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.

COMPATIBLE AND WEAKLY COMPATIBLE MAPS IN A COMPLEX FUZZY METRIC SPACE

SHOBHA JAIN [(1)] AND SHISHIR JAIN [(2)]

Abstract. In this paper we introduce the notion of compatibility of self maps in a complex fuzzy metric space. Using this, we establish some common fixed point theorems employing a generalized contractive condition, which extend and generalize the existing results in fuzzy metric spaces to a complex fuzzy metric space.They also generalize the existing results of Singh et al.[13] in a complex fuzzy metric space.

1. Introduction

The concept of fuzzy complex number was first introduced by J. J Buckley in 1989 in [2]-[4].In [10] Ramot et al. characterized the complex fuzzy set by a membership function, whose range is not limited to $[0, 1]$ but extended to unit circle in the complex plane.The frame of fuzzy complex analysis theory has had its primary shape and tends to form a new branch of Mathematics step by step.This new branch subject will be widely applied in fuzzy system theory, specially in fuzzy dynamical system theory, and will also be widely applied in the field of computational intelligence. It

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is well known that the fixed point theory plays a very important role in Mathematical theory and applications. In particular, it has important place in finding roots of algebraic equations and numerical Mathematics. Recently, Singh et al.[13] proved some fixed point theorems for two self mapping in the setting of a complex valued fuzzy metric space. Sessa [12] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric space and Jungck [7] enlarged this concept to compatible maps.
The aim of this paper is to introduce the notion of compatible maps in complex fuzzy metric space.Using this notion we prove some fixed point theorems.Our results improve and extend the results of Singh et al.[13].

2. Preliminaries

Throughout this paper we use the symbols and basic definitions of Singh et al.[13]. In view of partial order relation defined by Azam et al.[1] and Singh et al.[13] we define the relation " $\preceq$ " in $[0,1]e^{i\theta}$, for a fixed $\theta \in [0, \frac{\pi}{2}]$(for comparing two complex number) as follows:

$$z_1, z_2 \in [0,1]e^{i\theta}, z_1 = r_1 e^{i\theta}, z_2 = r_2 e^{i\theta}, z_1 \preceq z_2 \Leftrightarrow r_1 \leq r_2.$$

It follows that $z_1 \preceq z_2$ if one of the following conditions are satisfied :

- $Re(z_1) = Re(z_2)$, then $r_1 = r_2$.
- $Re(z_1) < Re(z_2)$, then $r_1 < r_2$.

Thus we define

- $\min\{z_1, z_2\} = z_1$ if $z_1 \preceq z_2$.
- $\max\{z_1, z_2\} = z_1$ if $z_2 \preceq z_1$, etc.

Definition 2.1. [13]: A complex fuzzy set S, defined on a universe of discourse U, is characterized by a membership function $\mu_s(x)$ that assign every element $x \in U$, a complex valued grade of membership in S. The values $\mu_s(x)$ lie within the unit circle in the complex plane, and thus are of the form

$$\mu_s(x) = r_s(x)e^{iw_s(x)}, i = \sqrt{-1}$$

where $r_s(x)$ and $w_s(x)$ both real-valued with $r_s(x) \in [0, 1]$.

Definition 2.2. [13] : A binary operation $* : [0, 1]e^{i\theta} \times [0, 1]e^{i\theta} \to [0, 1]e^{i\theta}$, where $\theta \in [0, \frac{\pi}{2}]$ is fixed, is called a complex valued continuous t-norm if it satisfies the following conditions:

- $*$ is associative and commutative.
- $*$ is continuous.
- $a * e^{i\theta} = a$, for all $a \in [0, 1]e^{i\theta}$ (existence of identity element $e^{i\theta} = 1.e^{i\theta}$).
- $a * b \preceq c * d$, whenever $a \preceq c$ and $b \preceq d$, for all $a, b, c, d \in [0, 1]e^{i\theta}$.

Example 2.1. *of t-norm are*

- $a * b = \min\{a, b\}, a, b \in [0, 1]e^{i\theta}$,
- $a * b = \max\{a + b - e^{i\theta}, 0\}$,

for a fixed $\theta \in [0, \frac{\pi}{2}]$.

Note: Throughout this article, θ is taken to be fixed angle in $[0, \frac{\pi}{2}]$ with the assumption that the complex fuzzy sets $S = \{(x, \mu_s(x)) : x \in U\}$ interact with other complex valued fuzzy metric sets in view of the partial ordering due to Azam et al.[1]. Now,we define a complex valued fuzzy metric space as follows:

Definition 2.3. [13] : The 3-tuple $(X, M, *)$ is said to be a complex valued fuzzy metric space, if X is an arbitrary non-empty set, $*$ is a complex valued continuous t-norm and $M : X^2 \times (0, \infty) \to [0, 1]e^{i\theta}$, is a complex fuzzy set, satisfying the following conditions :

$(CFM-1)$ $M(x,y,0) \succ 0$;

$(CFM-2)$ $M(x,y,t) = e^{i\theta}, \forall t > 0$ if and only if $x = y$;

$(CFM-3)$ $M(x,y,t) = M(y,x,t)$;

$(CFM-4)$ $M(x,y,t) * M(y,z,s) \preceq M(x,z,t+s)$;

$(CFM-5)$ $M(x,y,.) : (0,\infty) \to [0,1]e^{i\theta}$, is continuous,

for all $x,y,z \in X$ and $s,t > 0$.

M is called a complex fuzzy metric in the complex fuzzy metric space $(X,M,*)$.

Remark 1. : If $\theta = 0$, then Complex fuzzy metric space becomes ordinary fuzzy metric space in sense of George and Veeramani.

Example 2.2. [13]*: Let* $X = N$*.Define* $a * b = \min\{a,b\}$*,for all* $a,b \in [0,1]e^{i\theta}$*, where* $\theta \in [0,\pi/2]$ *is fixed. Taking*
$M(x,y,t) = e^{i\theta}e^{-|x-y|/t}$*, for all* $x,y \in X$ *and* $t \in (0,\infty)$*. Then* $(X,M,*)$ *is a complex valued fuzzy metric space.*

Example 2.3. [13]*: Let* $X = N$*. Define* $a * b = \min\{a,b\}$*, for all* $a,b \in [0,1]e^{i\theta}$*, where* $\theta \in [0,\frac{\pi}{2}]$ *is fixed. We define*

$$M(x,y,t) = \begin{cases} e^{i\theta\frac{x}{y}}, & \textit{if } x \leq y; \\ e^{i\theta\frac{y}{x}}, & \textit{if } y \leq x; \end{cases}$$

for all $x,y \in X$ *and* $t \in (0,\infty)$*.*
Then $(X,M,*)$ *is not a complex valued fuzzy metric space.*

In view of partial ordering due to Azam et al.[1] increasing and decreasing functions for the set of complex number, are defined.

Definition 2.4. [13]: Let X be any non-empty ordered set. A function $f : X \to C$ is called an increasing function if $f(x_1) \succ f(x_2)$, whenever $x_1 > x_2$ for $x_1, x_2 \in X$.

Example 2.4. [13]*: Let $X = N$ with usual partial order. Define $f : X \to C$ by $f(x) = xe^{\frac{\pi i}{3}}$, for all $x \in X$. Then $f(x)$ is an increasing function.*

Definition 2.5. [13] Let X be any non-empty ordered set. A function $f : X \to C$ is called a decreasing function if $f(x_1) \prec f(x_2)$, whenever $x_1 > x_2$ for $x_1, x_2 \in X$.

Example 2.5. [13] *Let $X = N$ with usual partial order. Define $f : X \to C$ by $f(x) = xe^{\frac{4\pi i}{3}}$, for all$x \in X$. Then $f(x)$ is a decreasing function.*

Lemma 2.1. *([13]): $M(x, y, .)$ is non-decreasing function for all $x, y \in X$.*

Theorem 2.1. *(Theorem 3.2* [13] **Convergence of a sequence***): Let $(X, M, *)$ be a Complex valued fuzzy metric space and τ be the topology induced by complex valued fuzzy metric. Then for a sequence $\{x_n\} \in X$ we have $x_n \to x$ if and only if $M(x_n, x, t) \to e^{i\theta}$, as $n \to \infty$ or $|M(x_n, x, t)| \to 1$, as $n \to \infty$.*

Remark 2. : Let $\{x_n\}$ and $\{y_n\}$ be two sequences in complex fuzzy metric $(X, M, *)$ such that $x_n \to x$ and $y_n \to y$ then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$.

Definition 2.6. (Definition 3.5 [13] **Cauchy sequence**): A sequence $\{x_n\} \in X$ in a complex valued fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = e^{i\theta}, p > 0, t > 0$ or $\lim_{n\to\infty} |M(x_{n+p}, x_n, t)| = 1, p > 0, t > 0$.

Here we are defining Cauchy sequence in a complex fuzzy metric space due to George and Veeramani [5].

Definition 2.7. (**Cauchy sequence**) : A sequence $\{x_n\} \in X$ in a complex valued fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if and only if for every $\epsilon > 0, t > 0$, there exists an integer n_0 such that$M(x_n, x_m, t) \succ e^{i\theta} - \epsilon$, for all $n, m \geq n_0$.

Definition 2.8. (Definition 3.6 [13]): A complex valued fuzzy metric space in which every Cauchy sequence is convergent is called a complete complex valued fuzzy metric space.

Lemma 2.2. *(Lemma 4.1* [13]*): Let* $(X, M, *)$ *be a complex valued fuzzy metric space such that* $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$*, for all* $x, y \in X$*, if*

$$M(x, y, kt) \succeq M(x, y, t), 0 < k < 1, t \in (0, \infty),$$

then $x = y$*.*

Lemma 2.3. *(Lemma 4.2* [13]*): Let* $\{y_n\}$ *be a sequence in a complex valued fuzzy metric space such that* $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$*, for all* $x, y \in X$*. If there exists a number* $k \in (0, 1)$ *such that*

$$M(y_{n+1}, y_{n+2}, kt) \succeq M(y_n, y_{n+1}, t), 0 < k < 1, t \in (0, \infty),$$

then $\{y_n\}$ *is a Cauchy sequence in* X*.*

Compatibility in complex fuzzy metric space

Definition 2.9. : Let S, T be two self mappings in a non-empty set X. The pair (S, T) is said to be compatible if $M(STx_n, TSx_n, t) \to e^{i\theta}, \forall t > 0$, as $n \to \infty$, whenever $M(Sx_n, x, t) \to e^{i\theta}$, and $M(Tx_n, x, t) \to e^{i\theta}, \forall t > 0$, as $n \to \infty$.

Definition 2.10. : A pair of self mappings (S, T) in X is called weakly compatible if for $v \in X$ and $Sv = Tv$, then $TSv = STv$.

Remark 3. : Every compatible pair of self maps is weakly compatible. Let the the pair of self maps (S, T) in X be compatible and $Sv = Tv = w$, for some $v, w \in X$. As $Sv = w$ and $Tv = w$, using $(CFM - 2)$ we have

$$M(Sv, w, t) = e^{i\theta} \text{ and } M(Tv, w, t) = e^{i\theta}.$$

Taking $\{x_n\} = v, \forall n$, we have

$$M(Sx_n, w, t) \to e^{i\theta} \text{ and } M(Tx_n, w, t) \to e^{i\theta}.$$

Using Compatibility of the pair (S, T) we get

$$M(TSx_n, STx_n, t) \to e^{i\theta}, \forall t > 0 \implies M(TSv, STv, t) \to e^{i\theta}, \forall t > 0.$$

i.e.

$$M(TSv, STv, t) = e^{i\theta}, \forall t > 0, \text{as L.H.S. is a constant sequence.}$$

So, we have $TSv = STv$ by $(CFM-2)$. So the pair (S, T) is weakly compatible.

3. Main Results

Theorem 3.1. *: Let $(X, M, *)$ be a complete complex fuzzy metric space with continuous t-norm $t * t \geq t$ such that $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \forall x, y \in X, \forall t > 0$. Let A, B, S and T be self mappings in X satisfying:*

(3.11) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(3.12) *The pair (A, S) is compatible and the pair (B, T) is weakly compatible;*

(3.13) *One of the map A or S is continuous;*

(3.14) *the mappings A, B, S and T satisfy :*

$$M(Ax, By, kt) \succeq \min \left\{ \begin{array}{l} M(Sx, Ty, t) * M(Sx, Ax, t) * M(By, Ty, t) \\ *M(Sx, By, 2t) * M(Ax, Ty, 2t) \end{array} \right\}.$$

for all $x, y \in X$ and some fixed $k \in (0, 1)$, for all t.

Then A, B, S and T have unique common fixed point in X.

Proof. : Let $x_0 \in X$ be any arbitrary point. As $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Letting $Ax_0 = Tx_1 = y_1$ and $Bx_1 = Sx_2 = y_2$, etc., we have a sequence $\{y_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$. Now using (3.14) with $x =$

$x_{2n}, y = x_{2n+1}$ we get,

$$\begin{aligned}
M(y_{2n+1}, y_{2n+2}, kt) &= M(Ax_{2n}, Bx_{2n+1}, kt) \\
&\succeq \min \left\{ \begin{array}{c} M(Sx_{2n}, Tx_{2n+1}, t) * M(Sx_{2n}, Ax_{2n}, t) \\ *M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Sx_{2n}, Bx_{2n+1}, 2t) \\ *M(Ax_{2n}, Tx_{2n+1}, 2t) \end{array} \right\}; \\
&= \min \left\{ \begin{array}{c} M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, t), \\ *M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n}, y_{2n+2}, 2t) \\ *M(y_{2n+1}, y_{2n+1}, 2t) \end{array} \right\}; \\
&\succeq \min \left\{ \begin{array}{c} M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * \\ M(y_{2n}, y_{2n+1}, t/2) * M(y_{2n+1}, y_{2n+2}, t/2) \end{array} \right\}, \\
&\succeq \min \left\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t) \right\},
\end{aligned}$$

as both the last two factors appear in the first two elements.

Thus

$$M(y_{2n+1}, y_{2n+2}, kt) \succeq \min \left\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t) \right\}. \tag{3.1}$$

Similarly, by taking $x = x_{2n}$ and $y = x_{2n-1}$ in (3.14) we get

$$M(y_{2n}, y_{2n+1}, kt) \succeq \min \left\{ M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t) \right\}, \forall n. \tag{3.2}$$

Combining equations(3.1) and (3.2) we get,

$$M(y_n, y_{n+1}, kt) \succeq \min \left\{ M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t) \right\}, \forall n. \tag{3.3}$$

Case I: Suppose $\min\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)\} = M(y_n, y_{n+1}, t)$. Then from (3.3) we have,

$M(y_n, y_{n+1}, kt) \succeq M(y_n, y_{n+1}, t), \forall n.$

So by Lemma 2.2, $y_n = y_{n+1}, \forall n$. Thus the sequence $\{y_n\}$ becomes a constant sequence. Hence it is a Cauchy sequence in X.

Case II: Suppose $\min\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)\} = M(y_{n-1}, y_n, t)$. Then from (3.3) we have,

$M(y_n, y_{n+1}, kt) \succeq M(y_{n-1}, y_n, t), \forall n$.

So by Lemma 2.3, the sequence $\{y_n\}$ is a Cauchy sequence in X. As X is complete, $\{y_n\} \to z$, for some $z \in X$. Hence

$$\{Ax_{2n}\} \to z, \quad \{Sx_{2n}\} \to z. \tag{3.4}$$

i.e.

$$M(Ax_{2n}, z, t) \to e^{i\theta}, \quad M(Sx_{2n}, z, t) \to e^{i\theta}.$$

$$\{Tx_{2n+1}\} \to z, \quad \{Bx_{2n+1}\} \to z. \tag{3.5}$$

i.e.

$$M(Tx_{2n+1}, z, t) \to e^{i\theta}, \quad M(Bx_{2n+1}, z, t) \to e^{i\theta}.$$

CASE 1: Assume S is continuous.

$$\{SAx_{2n}\} \to Sz, \quad \{S^2x_{2n}\} \to Sz. \tag{3.6}$$

i.e.

$$M(SAx_{2n}, Sz, t) \to e^{i\theta}, \quad M(S^2x_{2n}, Sz, t) \to e^{i\theta}.$$

The pair (A, S) is compatible, using equation (3.4) we have

$\lim_{n\to\infty} M(SAx_{2n}, ASx_{2n}, t) = e^{i\theta}$.

$$As \ \ \{ASx_{2n}\} \to Sz, \qquad and \quad \lim_{n\to\infty} M(SAx_{2n}, Sz, t) = e^{i\theta}, \forall t > 0. \tag{3.7}$$

Step 1: Taking $x = Sx_{2n}, y = x_{2n+1}$ in (3.14) we get,

$$M(ASx_{2n}, Bx_{2n+1}, kt) \succeq \min \left\{ \begin{array}{l} M(SSx_{2n}, Tx_{2n+1}, t), M(SSx_{2n}, ASx_{2n}, t), \\ M(Bx_{2n+1}, Tx_{2n+1}, t), M(SSx_{2n}, Bx_{2n+1}, 2t), \\ M(ASx_{2n}, Tx_{2n+1}, 2t) \end{array} \right\}.$$

Letting $n \to \infty$ and using equations (3.5),(3.6), (3.7) we get,

$$\begin{aligned} M(Sz, z, kt) &\succeq \min \left\{ \begin{array}{c} M(Sz, z, t), M(Sz, Sz, t), M(z, z, t), \\ M(Sz, z, 2t), M(Sz, z, 2t) \end{array} \right\}, \\ &\succeq \min \left\{ M(Sz, z, t), M(Sz, z, t) * M(z, z, t) \right\}, \\ &= \min \left\{ M(Sz, z, t), M(Sz, z, t) * e^{i\theta} \right\}, \\ &= M(Sz, z, t), \forall t > 0. \end{aligned}$$

Thus $M(Sz, z, kt) \geq M(Sz, z, t), \forall t > 0$.

So by Lemma 2.2, we have $Sz = z$.

Step2 : Taking $x = z, y = x_{2n+1}$ in (3.14) we get,

$$M(Az, Bx_{2n+1}, kt) \succeq \min \left\{ \begin{array}{l} M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Bx_{2n+1}, Tx_{2n+1}, t), \\ M(Sz, Bx_{2n+1}, 2t), M(Az, Tx_{2n+1}, 2t) \end{array} \right\}.$$

Letting $n \to \infty$ and using equation (3.5) and using $Sz = z$ we get,

$$\begin{aligned} M(Az, z, kt) &\succeq \min \left\{ \begin{array}{c} M(z, z, t), M(z, Az, t), M(z, z, t), \\ M(z, z, 2t), M(Az, z, 2t) \end{array} \right\}, \\ &= M(Az, z, t). \end{aligned}$$

Thus,

$$M(Az, z, kt) \succeq M(Az, z, t), \forall t > 0.$$

By Lemma 2.2, we have $Az = z$. Thus we have $Az = Sz = z$.

As $A(X) \subseteq T(X)$ there exists some $v \in X$ such that $Az = Tv$. So we get $z = Sz =$

$Az = Tv$.

Step 3 : Taking $x = z$ and $y = v$ in (3.14) we get

$$M(Az, Bv, kt) \succeq \min \left\{ \begin{array}{l} M(Sz, Tv, t), M(Sz, Az, t), M(Bv, Tv, t), \\ M(Sz, Bv, 2t), M(Az, Tv, 2t) \end{array} \right\}.$$

Using $z = Sz = Az = Tv$ we get,

$$\begin{aligned} M(z, Bv, kt) &\succeq \min \left\{ \begin{array}{c} M(z, z, t), M(z, z, t), M(Bv, z, t), \\ M(z, Bv, 2t), M(z, z, 2t) \end{array} \right\}, \\ &\succeq \min \left\{ e^{i\theta}, M(Bv, z, t), M(z, z, t) * M(z, Bv, t) \right\}, \\ &= \min \left\{ e^{i\theta}, M(Bv, z, t), e^{i\theta} * M(z, Bv, t), \right\}, \\ &= M(z, Bv, t), \forall t > 0. \end{aligned}$$

Thus

$$M(z, Bv, kt) \succeq M(z, Bv, t), \forall t > 0.$$

So by Lemma 2.2, we have $Bv = z$. Thus we get $Tv = Bv = z$. As the pair (B, T) is weakly compatible we get $Bz = Tz$.

Step 4: Taking $x = z$ and $y = z$ in (3.14) we get

$$M(Az, Bz, kt) \succeq \min \left\{ \begin{array}{l} M(Sz, Tz, t), M(Sz, Az, t), M(Bz, Tz, t), \\ M(Sz, Bz, 2t), M(Az, Tz, 2t) \end{array} \right\}.$$

Using $z = Sz = Az$ and $Tz = Bz$ we get,

$$\begin{aligned} M(z, Bz, kt) &\succeq \min \left\{ \begin{array}{c} M(z, Bz, t), M(z, z, t), M(Bz, Bz, t), \\ M(z, Bz, 2t), M(z, Bz, 2t) \end{array} \right\}, \\ &\succeq \min \left\{ M(Bz, z, t), e^{i\theta}, M(z, z, t) * M(z, Bz, t) \right\}, \\ &= \min \left\{ M(z, Bz, t), e^{i\theta} * M(z, Bz, t) \right\}, \\ &= M(z, Bz, t). \end{aligned}$$

Thus

$$M(z,Bz,kt) \succeq M(z,Bz,t), \forall t>0.$$

So by Lemma 2.2 we have $Bz=z$ and so $Bz=Tz=z$. Combining the results of step 2 and step 4, we get$Az=Sz=Bz=Sz=z$.

CASE 2: Assume A is continuous.

$$\{A^2x_{2n}\} \to Az, \quad \{ASx_{2n}\} \to Az. \tag{3.8}$$

i.e.

$$M(A^2x_{2n},Az,t) \to e^{i\theta}, \quad M(ASx_{2n},Az,t) \to e^{i\theta}, \forall t>0.$$

The pair (A,S) is compatible using equation (3.4) we have $\lim_{n\to\infty} M(ASx_{2n},SAx_{2n},t) = e^{i\theta}$. Using equation (3.8) we have

$$\lim_{n\to\infty} M(SAx_{2n},Az,t) = e^{i\theta}.$$

Thus

$$\{SAx_{2n}\} \to Az. \tag{3.9}$$

Step 5: Taking $x=Ax_{2n}, y=x_{2n+1}$ in (3.14) we get,

$$M(A^2x_{2n},Bx_{2n+1},kt) \succeq \min \left\{ \begin{array}{l} M(SAx_{2n},Tx_{2n+1},t), M(SAx_{2n},AAx_{2n},t), \\ M(Bx_{2n+1},Tx_{2n+1},t), M(SAx_{2n},Bx_{2n+1},2t), \\ M(A^2x_{2n},Tx_{2n+1},2t) \end{array} \right\}.$$

Letting $n\to\infty$ and using equations(3.5),(3.8) (3.9)we get,

$$\begin{aligned} M(Az,z,kt) &\succeq \min \left\{ \begin{array}{c} M(Az,z,t), M(Az,Az,t), M(z,z,t), \\ M(Az,z,2t), M(Az,z,2t) \end{array} \right\}, \\ &\succeq \min \left\{ M(Az,z,t), M(Az,z,t) * M(z,z,t) \right\}, \\ &= \min \left\{ M(Az,z,t), M(Az,z,t) * e^{i\theta} \right\}, \\ &= M(Az,z,t). \end{aligned}$$

Thus

$M(Az, z, kt) \geq M(Az, z, t), \forall t > 0$. So by Lemma 2.2, we have $Az = z$.

As$A(X) \subseteq T(X)$ there exists some $u \in X$ such that $Az = Tu$. So we get

$$z = Az = Tu. \tag{3.10}$$

Step 6 : Taking $x = x_{2n}$ and $y = u$ in (3.14) we get

$$M(Ax_{2n}, Bu, kt) \succeq \min \left\{ \begin{array}{l} M(Sx_{2n}, Tu, t), M(Sx_{2n}, Ax_{2n}, t), M(Bu, Tu, t), \\ M(Sx_{2n}, Bu, 2t), M(Ax_{2n}, Tu, 2t) \end{array} \right\}.$$

Letting $n \to \infty$ and using equations (3.4),(3.10) we get,

$$\begin{aligned} M(z, Bu, kt) &\succeq \min \left\{ \begin{array}{c} M(z, z, t), M(z, z, t), M(Bu, z, t), \\ M(z, Bu, 2t), M(z, z, 2t) \end{array} \right\}, \\ &\succeq \min \left\{ e^{i\theta}, M(Bu, z, t), M(z, z, t) * M(z, Bu, t) \right\}, \\ &= \min \left\{ e^{i\theta}, M(Bu, z, t), e^{i\theta} * M(z, Bu, t) \right\}, \\ &= M(z, Bu, t). \end{aligned}$$

Thus

$$M(z, Bu, kt) \succeq M(z, Bu, t), \forall t > 0.$$

So by Lemma 2.2, we have $Bu = z$. Thus we get $Tu = Bu = z$. As the pair (B, T) is weakly compatible we get

$$Bz = Tz. \tag{3.11}$$

Step 7: Taking $x = x_{2n}$ and $y = z$ in (3.14) we get

$$M(Ax_{2n}, Bz, kt) \succeq \min \left\{ \begin{array}{l} M(Sx_{2n}, Tz, t), M(Sx_{2n}, Ax_{2n}, t), M(Bz, Tz, t), \\ M(Sx_{2n}, Bz, 2t), M(Ax_{2n}, Tz, 2t) \end{array} \right\}.$$

Letting $n \to \infty$ and using (3.4), (3.11) we get,

$$\begin{aligned} M(z, Bz, kt) &\succeq \min\left\{ \begin{array}{c} M(z, Bz, t), M(z, z, t), M(Bz, Bz, t), \\ M(z, Bz, 2t), M(z, Bz, 2t) \end{array} \right\}, \\ &\succeq \min\left\{ M(z, Bz, t), e^{i\theta}, M(z, z, t) * M(z, Bz, t) \right\}, \\ &= \min\left\{ e^{i\theta}, M(Bz, z, t), e^{i\theta} * M(z, Bz, t) \right\}, \\ &= M(z, Bz, t). \end{aligned}$$

Thus

$$M(z, Bz, kt) \succeq M(z, Bz, t), \forall t > 0.$$

So by Lemma 2.2, we have $Bz = z$. Thus we get $Tz = Bz = z = Az$.

As $B(X) \subseteq S(X)$ there exists $z_1 \in X$ such that $z = Bz = Sz_1$. So we have $Az = Tz = z = Bz = Sz_1$.

Step 8: Taking $x = z_1$ and $y = z$ in (3.14) we get

$$M(Az_1, Bz, kt) \succeq \min\left\{ \begin{array}{l} M(Sz_1, Tz, t), M(Sz_1, Az_1, t), M(Bz, Tz, t), \\ M(Sz_1, Bz, 2t), M(Az_1, Tz, 2t) \end{array} \right\}.$$

Using $z = Tz = Bz = Sz_1$ we get

$$M(Az_1, z, kt) \succeq \min\left\{ \begin{array}{l} M(z, z, t), M(z, Az_1, t), M(z, z, t), \\ M(z, z, 2t), M(Az_1, z, 2t) \end{array} \right\}.$$

As before we have

$$M(Az_1, z, kt) \succeq M(Az_1, z, t), \forall t > 0.$$

Again by Lemma 2.2, we have $Az_1 = z$. Thus we get $Sz_1 = Az_1 = z$.

As the pair (A, S) is compatible, it is also weakly compatible. So we get $Az = Sz$.

Combining the result obtained in step 7 to step 8 we get $z = Az = Bz = Sz = Tz$.

Uniqueness: Let w be another common fixed point of maps A, B, S and T i. e.

$Aw = Bw = Sw = Tw = w$.

Step 9: Taking $x = w$ and $y = z$ in (3.14) we get

$$M(Aw, Bz, kt) \succeq \min \left\{ \begin{array}{l} M(Sw, Tz, t), M(Sw, Aw, t), M(Bz, Tz, t), \\ M(Sw, Bz, 2t), M(Aw, Tz, 2t) \end{array} \right\}.$$

Using $z = Tz = Bz$ and $Aw = Sw = z$ we get

$$M(w, z, kt) \succeq \min \left\{ \begin{array}{l} M(w, z, t), M(w, z, t), M(z, z, t), \\ M(w, z, 2t), M(w, z, 2t) \end{array} \right\}.$$

So,

$$M(w, z, kt) \succeq M(w, z, t), \forall t > 0.$$

Again by Lemma 2.2, we have $w = z$. So z is the unique common fixed point of maps A, B, S and T. □

Restricting the contractive condition (3.14) of above theorem to only first factor we have:

Corollary 3.1. *: Let $(X, M, *)$ be a complete complex fuzzy metric space with continuous t-norm $a * b = min\{a, b\}$ such that $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \forall x, y \in X, \forall t > 0$. Let A, B, S and T be self mappings in X satisfying:*

(3.11.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(3.11.2) *The pair (A, S) is compatible and the pair (B, T) is weakly compatible;*

(3.11.3) *one of the map A or S is continuous;*

(3.11.4) *the mappings A, B, S and T satisfy :*

$$M(Ax, By, kt) \succeq M(Sx, Ty, t).$$

for all $x, y \in X$ some fixed $k \in (0, 1)$.

Then A, B, S and T have unique common fixed point in X.

Taking $A = B = f$ and $S = T = I$, the identity map in X, in Corollary 3.1 we get:

Corollary 3.2. *:Let $(X, M, *)$ be a complete complex fuzzy metric space with continuous t-norm $a * b = min\{a, b\}$ such that $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \forall x, y \in X, \forall t > 0$. Let f be a self map in X satisfying:*

$$M(fx, fy, kt) \succeq M(x, y, t).$$

for all $x, y \in X$ some fixed $k \in (0, 1)$.
Then f has the unique fixed point in X, (which is precisely the Theorem 4.1 of [13]*.)*

Remark 4. : Thus Corollary 3.2, is a generalization of Theorem 4.1 of [13].

Corollary 3.3. *:Let $(X, M, *)$ be a complete complex fuzzy metric space with continuous t-norm $a * b = min\{a, b\}$ such that $lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \forall x, y \in X, \forall t > 0$. Let f be a self map in X satisfying:*

$$M(fx, fy, kt) \succeq M(x, y, t).$$

for all $x, y \in X$ some fixed $k \in (0, 1)$.
Then f has the unique fixed point in X, (which is precisely the Theorem 4.1 of [13]*.)*

Remark 5. : Thus Corollary 3.3, is a generalization of Theorem 4.1 of [13].

Again, taking $B = A$ and $T = S$ in Corollary 3.1 we get:

Corollary 3.4. *: Let $(X, M, *)$ be a complete complex fuzzy metric space with continuous t-norm $a * b = \min\{a, b\}$ such that $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \forall x, y \in X, \forall t > 0$. Let A and S be self mappingsin X satisfying:*

(3.41) $A(X) \subseteq S(X)$;

(3.42) *The pair* (A, S) *is compatible;*

(3.43) *either the map* A *or else the map* S *is continuous;*

(3.44) *the mappings* A *and* S *satisfy :*

$$M(Ax, Ay, kt) \succeq M(Sx, Sy, t).$$

for all $x, y \in X$ *some fixed* $k \in (0, 1)$.
Then A *and* S *have unique common fixed point in* X *(which is precisely the Theorem* 6.2 *of* [13]*).*

Remark 6. :Thus Corollary 3.3 is a generalization of Theorem 6.2 of [13], with the remark that condition (3.53) needs continuity of either of the self maps A or S, not that of only S, as required in [13].

Example 3.1. *(of Theorem 3.1): Let* $X = \{0, 1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ *with the metric* d *defined by*

$$d(x, y) = |x - y|, \forall x, y \in X.$$

Define

$$M(x, y, t) = e^{i\theta} \frac{t}{t + d(x, y)}$$

Then $(X, M, *)$ *is a complex valued complete metric space with t-norm* '$*$' *defined as* $a * b = min\{a, b\}$, *for* $a, b \in [0, 1]e^{i\theta}$, *and a fixed* $\theta \in [0, \frac{\pi}{2}]$.
Here $\lim_{n\to\infty} M(x, y, t) = e^{i\theta}, \forall x, y \in X, t \in (0, \infty)$. *Define self maps* A, B, S *and* T *as follows:*

$$A(x) = B(x) = \frac{x}{6},$$

$$S(x) = T(x) = \frac{x}{2}.$$

Then conditions (3.11)(3.12) *and* (3.13) *hold good. Also condition* (3.14) *holds for* $k = \frac{2}{5}$. *Therefore, by Theorem 3.1 , the self maps* A, B, S *and* T *have a unique common fixed point in* X. *Here* 0 *is the unique common fixed point.*

Note: In the above corollaries we have considered only first factor of the right hand side of contractive condition (3.14). By taking other combinations of the five factors of right hand side we have about more than $2^5 - 1$ new interesting results in a complex fuzzy metric space.

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Shri Vaishnav Vidyapeeth Vishwavidyalaya, Indore(M.P.), India

Email address: `(1) shobajain1@yahoo.com`

Email address: `(2)jainshishir11@rediffmail.com`