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REMARKS ON ABSTRACT SUBDIFFERENTIAL

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Abstract. In this paper, the notion of subdifferential is extended in the framework of abstract convexity and some basic properties of this new concept is considered. Also, a representation based on maximal elements is given. Our results is supported by some examples.

1. Introduction

Abstract convexity was presented by Alexander Rubinov in [6]. Many applications of abstract convex analysis in global optimization problems are certainly based on the description of support sets or, at least, its maximal elements. The most useful tools for this theory are abstract subdifferentials and abstract conjugate function. The abstract subdifferential allows one to simplify description of the support functions which coincide with the initial function at a given point. More details can be found in [1–8] and the references cited therein.

Let X be a set and L be a set of real valued functions $l : X \to \mathbb{R}$. For each $l \in L$ and $c \in \mathbb{R}$, consider the shift $h_{l,c}$ of l by the constant c

$$h_{l,c}(x) := l(x) - c, \ \ (x \in X).$$

The set L is called a set of abstract linear functions if $h_{l,c} \notin L$, for all $l \in L$ and all $c \in \mathbb{R}\backslash\{0\}$. The set of all $h_{l,c}$ functions will be denoted by H_L or for simplicity H. Let

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$\mathcal{F}(X)$ be the family of all functions $f : X \to \mathbb{R}_{+\infty}$ and the function $-\infty$, where the function $-\infty$ is defined by $-\infty(x) = -\infty$, for every $x \in X$, and $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$.

Recall (see [6]) that a function $f \in \mathcal{F}(X)$ is called *H-convex*, if

$$f(x) = \sup\{h(x) : h \in \operatorname{supp}(f, H)\}, \ \forall x \in X,$$

where

$$\operatorname{supp}(f, H) := \{h \in H : h \leq f\}$$

is called the *support set* of the function f and $h \leq f$ if and only if $h(x) \leq f(x)$ for all $x \in X$. The *Fenchel-Moreau conjugate function* to H-convex function $f : X \to \overline{\mathbb{R}}$ is defined by $f^* : L \to \overline{\mathbb{R}}$

$$f^*(l) := \sup_{x \in X}(l(x) - f(x)), \ \forall l \in L, \tag{1.1}$$

where, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Let $f : X \to \mathbb{R}_{+\infty}$, the abstract subdifferential (or L-subdifferential) of f with respect to a subset $Z \subseteq X$ at the point $x_0 \in Z$ [6, Page 281] is defined as

$$\partial_{L,Z} f(x_0) := \{l \in L : l(x) - l(x_0) \leq f(x) - f(x_0), \ \forall x \in Z\}.$$

Generally, if $Z = X$ we will set $\partial_L f(x_0)$ instead of $\partial_{L,X} f(x_0)$. The abstract subdifferential is a subset of L, its counterpart in the set H has been introduced in [6, Section 8.2.3] by

$$\mathcal{D}_{L,Z} f(x_0) := \{h \in H : h(x) = l(x) - l(x_0), \ l \in \partial_{L,Z} f(x_0)\}.$$

Since there will be no ambiguity regarding the choice of L and Z, we simply denote $\mathcal{D}_{L,Z} f$ by $\mathcal{D} f$.

2. Abstract $\hat{\epsilon}_L$-Subdifferential

In this section, we define a new abstract subdifferential which is an extension of L-subdifferential and then we establish some results about this new subdifferential. Our results extend some well-known results of L-subdifferential theory in [6]. In the sequel, we assume that X is a Banach space and Z a subset of X. Let $\epsilon \geq 0$, for $x_0 \in \operatorname{dom} f$, we define $\hat{\epsilon}_L$-*subdifferential* as

$$\widehat{\partial}_{L,Z,\epsilon} f(x_0) := \{l \in L : l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon\|x - x_0\|, \ \forall\, x \in Z\}. \tag{2.1}$$

Proposition 2.1. *The following statements hold:*

(i) *If $Z_1 \subseteq Z_2$, then $\widehat{\partial}_{L,Z_1,\epsilon} f(x_0) \supseteq \widehat{\partial}_{L,Z_2,\epsilon} f(x_0)$.*
(ii) *If $L_1 \subseteq L_2$, then $\widehat{\partial}_{L_1,Z,\epsilon} f(x_0) \subseteq \widehat{\partial}_{L_2,Z,\epsilon} f(x_0)$.*
(iii) *If $0 \leq \gamma < \epsilon$, then $\widehat{\partial}_{L,Z,\gamma} f(x_0) \subseteq \widehat{\partial}_{L,Z,\epsilon} f(x_0)$.*

Proof. Let $l \in \widehat{\partial}_{L,Z_2,\epsilon} f(x_0)$. Then

$$l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon||x - x_0||, \ \forall\, x \in Z_2.$$

Since $Z_1 \subseteq Z_2$, we get:

$$l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon||x - x_0||, \ \forall\, x \in Z_1.$$

Therefore, $l \in \widehat{\partial}_{L,Z_1,\epsilon} f(x_0)$; i.e., (i) is proved. For (ii), let $l \in \widehat{\partial}_{L_1,Z,\epsilon} f(x_0)$. Then

$$l \in L_1, \ l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon||x - x_0||, \ \forall\, x \in Z.$$

Since $L_1 \subseteq L_2$, we obtain that:

$$l \in L_2, \ l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon||x - x_0||, \ \forall\, x \in Z.$$

So $l \in \widehat{\partial}_{L_2,Z,\epsilon} f(x_0)$. This proves (ii). Finally, we assume $l \in \widehat{\partial}_{L,Z,\gamma} f(x_0)$ with $0 \leq \gamma < \epsilon$. Then

$$\begin{aligned} l(x) - l(x_0) &\leq f(x) - f(x_0) + \gamma||x - x_0|| \\ &\leq f(x) - f(x_0) + \epsilon||x - x_0||. \end{aligned}$$

Therefore, $l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$ and hence (iii) is proved. □

Proposition 2.2. *Let L be a set of abstract linear functions on X. Let f be an H_L-convex function, $\epsilon \geq 0$ and $x_0 \in X$. Set*

$$B_L(x_0) := \{l \in L : f^*(l) + f(x_0) - l(x_0) \leq \epsilon\|x - x_0\|, \ \forall\, x \in X\}.$$

Then $B_L(x_0) \subseteq \widehat{\partial}_{L,\epsilon} f(x_0)$.

Proof. Take $l \in B_L(x_0)$. Therefore

$$\begin{aligned} l(x_0) - f(x_0) + \epsilon\|x - x_0\| &\geq f^*(l) \\ &= \sup_{x \in X}(l(x) - f(x)) \\ &\geq l(x) - f(x). \end{aligned}$$

Hence

$$l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon\|x - x_0\|, \ \forall\, x \in X;$$

i.e., $l \in \widehat{\partial}_{L,\epsilon} f(x_0)$. □

The converse of the above proposition does not hold, for a counterexample see Example 4.3.

Proposition 2.3. *Let $Z \subseteq X$, g be a function on Z, $x_0 \in Z$ and $\epsilon \geq 0$. Let*

$$f(x) = \sup\{h(x) - \epsilon\|x - x_0\| : h \in H_L \text{ and } h(x) - \epsilon\|x - x_0\| \leq g(x), \ \forall\, x \in Z\}.$$

If $\widehat{\partial}_{L,Z,\epsilon} f(x_0) \neq \emptyset$, then the following statements are equivalent:

(i) $g(x_0) = f(x_0)$.

(ii) $\widehat{\partial}_{L,Z,\epsilon} f(x_0) = \widehat{\partial}_{L,Z,\epsilon} g(x_0)$.

(iii) $\widehat{\partial}_{L,Z,\epsilon} g(x_0) \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$**:** Let $f(x_0) = g(x_0)$. By the definition of f, we have $f(x) \leq g(x)$. Assume that $l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$. Then

$$\begin{aligned} l(x) - l(x_0) &\leq f(x) - f(x_0) + \epsilon\|x - x_0\|, \ \forall\, x \in Z \\ &\leq g(x) - g(x_0) + \epsilon\|x - x_0\|; \end{aligned}$$

i.e., $l \in \widehat{\partial}_{L,Z,\epsilon} g(x_0)$.

Now, suppose that $l \in \widehat{\partial}_{L,Z,\epsilon} g(x_0)$. Then

$$l(x) - l(x_0) \leq g(x) - g(x_0) + \epsilon\|x - x_0\|, \ \forall\, x \in Z,$$

or equivalently,

$$l(x) - l(x_0) + g(x_0) - \epsilon\|x - x_0\| \leq g(x), \ \forall\, x \in Z.$$

Let $h(x) := l(x) - l(x_0) + g(x_0)$. Then

$$h(x) - \epsilon\|x - x_0\| \leq g(x), \ \forall\, x \in Z.$$

The function h is L-affine, so

$$\begin{aligned} f(x) &= \sup\{h(x) : h(x) - \epsilon\|x - x_0\| \leq g(x), \ \forall\, x \in Z\} \\ &\geq h(x) - \epsilon\|x - x_0\| \\ &= l(x) - l(x_0) + g(x_0) - \epsilon\|x - x_0\| \\ &= l(x) - l(x_0) + f(x_0) - \epsilon\|x - x_0\|. \end{aligned}$$

Therefore, $l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$.

$(ii) \Rightarrow (iii)$**:** Since $\widehat{\partial}_{L,Z,\epsilon} f(x_0) \neq \emptyset$, by considering (ii) we get (iii).

$(iii) \Rightarrow (i)$**:** Let $l \in \widehat{\partial}_{L,Z,\epsilon} g(x_0)$. Thus $l(x) - l(x_0) + g(x_0) - \epsilon\|x - x_0\| \leq g(x)$. Since $h(x) = l(x) - l(x_0) + g(x_0)$, is L-affine and $h(x) - \epsilon\|x - x_0\| \leq g(x)$ for all $x \in Z$, we have:

$$h(x) - \epsilon\|x - x_0\| \leq \sup\{h : h(x) \leq g(x) + \epsilon\|x - x_0\|,\ \forall\, x \in Z\} = f(x),$$

Since $h(x_0) = g(x_0)$, we have $g(x_0) \leq f(x_0)$. On the other hand, $f(x) \leq g(x)$ implies that $f(x_0) \leq g(x_0)$. Thus

$$g(x_0) \leq f(x_0) \leq g(x_0) \ \Rightarrow\ f(x_0) = g(x_0).$$

The proof is completed. □

3. Some Characterization of $\hat{\epsilon}_L$-Subdifferential

Denote by H the set of all L-affine functions. Fix a subset $Z \subseteq X$ and $x_0 \in Z$. The set of L-subdifferentials of f may be very large, so it is useful to identify special members of it, and relate them with special members of the support set on Z. In [6, Proposition 7.1] it is proved that there exists a bijective correspondence between the set of L-subdifferentials of f at x_0 and the set

$$\partial^*_{H,Z} f(x_0) := \{h \in H : h(x) \leq f(x),\ \forall x \in Z, h(x_0) = f(x_0)\}. \tag{3.1}$$

This set is nonempty if and only if f is (H, Z)-convex at x_0. In particular, Proposition 7.1 of [6] allows us to determine whether the abstract subdifferential is empty or nonempty. In some cases, for instance in [6, Example 8.3], the abstract subdifferential at given point is empty, and hence we are forced to look at the abstract $\hat{\epsilon}_L$-subdifferentials of f at x_0 with respect to the set Z. It is clear that for $\epsilon = 0$ we recover the set of abstract subdifferentials. Note also that every H-convex function on Z has $\hat{\epsilon}_L$-subdifferentials (for a proof of this fact, see page 286 in [6]). Our first result extends [6, Proposition 7.1] to the framework of abstract limiting subdifferentials. The natural replacement for the set (3.1) is

$$\partial^*_{H,Z,\epsilon} f(x_0) := \{h \in H : h(x) \leq f(x) + \epsilon\|x - x_0\|,\ \forall\, x \in Z, h(x_0) = f(x_0)\}. \tag{3.2}$$

Note that, when $\epsilon = 0$, we recover the set $\partial^*_{H,Z} f(x_0)$. The proof of the next result follows the same steps as in [6, Proposition 7.1] with some necessary changes.

Proposition 3.1. *Let $f : X \to \mathbb{R}_{+\infty}$, $Z \subseteq X$ and $x_0 \in Z$. Suppose that $l \in L$, $\epsilon \geq 0$ and $c := l(x_0) - f(x_0)$. Then the following statements are equivalent:*

(i) $l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$.

(ii) $h_{l,c} \in \partial^*_{H_L,Z,\epsilon} f(x_0)$.

Proof. For $\epsilon = 0$ the equivalence of the above statements are established in [6, Proposition 7.1]. Therefore we only consider the case that $\epsilon > 0$. By our assumption, we have:

$$h_{l,c}(x_0) = l(x_0) - c = f(x_0).$$

Then

$$\begin{aligned} h_{l,c} \in \partial^*_{H_L,Z,\epsilon} f(x_0) &\Leftrightarrow \forall x \in Z,\ h_{l,c}(x) \leq f(x) + \epsilon\|x - x_0\|, \\ &\Leftrightarrow \forall\, x \in Z,\ l(x) - c \leq f(x) + \epsilon\|x - x_0\|, \\ &\Leftrightarrow \forall\, x \in Z,\ l(x) - l(x_0) + f(x_0) \leq f(x) + \epsilon\|x - x_0\|, \\ &\Leftrightarrow \forall\, x \in Z,\ l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon\|x - x_0\|, \\ &\Leftrightarrow l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0). \end{aligned}$$

□

Proposition 3.2. *Let Z be a subset of a Banach space and let L be a set of abstract linear functions. Suppose that $f : Z \to \mathbb{R}$ be a Lipschitz function and $x_0 \in Z$. Set*

$K := \{l \in L : l(x) = a\|x - x_0\|, a \leq \min\{0, \widehat{\beta}_\epsilon(f, x_0)\}\}$, *where*

$$\widehat{\beta}_\epsilon(f, x_0) := \beta(f, x_0) + \epsilon \text{ and } \beta(f, x_0) := \inf_{x \neq x_0, x \in Z} \frac{f(x) - f(x_0)}{\|x - x_0\|}.$$

Then $\widehat{\partial}_{L,Z,\epsilon} f(x_0) \neq \emptyset$ *and* $K \subseteq \widehat{\partial}_{L,Z,\epsilon} f(x_0)$. *Moreover, equality holds for*

$$L := \{a\|x - x_0\| : a \leq 0\}.$$

Proof. It follows from $\partial_{L,Z} f(x_0) \subseteq \widehat{\partial}_{L,Z,\epsilon} f(x_0)$ and $\partial_{L,Z} f(x_0) \neq \emptyset$ that $\widehat{\partial}_{L,Z,\epsilon} f(x_0) \neq \emptyset$ (see Proposition 7.2 in [6]). To prove the inclusion let $l \in K$ so $l(x) = a\|x - x_0\|$ with $a \leq \widehat{\beta}_\epsilon(f, x_0)$. Therefore,

$$a \leq \inf_{x \neq x_0, x \in Z} \frac{f(x) - f(x_0)}{\|x - x_0\|} + \epsilon.$$

That is for each $x \in Z$ with $x \neq x_0$ we have $a \leq \frac{f(x) - f(x_0)}{\|x - x_0\|} + \epsilon$, or equivalently

$$a\|x - x_0\| \leq f(x) - f(x_0) + \epsilon\|x - x_0\|, \ \forall\, x \in Z,$$

and consequently,

$$l(x) - l(x_0) \leq f(x) - f(x_0) + \epsilon\|x - x_0\|, \ \forall\, x \in Z.$$

Therefore $l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$.

Suppose that $L := \{a\|x - x_0\| : a \leq 0\}$ and $l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$. From Proposition 3.1, we have $h \in \partial^*_{H_L,Z,\epsilon} f(x_0)$, where $h(x) = a\|x - x_0\| - c$ with $c = -f(x_0)$. Hence,

$$h(x) \leq f(x) + \epsilon\|x - x_0\|, \ \forall\, x \in Z.$$

Therefore

$$a\|x - x_0\| + f(x_0) \leq f(x) + \epsilon\|x - x_0\|, \ \forall\, x \in Z.$$

Equivalently,

$$a \leq \frac{f(x) - f(x_0)}{\|x - x_0\|} + \epsilon, \ \forall\, x \in Z,$$

and consequently,

$$a \leq \inf_{x \neq x_0, x \in Z} \frac{f(x) - f(x_0)}{\|x - x_0\|} + \epsilon = \widehat{\beta}_\epsilon(f, x_0).$$

The proof is completed. □

Let $Z \subseteq X$, and let L be a set of elementary functions defined on X and H the set of L-affine functions. For a fixed $x_0 \in Z$ the set

$$\mathcal{D}_\epsilon f(x_0) := \{h \in H : h(x) = l(x) - l(x_0),\ \forall x \in Z,\ l \in \partial_{L,Z,\epsilon} f(x_0)\}. \tag{3.3}$$

is defined in [6, Section 8.3.2]. Given a subset U of functions defined on Z, we say that $g \in U$ is a maximal element of the set U when $g' \in U$, $g'(x) \geq g(x)$ for all $x \in Z$ implies that $g' = g$. Proposition 8.4 in [6] establishes a bijection between maximal elements of $\partial^*_{H,Z} f(x_0)$ and maximal elements of $Df(x_0)$. Inspired by this result, we will establish a similar connection between maximal elements of $\partial^*_{H,Z,\epsilon} f(x_0)$ and maximal elements of the set

$$\widehat{\mathcal{D}}_\epsilon f(x_0) := \{h - \epsilon\| \cdot - x_0\| : h \in H, h(x) = l(x) - l(x_0),\ \forall\, x \in Z,\ l \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)\}. \tag{3.4}$$

A careful inspection of Proposition 8.4 in [6] shows that maximal elements in $Df(x_0)$ are in bijective correspondence with those maximal elements $h \in \partial^*_{H,Z} f$ which verify $h(x_0) = f(x_0)$. In other words, there is a one-to-one correspondence between maximal elements of the sets $\partial^*_{H,Z} f(x_0)$ and $Df(x_0)$. With this stronger statement in mind, our next result becomes Proposition 8.4 in [6] when $\epsilon = 0$.

Proposition 3.3. *Let f be H_L-convex on $Z \subseteq X$, $x_0 \in Z$, $l \in L$ and $\epsilon \geq 0$. Then for all $x \in Z$ the following statements are equivalent:*

(i) *$h(x) - \epsilon\|x - x_0\|$ is a maximal element of $\widehat{\mathcal{D}}_\epsilon f(x_0)$*

(ii) *$h'(x) = l(x) - (l(x_0) - f(x_0))$ is a maximal element of $\partial^*_{H_L,Z,\epsilon} f(x_0)$.*

Proof. First assume that $h(x) - \epsilon\|x - x_0\|$, where $h(x) = l(x) - l(x_0)$ is a maximal element of $\widehat{\mathcal{D}}_\epsilon f(x_0)$ and let $g \in \partial^*_{H_L,Z,\epsilon} f(x_0)$ be such that

$$g(x) \geq l(x) - (l(x_0) - f(x_0)) = h'(x). \tag{3.5}$$

We show that $g(x) = h'(x)$ for all $x \in Z$. It follows from the definition of $\partial^*_{H_L,Z,\epsilon} f(x_0)$ that $g(x) = l'(x) - c'$ for some $l' \in L$, $c' \in \mathbb{R}$ and also we have:

$$g(x_0) = f(x_0) = l'(x_0) - c'.$$

Hence, $c' = l'(x_0) - f(x_0)$, which means that $g(x) = l'(x) - (l'(x_0) - f(x_0))$. Thus, by (3.5)

$$l'(x) - l'(x_0) \geq l(x) - l(x_0).$$

Therefore

$$l'(x) - l'(x_0) - \epsilon\|x - x_0\| \geq l(x) - l(x_0) - \epsilon\|x - x_0\| = h(x) - \epsilon\|x - x_0\|. \tag{3.6}$$

On the other hand, since $g \in \partial^*_{H_L,Z,\epsilon} f(x_0)$,

$$\begin{aligned} f(x) - f(x_0) - (l'(x) - l'(x_0)) &\geq f(x) - f(x_0) - (g(x) - f(x_0)) \\ &= f(x) - g(x) \geq -\epsilon\|x - x_0\|, \end{aligned}$$

which implies that $l' \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$. Maximality of $h - \epsilon\|. - x_0\|$ in $\widehat{\mathcal{D}}_\epsilon f(x_0)$ and inequality (3.6), imply that $l'(x) - l'(x_0) = l(x) - l(x_0)$. Therefore, $g(x) = l(x) - (l(x_0) - f(x_0))$ for all $x \in Z$ and so g is a maximal element of $\partial^*_{H_L,Z,\epsilon} f(x_0)$.
Conversely, let $h'(x) = l(x) - (l(x_0) - f(x_0))$ be a maximal element of $\partial^*_{H_L,Z,\epsilon} f(x_0)$ and let $g - \epsilon\|. - x_0\| \in \widehat{\mathcal{D}}_\epsilon f(x_0)$ be such that

$$g(x) - \epsilon\|x - x_0\| \geq l(x) - l(x_0) - \epsilon\|x - x_0\| = h(x) - \epsilon\|x - x_0\|.$$

We show that $g(x) = h(x)$ for all $x \in Z$. By the definition of $\widehat{\mathcal{D}}_\epsilon f(x_0)$, there exists $l' \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$ such that $g(x) = l'(x) - l'(x_0)$. Hence,

$$l'(x) - l'(x_0) \geq l(x) - l(x_0),\ \forall\, x \in Z,$$

and so

$$l'(x) - (l'(x_0) - f(x_0)) \geq l(x) - (l(x_0) - f(x_0)) = h'(x),\ \forall\, x \in Z. \tag{3.7}$$

Since $l' \in \widehat{\partial}_{L,Z,\epsilon} f(x_0)$,

$$\begin{aligned} f(x) + \epsilon\|x - x_0\| &\geq f(x_0) + l'(x) - l'(x_0), \\ &\geq f(x_0) + l(x) - l(x_0) = h'(x). \end{aligned}$$

Moreover, it follows from $h'(x_0) = f(x_0)$ that $h' \in \partial^*_{H_L,Z,\epsilon} f(x_0)$. Therefore, by the fact that h' is a maximal element of $\partial^*_{H_L,Z,\epsilon} f(x_0)$ and (3.7), we get

$$l'(x) - l'(x_0) = l(x) - l(x_0).$$

Thus $g(x) = h(x)$ for all $x \in Z$ and $g - \epsilon\|.-x_0\|$ is a maximal element of $\widehat{\mathcal{D}}_\epsilon f(x_0)$. □

4. Examples

The following example completes the Proposition 3.2.

Example 4.1. (i) *Let* $L = \{ax^2 + bx : a < 0, b \in \mathbb{R}\}$, $Z_1 = [-1/4, 1/4]$ *and* $Z_2 = [-1, 1]$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = -|x - 1/2|$ *for all* $x \in \mathbb{R}$. *Consider the abstract linear function* $l \in L$ *defined by* $l(x) = -x^2 + x$. *Then one can verify that* $\widehat{\partial}_{L,Z_1,\epsilon} f(1/2) \not\subseteq \widehat{\partial}_{L,Z_2,\epsilon} f(1/2)$.

(ii) *Let* $L_1 = \{a\|x - x_0\| : a \leq \min\{0, \widehat{\beta}_\epsilon(f, x_0)\}$ *and* $L_2 = \{a\|x - x_0\| : a \leq 0\}$. *Let* Z *be a subset of* $\mathbb{R}$ *and* $f(x) = \|x - 1/2\|$ *or any other lipschitz function. Then by Proposition 3.2 for* $x_0 \in Z$ *we have* $\widehat{\partial}_{L_1,Z,\epsilon} f(x_0) = \widehat{\partial}_{L_2,Z,\epsilon} f(x_0)$.

For a counterexample, let $L_1 = \{a\|x-x_0\| : a \leq -1\}$ *and* $L_2 = \{a\|x-x_0\| : a \leq 0\}$. *Let* $Z = \mathbb{B}(0,1)$ *and* $f(x) = -1/2\|x\|^2$. *Take* $x_0 = 0$, *then by Proposition 3.2 we have*

$$\widehat{\partial}_{L_2,Z,\epsilon} f(0) = \{l \in L : l(x) = a\|x - x_0\|, a \leq \min\{0, \widehat{\beta}_\epsilon(f,0)\}\},$$

where

$$\begin{aligned}\widehat{\beta}_\epsilon(f,0) :&= \inf_{x \neq x_0, x\in Z} \frac{f(x) - f(x_0)}{\|x - x_0\|} + \epsilon \\ &= \inf_{x \neq x_0, x\in Z} \frac{-1/2\|x\|^2}{\|x\|} + \epsilon = -1/2 + \epsilon.\end{aligned}$$

We have $\widehat{\beta}_\epsilon(f,0) > 0$, *for every* $\epsilon > 1/2$. *Let* $l \in L_2$ *be defined by* $l(x) = -1/2\|x\|$. *Then for all* $\epsilon > 1/2$, *it is easy to see that* $\widehat{\partial}_{L_2,Z,\epsilon} f(0) \nsubseteq \widehat{\partial}_{L_1,Z,\epsilon} f(0)$.

(iii) *Let* $L = \{ax^2 + bx : a < 0, b \in \mathbb{R}\}$, $Z = [-1/4, 1/4]$, $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = -|x - 1/2|$ *for all* $x \in \mathbb{R}$ *and let* $l \in L$ *be defined by* $l(x) = -x^2 + x$. *Then it is easy to check that* $l \in \widehat{\partial}_{L,Z,3/4} f(1/2) \setminus \widehat{\partial}_{L,Z,1/4} f(1/2)$.

Example 4.2. *Let* $L := \{a\|x-x_0\| : a \leq 0\}$ *as in the second part of Proposition 3.2. Let* $f(x) = -\|x\|^2$ *and take* $x_0 = 0$. *Notice that* $\beta(f,0) = \inf\{-\|x\| : x \neq 0, x \in Z\}$ *and* $\hat{\beta}_\epsilon(f,0) := \beta(f,0) + \epsilon$. *So when* Z *is unbounded, both of them is* $-\infty$ *and so* $\widehat{\partial}_{L,Z,\epsilon} f(0)$ *will be the empty set.*

Assume that $Z = \mathbb{B}(0,1)$ *is the unit ball of* X. *Then we have* $\beta(f,0) = -1$ *and* $\hat{\beta}_\epsilon(f,0) = \epsilon - 1$. *Hence, in order to have the inclusion* $\widehat{\partial}_{L,Z,\epsilon} f(0) \subsetneq L$ *hold, suppose that* $\epsilon < 1$. *Since* $l(0) = 0$, *by Proposition 3.3 and Proposition 3.2, the maximal element of* $\widehat{\mathcal{D}}_\epsilon f(0)$ *is equal to*

$$\hat{\beta}_\epsilon(f,0)\|x - x_0\| - \epsilon\|x - x_0\| = (\epsilon - 1)\|x\| - \epsilon\|x\| = -\|x\|.$$

The corresponding element in $\partial^*_{H_L,Z,\epsilon} f(0)$ *is* $h'(x) = (\epsilon - 1)\|x\|$.

Example 4.3. *Let* $L := \{a\|x - x_0\| : a \leq 0\}$ *as in the second part of Proposition 3.2. Let* $f(x) = -2\|x\|$ *and take* $x_0 = 0$*. Assume that* $Z = \mathbb{B}(0,1)$*, the unit ball of* X*. Suppose that* $l \in \widehat{\partial}_{L,\epsilon} f(x_0)$ *and* $\epsilon < 1$*.*

Since $l(0) = f(0) = 0$*, by Proposition 3.2, we have* $\beta(f,0) = -1$*,* $\hat{\beta}_\epsilon(f,0) = \epsilon - 1$ *and*

$$\begin{aligned} f^*(l) + f(x_0) - l(x_0) &= \sup_{x \in X}(l(x) - f(x)) \\ &\geq l(x) - f(x) = a\|x\| + 2\|x\|. \end{aligned}$$

Now take $a = \hat{\beta}_\epsilon(f,0)$*, so*

$$f^*(l) + f(x_0) - l(x_0) \geq (\epsilon - 1)\|x\| + 2\|x\| > \epsilon\|x\| = \epsilon\|x - x_0\|.$$

That is the converse of Proposition 2.2 does not hold.

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