14}$
14	$5.21899710998 \times 10^{-15}$	$1.30474927749 \times 10^{-15}$	$-2.60949855499 \times 10^{-15}$
15	$3.55711103334 \times 10^{-16}$	$8.89277758336 \times 10^{-17}$	$-1.77855551667 \times 10^{-16}$
16	$2.41189145104 \times 10^{-17}$	$6.02972862761 \times 10^{-18}$	$-1.20594572552 \times 10^{-17}$
17	$1.62786500345 \times 10^{-18}$	$4.06966250863 \times 10^{-19}$	$-8.13932501726 \times 10^{-19}$
18	$1.09418426084 \times 10^{-19}$	$2.73546065211 \times 10^{-20}$	$-5.47092130421 \times 10^{-20}$
19	$7.3274638586 \times 10^{-21}$	$1.83186596465 \times 10^{-21}$	$-3.6637319293 \times 10^{-21}$
20	$4.89060060896 \times 10^{-22}$	$1.22265015224 \times 10^{-22}$	$-2.44530030448 \times 10^{-22}$

Next, the following graph shows the convergence behaviour of iteration (3.1) for the above example.

5. Conclusion

It is clear from the graph as well as iteration table that our sequence (3.1) converges to 0 which is the fixed point of T_1, T_2, T_3 and minimizer



of $f(x)$. Our main results extend the corresponding main results of Cholamjiak et al. [9] and Sombut et al. [35]. Indeed, we present a new modified proximal point algorithm for solving convex minimization problem as well as common fixed point problem for three nonexpansive mappings in CAT(0) spaces.

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