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TRANSMUTED ARADHANA DISTRIBUTION: PROPERTIES AND APPLICATION

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ABSTRACT. In this paper, we introduce a new continuous distribution called transmuted Aradhana distribution (TAD). It is a generalization of Aradhana distribution based on the quadratic rank transmutation map. Properties of the proposed distribution including shape of the density, reliability and hazard rate functions, mean residual life function, rth moment, moment generating function, order statistics, Renyi entropy, quantile function, are explored. We use maximum likelihood method and method of moments to estimate the parameters of the TAD. We present an application to real life data set to illustrate the usefulness of the proposed distribution. It is shown that the TAD is more adequate for modeling this data than Aradhana distribution and some other available distributions.

1. Introduction

Generalizing classical distributions by adding one or more parameters provides more flexibility for modeling many real life data. [20] innovated the quadratic rank transmutation map to suggest a new family of distributions called transmuted family of distributions. Many authors considered this transmutation map to generalize some existing distributions. For example, [13,14] used the quadratic rank transmutation map to introduce the transmuted Lindley distribution and transmuted Rayleigh distribution. [6] proposed the transmuted Lomax distribution. [4,5] suggested transmuted Weibull and transmuted log-logistic distributions. [15] generalized Pareto distribution to present the transmuted Pareto distribution. Some other transmuted distributions that have been proposed in the literature include the transmuted Weibull

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Fr$\acute{e}$chet distribution [1], transmuted exponentiated Fr$\acute{e}$chet distribution [9], transmuted new modified Weibull distribution [21], transmuted Mukherjee-Islam distribution [3], transmuted Janardan distribution [2], transmuted Ishita distribution [10], and transmuted Generalized Gamma distribution [12].

In this paper, we adapt the quadratic rank transmutation map to propose a new distribution called transmuted Aradhana distribution (TAD) as a generalization of the Aradhana distribution [18]. The TAD extends Aradhana distribution and provides great flexibility in modeling data in practice as will be illustrated in Section 8.

The outline of this paper is as follows. In Section 2, we define the probability density function (*pdf*) and cumulative distribution function (*cdf*) of the TAD. In Section 3, we study some mathematical properties of the TAD including the moment generating function, r^{th} moment, mean, variance, skewness, kurtosis, coefficient of variation, quantile function. The reliability, hazard rate, cumulative hazard, reversed hazard, odds, and mean residual life functions of the TAD are investigated in Section 4. The distributions of order statistics are presented in Section 5. Maximum likelihood and method of moments estimators for the unknown parameter of the TAD are given in Section 6. In Section 7, we derive the Renyi entropy. In Section 8, we present an application to real lifetime data. We conclude the paper in Section 9.

2. Transmuted Aradhana Distribution (TAD)

The probability density function (*pdf*) and cumulative distribution function (*cdf*) of Aradhana distribution, introduced by [18], are given ,respectively, by

$$f(x) = \frac{\theta^3}{\theta^2 + 2\theta + 2}(1+x)^2 e^{-\theta x}; x > 0, \theta > 0, \tag{2.1}$$

and

$$F(x) = 1 - \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x}; x > 0, \theta > 0. \tag{2.2}$$

Definition 2.1. A random variable X is said to have a transmuted distribution, see [7], if its *cdf* and *pdf* are given ,respectively, by

$$G(x) = (1+\lambda)F(x) - \lambda[F(x)]^2, \quad \mid \lambda \mid \leq 1 \tag{2.3}$$

and

$$g(x) = f(x)\left[1 + \lambda - 2\lambda F(x)\right] \tag{2.4}$$

where $F(x)$ and $f(x)$ are the *cdf* and the *pdf* of the base distribution.

Definition 2.2. Based on Definition 2.1 and using (2.1) and (2.2) , a random variable X is said to have a transmuted Aradhana distribution (TAD) if its *cdf* is defined as

$$\begin{aligned} G(x) &= (1+\lambda)\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)-\lambda\left[1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right]^2 \\ &= 1+\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\left[(\lambda-1)-\lambda\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right] \end{aligned} \tag{2.5}$$

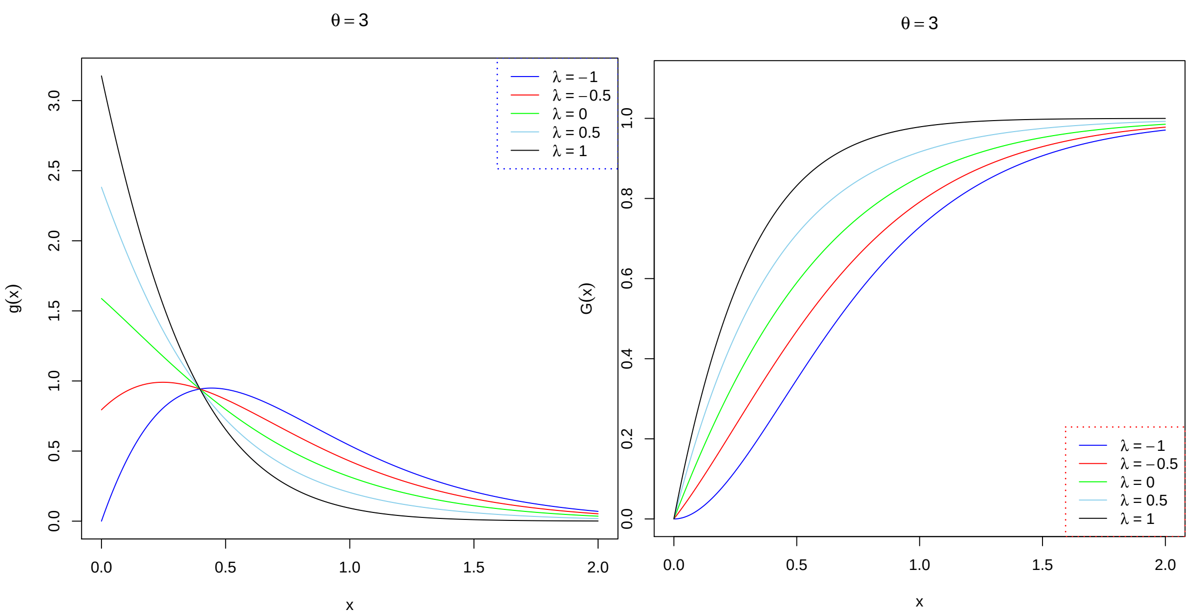
with a corresponding *pdf* given by

(2.6)

$$g(x)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-\theta x}\left((1-\lambda)+2\lambda e^{-\theta x}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)\right);x>0,\theta>0.$$

Figure 1 shows the *pdf* and *cdf* of the TAD with different values of the distribution parameters.

FIGURE 1. The *pdf* (A) and *cdf* (B) of the TAD distribution with $\theta = 3$ and variant values of λ



(A) *pdf* of TAD (B) *cdf* of TAD

3. Mathematical properties of the TAD

In this section, we derive the moment generating function, the r^{th} moment and quantile function for the TAD random variable. Also, mean, variance, coefficient of variation, kurtosis, and skewness are derived.

3.1. **Moment Generating Function.**

Theorem 3.1. *The moment generating function of the TAD random variable is given by*

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{4(2\theta)^k(\theta^2+2\theta+2)^2}\Bigg[4(\theta^2+2\theta+2)\Big[2^k(1-\lambda)\left(\theta^2+(k+1)(2\theta+(k+2))\right)+\lambda\theta^2\Big]$$

$$+\lambda(k+1)\left(2\theta(4\theta^2+6\theta+4)+(k+2)\left((6\theta^2+6\theta+2)+(k+3)\left(\frac{(k+4)}{4}+(2\theta+1)\right)\right)\right)\Bigg];|t|<\theta. \tag{3.1}$$

Proof. The moment generating function of TAD ,with *pdf* $g(x)$ in (2.6), can be proved as

$$\begin{aligned}
M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} g(x)dx \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\int_0^\infty (1+x)^2\left((1-\lambda)e^{-(\theta-t)x}+2\lambda e^{-(2\theta-t)x}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)\right)dx \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\Bigg[(1-\lambda)\int_0^\infty (1+2x+x^2)e^{-(\theta-t)x}dx+2\lambda\int_0^\infty(1+2x+x^2)e^{-(2\theta-t)x}dx \\
&\qquad +\frac{2\lambda\theta}{\theta^2+2\theta+2}\int_0^\infty x(1+2x+x^2)(\theta x+2\theta+2)e^{-(2\theta-t)x}dx\Bigg] \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\Bigg[(1-\lambda)\left(\frac{1}{\theta-t}+\frac{2}{(\theta-t)^2}+\frac{2}{(\theta-t)^3}\right)+2\lambda\left(\frac{1}{2\theta-t}+\frac{2}{(2\theta-t)^2}+\frac{2}{(2\theta-t)^3}\right) \\
&\qquad +\frac{2\lambda\theta}{\theta^2+2\theta+2}\left(\frac{24\theta}{(2\theta-t)^5}+\frac{6(4\theta+2)}{(2\theta-t)^4}+\frac{2(5\theta+4)}{(2\theta-t)^3}+\frac{2\theta+2}{(2\theta-t)^2}\right)\Bigg] \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\Bigg[(1-\lambda)\left(\frac{1}{\theta(1-\frac{t}{\theta})}+\frac{2}{\theta^2(1-\frac{t}{\theta})^2}+\frac{2}{\theta^3(1-\frac{t}{\theta})^3}\right)+\frac{\lambda}{\theta(1-\frac{t}{2\theta})} \\
&\qquad +\frac{2\lambda}{\theta^2+2\theta+2}\left(\frac{4\theta^2+6\theta+4}{4\theta^2(1-\frac{t}{2\theta})^2}+\frac{12\theta^2+12\theta+4}{8\theta^3(1-\frac{t}{2\theta})^3}+\frac{3}{4\theta^3(1-\frac{t}{2\theta})^5}+\frac{3(4\theta+2)}{8\theta^3(1-\frac{t}{2\theta})^4}\right)\Bigg]
\end{aligned}$$

Using binomial series ($\frac{1}{(1-z)^\alpha} = \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k} z^k$; $|z|<1$), we have

$$
\begin{aligned}
M_X(t) &= \frac{\theta^3}{\theta^2+2\theta+2}\Bigg[(1-\lambda)\left(\frac{1}{\theta}\sum_{k=0}^{\infty}(\frac{t}{\theta})^k + \frac{2}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{t}{\theta})^k + \frac{2}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}(\frac{t}{\theta})^k\right) + \frac{\lambda}{\theta}\sum_{k=0}^{\infty}(\frac{t}{2\theta})^k \\
&\quad + \frac{2\lambda}{\theta^2+2\theta+2}\left(\frac{4\theta^2+6\theta+4}{4\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{t}{2\theta})^k + \frac{12\theta^2+12\theta+4}{8\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}(\frac{t}{2\theta})^k\right. \\
&\quad \left.+\frac{3}{4\theta^3}\sum_{k=0}^{\infty}\binom{k+4}{k}(\frac{t}{2\theta})^k + \frac{3(4\theta+2)}{8\theta^3}\sum_{k=0}^{\infty}\binom{k+3}{k}(\frac{t}{2\theta})^k\right)\Bigg]; |t|<\theta \\
&= \sum_{k=0}^{\infty}\frac{t^k}{4(2\theta)^k(\theta^2+2\theta+2)^2}\Bigg[4(\theta^2+2\theta+2)\left[2^k(1-\lambda)\left(\theta^2+(k+1)(2\theta+(k+2))\right)+\lambda\theta^2\right] \\
&\quad +\lambda(k+1)\left(2\theta(4\theta^2+6\theta+4)+(k+2)\left((6\theta^2+6\theta+2)+(k+3)\left(\frac{(k+4)}{4}+(2\theta+1)\right)\right)\right)\Bigg]; |t|<\theta.
\end{aligned}
$$

□

3.2. **Moments and associated measures.** The r^{th} moment of the TAD random variable can be obtained as the coefficient of $\frac{t^r}{r!}$ in the moment generating function $M_X(t)$ in (3.1). That is

$$
\begin{aligned}
E(X^r) &= \frac{r!}{4(2\theta)^r(\theta^2+2\theta+2)^2}\Bigg[4(\theta^2+2\theta+2)\left[2^r(1-\lambda)\left(\theta^2+(r+1)(2\theta+(r+2))\right)+\lambda\theta^2\right] \\
&\quad +\lambda(r+1)\left(2\theta(4\theta^2+6\theta+4)+(r+2)\left((6\theta^2+6\theta+2)+(r+3)\left(\frac{(r+4)}{4}+(2\theta+1)\right)\right)\right)\Bigg]. \qquad (3.2)
\end{aligned}
$$

Hence, the first four moments of TAD are given by

$$E(X) = \frac{1}{4\theta(\theta^2+2\theta+2)^2}\left[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\right], \qquad (3.3)$$

$$E(X^2) = \frac{1}{4\theta^2(\theta^2+2\theta+2)^2}\left[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda\left(6\theta^4+48\theta^3+150\theta^2+180\theta+105\right)\right], \qquad (3.4)$$

$$E(X^3) = \frac{3}{8\theta^3(\theta^2+2\theta+2)^2}\left[16\left(\theta^4+10\theta^3+38\theta^2+56\theta+40\right)-\lambda\left(14\theta^4+140\theta^3+520\theta^2+700\theta+455\right)\right], \qquad (3.5)$$

$$E(X^4) = \frac{3}{4\theta^4(\theta^2+2\theta+2)^2}\left[32\left(\theta^4+12\theta^3+52\theta^2+80\theta+60\right)-5\lambda\left(6\theta^4+72\theta^3+308\theta^2+448\theta+315\right)\right]. \qquad (3.6)$$

Therefore, variance of the TAD random variable is given by

$$\begin{aligned}\sigma^2 &= var(X) = E(X^2) - (E(X))^2\\ &= \frac{1}{16\theta^2(\theta^2+2\theta+2)^4}\Bigg(4(\theta^2+2\theta+2)^2\Big[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda(6\theta^4+48\theta^3+150\theta^2+180\theta+105)\Big]\\ &\qquad -\Big[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\Big]^2\Bigg). \end{aligned} \tag{3.7}$$

Using (3.3) and (3.7), the coefficient of variation $(C.V)$ of the TAD random variable can be defined as

$$C.V=\frac{\sigma}{E(X)}=\frac{\Bigg(4(\theta^2+2\theta+2)^2\Big[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda(6\theta^4+48\theta^3+150\theta^2+180\theta+105)\Big]-\Big[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\Big]^2\Bigg)^{1/2}}{4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)}.$$

The skewness and the kurtosis of a random variable are defined as:

$$\begin{aligned} sk(X) &= \frac{E(X^3)-3E(X)E(X^2)+2(E(X))^3}{\sigma^3}\\ kur(X) &= \frac{E(X^4)-4E(X)E(X^3)+6(E(X))^2E(X^2)-3(E(X))^4}{\sigma^4}. \end{aligned}$$

Based on these formulas, the skewness and the kurtosis of the TAD random variable are given, respectively, by:

$$sk(X)=\frac{\left[\begin{aligned}&24(\theta^2+2\theta+2)^4\Big[16\left(\theta^4+10\theta^3+38\theta^2+56\theta+40\right)-\lambda(14\theta^4+140\theta^3+520\theta^2+700\theta+455)\Big]\\ &-12(\theta^2+2\theta+2)^2\Big[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\Big]\\ &\times\Big[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda(6\theta^4+48\theta^3+150\theta^2+180\theta+105)\Big]\\ &+2\Big[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\Big]^3\end{aligned}\right]}{\left[\begin{aligned}&4(\theta^2+2\theta+2)^2\Big[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda(6\theta^4+48\theta^3+150\theta^2+180\theta+105)\Big]\\ &-\Big[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\Big]^2\end{aligned}\right]^{3/2}},$$

$$
kur(X)=\frac{\left[\begin{array}{l}
192(\theta^2+2\theta+2)^6\left[32\left(\theta^4+12\theta^3+52\theta^2+80\theta+60\right)-5\lambda\left(6\theta^4+72\theta^3+308\theta^2+448\theta+315\right)\right]\\
-\,96(\theta^2+2\theta+2)^4\left[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\right]\\
\times\left[16\left(\theta^4+10\theta^3+38\theta^2+56\theta+40\right)-\lambda\left(14\theta^4+140\theta^3+520\theta^2+700\theta+455\right)\right]\\
+\,24(\theta^2+2\theta+2)^2\left[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\right]^2\\
\times\left[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda\left(6\theta^4+48\theta^3+150\theta^2+180\theta+105\right)\right]\\
-\,3\left[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\right]^4
\end{array}\right]}{\left[\begin{array}{l}
4(\theta^2+2\theta+2)^2\left[8\theta^4+64\theta^3+208\theta^2+288\theta+192-\lambda\left(6\theta^4+48\theta^3+150\theta^2+180\theta+105\right)\right]\\
\qquad-\left[4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)\right]^2
\end{array}\right]^2}.
$$

In Table 1, we present the values of the mean, variance, coefficient of variation, skewness, and kurtosis of the TAD random variable with different values of λ and θ.

TABLE 1. The mean, variance, skewness, kurtosis and the coefficient of variation of the TAD for variant values of λ and θ

	$\theta=2$					$\theta=4$					$\theta=5$					$\theta=8$				
λ	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**
-1	1.304	0.666	0.626	1.204	5.209	0.541	0.139	0.689	1.376	5.866	0.41	0.083	0.704	1.426	6.093	0.232	0.028	0.725	1.51	6.506
-0.9	1.263	0.673	0.649	1.196	5.174	0.523	0.139	0.714	1.376	5.859	0.396	0.083	0.729	1.428	6.093	0.225	0.028	0.75	1.514	6.516
-0.8	1.223	0.677	0.673	1.199	5.165	0.506	0.139	0.738	1.386	5.881	0.383	0.083	0.753	1.439	6.122	0.217	0.028	0.775	1.526	6.556
-0.7	1.183	0.677	0.696	1.211	5.182	0.488	0.139	0.763	1.404	5.931	0.369	0.083	0.778	1.459	6.181	0.209	0.028	0.8	1.547	6.627
-0.6	1.142	0.675	0.719	1.232	5.225	0.471	0.137	0.787	1.43	6.011	0.356	0.082	0.803	1.486	6.271	0.202	0.028	0.825	1.575	6.732
-0.5	1.102	0.669	0.742	1.26	5.296	0.453	0.135	0.812	1.464	6.123	0.343	0.08	0.828	1.52	6.394	0.194	0.027	0.851	1.611	6.871
-0.4	1.062	0.659	0.765	1.295	5.397	0.435	0.133	0.837	1.504	6.269	0.329	0.079	0.853	1.561	6.552	0.186	0.027	0.876	1.653	7.048
-0.3	1.021	0.647	0.788	1.337	5.528	0.418	0.13	0.861	1.55	6.451	0.316	0.077	0.878	1.609	6.748	0.179	0.026	0.902	1.702	7.267
-0.2	0.981	0.631	0.81	1.384	5.692	0.4	0.126	0.886	1.603	6.674	0.302	0.075	0.903	1.663	6.987	0.171	0.025	0.927	1.758	7.531
-0.1	0.94	0.612	0.832	1.437	5.894	0.383	0.121	0.91	1.663	6.942	0.289	0.072	0.928	1.724	7.273	0.163	0.024	0.952	1.82	7.847
0	0.9	0.59	0.853	1.496	6.135	0.365	0.116	0.934	1.728	7.26	0.276	0.069	0.952	1.791	7.613	0.155	0.023	0.977	1.89	8.223
0.1	0.86	0.564	0.874	1.56	6.422	0.348	0.111	0.957	1.8	7.636	0.262	0.065	0.976	1.865	8.015	0.148	0.022	1.001	1.966	8.665
0.2	0.819	0.536	0.893	1.629	6.758	0.33	0.105	0.979	1.879	8.078	0.249	0.062	0.998	1.945	8.487	0.14	0.021	1.024	2.05	9.187
0.3	0.779	0.504	0.911	1.702	7.148	0.313	0.098	1	1.963	8.596	0.236	0.058	1.019	2.032	9.041	0.132	0.019	1.046	2.141	9.799
0.4	0.738	0.468	0.927	1.777	7.597	0.295	0.09	1.018	2.052	9.197	0.222	0.053	1.038	2.125	9.686	0.125	0.018	1.065	2.238	10.516
0.5	0.698	0.43	0.939	1.854	8.102	0.278	0.082	1.033	2.144	9.886	0.209	0.048	1.053	2.22	10.429	0.117	0.016	1.081	2.339	11.346
0.6	0.658	0.388	0.947	1.925	8.646	0.26	0.074	1.043	2.234	10.654	0.195	0.043	1.063	2.315	11.261	0.109	0.014	1.091	2.44	12.284
0.7	0.617	0.343	0.949	1.979	9.173	0.243	0.064	1.045	2.31	11.443	0.182	0.038	1.065	2.395	12.127	0.102	0.012	1.093	2.527	13.273
0.8	0.577	0.295	0.941	1.991	9.511	0.225	0.054	1.036	2.342	12.047	0.169	0.032	1.056	2.432	12.809	0.094	0.01	1.083	2.57	14.081
0.9	0.537	0.243	0.919	1.897	9.127	0.208	0.044	1.009	2.251	11.737	0.155	0.025	1.027	2.341	12.517	0.086	0.008	1.052	2.476	13.811
1	0.496	0.188	0.874	1.51	6.106	0.19	0.033	0.952	1.778	7.506	0.142	0.019	0.967	1.839	7.88	0.079	0.006	0.986	1.925	8.448

Table 1 shows that as the value of θ increases, the values of the mean and variance decrease while the values of the coefficient of variation, skewness, and kurtosis increase. Also, it can be seen that the mean is decreasing as the value of λ increases. The positive values of skewness in Table 1 indicate that the TAD is right skewed as shown in Figure 1.

TABLE 2. The mean, variance, skewness, kurtosis and the coefficient of variation of Exponential, Ishita, Akash and Aradhana distributions with variant values of θ.

	$\theta = 2$					$\theta = 4$					$\theta = 5$					$\theta = 8$				
Distribution	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**	**E(X)**	**Var(X)**	**C.V**	**sk**	**kur**
Exponential	0.5	0.25	1	2	9	0.25	0.0625	1	2	9	0.2	0.04	1	2	9	0.125	0.0156	1	2	9
Ishita	1.263	0.673	0.649	1.196	5.174	0.265	0.074	1.023	2.107	9.750	0.206	0.044	1.014	2.072	9.537	0.126	0.016	1.004	2.022	9.172
Akash	0.833	0.639	0.959	1.614	6.391	0.306	0.101	1.041	2.064	9.121	0.230	0.057	1.039	2.114	9.601	0.133	0.018	1.023	2.107	9.750
Aradhana	0.9	0.59	0.854	1.496	6.135	0.365	0.117	0.934	1.728	7.260	0.276	0.069	0.952	1.791	7.613	0.156	0.023	0.977	1.890	8.223

For comparison, we provide in Table 2 the values of the mean, variance, coefficient of variation, skewness, and kurtosis of the exponential, Ishita, Akash and Aradhana distributions that will be used later in Section 8. It can be seen from Tables 1 and 2 that TAD has smaller values of skewness and kurtosis than those of Aradhana, exponential and Akash distributions when $\lambda < 0$. On the other hand, TAD has larger values of these two measures than those of Aradhana distribution when $\lambda > 0$, but it depends on values of λ and θ for other distributions.

3.3. **Quantile Function.**

Definition 3.1. The q^{th} quantile value, x_q, is a value of the random variable, X, with *cdf* $G(x)$ such that

$$G(x_q) = p(X \leq x_q) = q \quad ; 0 < q < 1. \tag{3.8}$$

Lemma 3.2. *The q^{th} quantile, x_q, of the TAD is the solution of*

$$\left(1 + \frac{\theta x_q(\theta x_q + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x_q} - \frac{\lambda - 1 + \sqrt{\lambda^2 + 2\lambda(1 + 2q) + 1}}{2\lambda} = 0, \qquad x_q > 0.$$

Proof. Using the *cdf* of TAD in (2.5) and plug it in (3.8), we have

$$1 + \left(1 + \frac{\theta x_q(\theta x_q + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x_q} \left[(\lambda - 1) - \lambda \left(1 + \frac{\theta x_q(\theta x_q + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x_q}\right] = q.$$

Let $z = \left(1 + \frac{\theta x_q(\theta x_q + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x_q}$, then we have

$$\lambda z^2 + (1 - \lambda)z - (1 - q) = 0.$$

We can use the general formula to solve the above quadratic equation, for $0 \leq z \leq 1$, to get

$$z = \frac{\lambda - 1 + \sqrt{(1 - \lambda)^2 + 4\lambda(1 - q)}}{2\lambda} = \frac{\lambda - 1 + \sqrt{\lambda^2 + 2\lambda(1 + 2q) + 1}}{2\lambda},$$

replacing z by its value $\left(1 + \frac{\theta x_q(\theta x_q + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x_q}$, we have

$$\left(1 + \frac{\theta x_q(\theta x_q + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x_q} = \frac{\lambda - 1 + \sqrt{\lambda^2 + 2\lambda(1 + 2q) + 1}}{2\lambda}$$

Therefore, the q^{th} quantile is the positive solution of

$$\left(1+\frac{\theta x_q(\theta x_q+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x_q}-\frac{\lambda-1+\sqrt{\lambda^2+2\lambda(1+2q)+1}}{2\lambda}=0$$

which can be found by numerical methods. □

The median of TAD can be obtained by solving the above equation with $q = 0.5$.

4. Reliability analysis

The reliability or survival function, $R(t)$, is the probability that an item will survive beyond a time t. The reliability function of the TAD, with *cdf* $G(x)$ in (2.5), is given by

$$R(t)=1-G(t)=\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\left[(1-\lambda)+\lambda\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\right].$$

The hazard or failure rate function, $h(t)$, of the TAD, with *pdf* $g(x)$ in (2.6), is defined as

$$h(t) = \frac{g(t)}{1-G(t)} = \frac{\frac{\theta^3}{\theta^2+2\theta+2}(1+t)^2\left((1-\lambda)+2\lambda e^{-\theta t}\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)\right)}{\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)\left[(1-\lambda)+\lambda\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\right]}.$$

The cumulative hazard function, $H(t)$, reversed hazard rate function, $h_{rev}(t)$, and odds function, $O(t)$, of the TAD are , respectively, defined as

$$\begin{aligned} H(t) &= -ln\,(1-G(t)) \\ &= \theta t - ln\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right) - ln\left((1-\lambda)+\lambda\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\right), \end{aligned}$$

$$h_{rev}(t) = \frac{g(t)}{G(t)} = \frac{\frac{\theta^3}{\theta^2+2\theta+2}(1+t)^2\left((1-\lambda)+2\lambda e^{-\theta t}\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)\right)}{e^{\theta t}+\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)\left[(\lambda-1)-\lambda\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\right]},$$

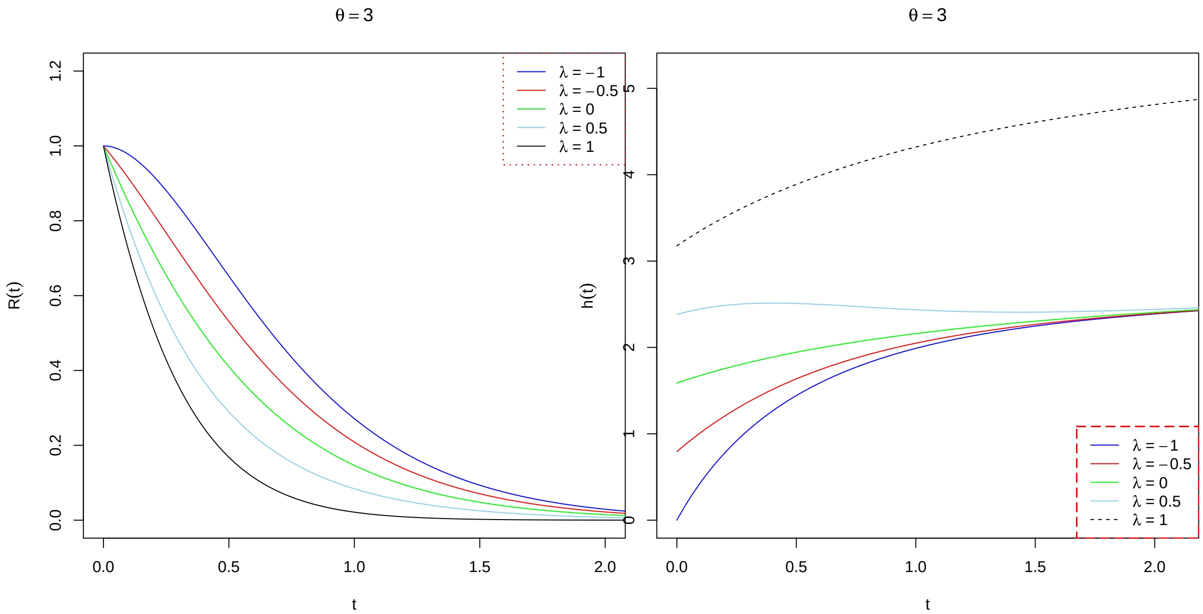
and

$$O(t) = \frac{G(t)}{1-G(t)} = \frac{e^{\theta t}}{\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)\left[(1-\lambda)+\lambda\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\right]} - 1.$$

Graphs of reliability, hazard, cumulative hazard, reversed hazard and odds functions of the TAD for some values of the distribution parameters are shown in Figures 2 and 3. It can be seen that as λ increases, values of hazard, cumulative hazard, and odds functions increases while the reliability and reversed hazard function values decreases.

The mean residual life function, $MRL(t)$, is defined as the expected value of the

FIGURE 2. The reliability (A) and hazard (B) of the TAD with $\theta = 3$ and variant values of λ



(A) Reliability (B) Hazard

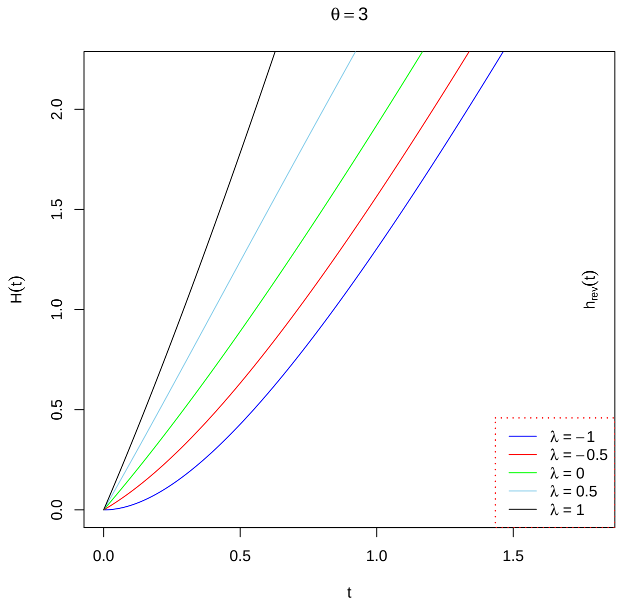
remaining lifetimes after a specified time t. The mean residual life function of the TAD, with *cdf* $G(x)$ in (2.5), is defined as

$$MRL(t) = \frac{1}{1-G(t)} \int_t^{\infty} (1 - G(x))\, dx$$

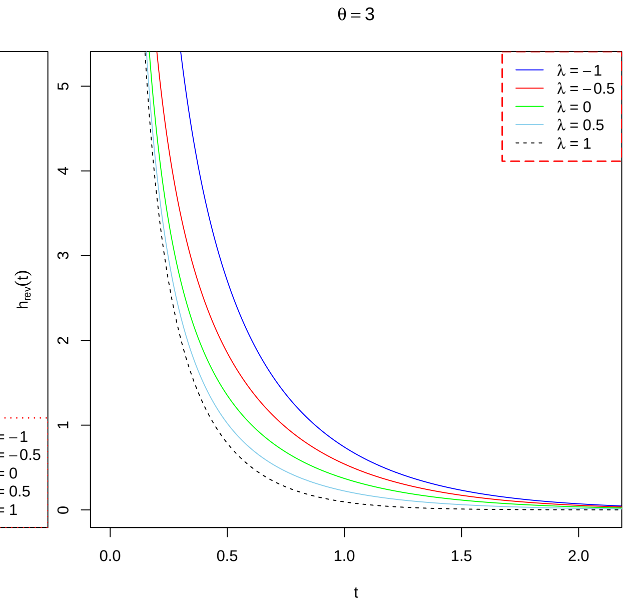
$$= \frac{\left[\begin{array}{l} \mathrm{e}^{-\theta t}\Big(4\left(\theta^2+2\theta+2\right)(1-\lambda)\left(\theta\left(t+1\right)\left(\theta\left(t+1\right)+4\right)+6\right)\mathrm{e}^{\theta t} \\ +\,2\theta^4\lambda t^4+4\theta^3\left(2\theta+3\right)\lambda t^3+2\theta^2\left(6\theta^2+18\theta+17\right)\lambda t^2 \\ +\,2\theta\left(4\theta^3+18\theta^2+34\theta+25\right)\lambda t+\left(2\theta^4+12\theta^3+34\theta^2+50\theta+33\right)\lambda\Big) \end{array}\right]}{\left[4\theta\left(\theta^2+2\theta+2\right)^2\right]\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)\left[(1-\lambda)+\lambda\left(1+\frac{\theta t(\theta t+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta t}\right]}.$$

Note that $MRL(0) = \frac{4\theta^4+24\theta^3+64\theta^2+80\theta+48-\lambda\left(2\theta^4+12\theta^3+30\theta^2+30\theta+15\right)}{4\theta(\theta^2+2\theta+2)^2} = E(X)$. Figure 4 shows the mean residual life function of the TAD for different values of λ and θ. It can be seen that the $MRL(t)$ is a decreasing function of t, θ, and λ.

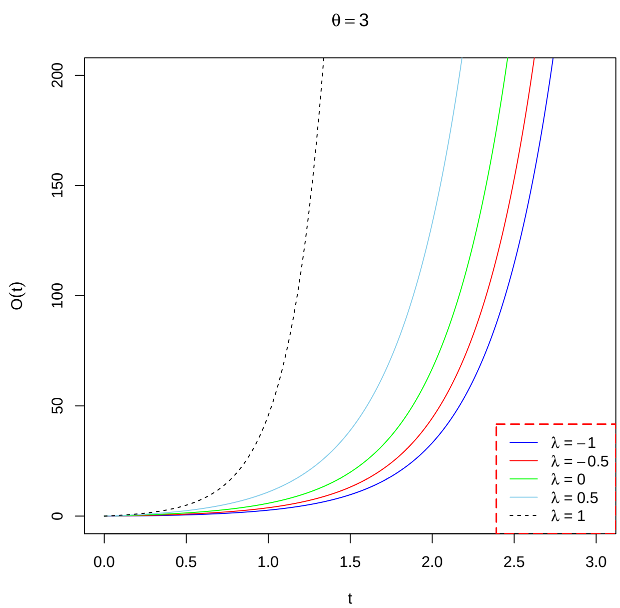
FIGURE 3. The cumulative hazard (A), reversed hazard (B), and odds (C) functions of the TAD with $\theta = 3$ and variant values of λ



(A) Cumulative hazard

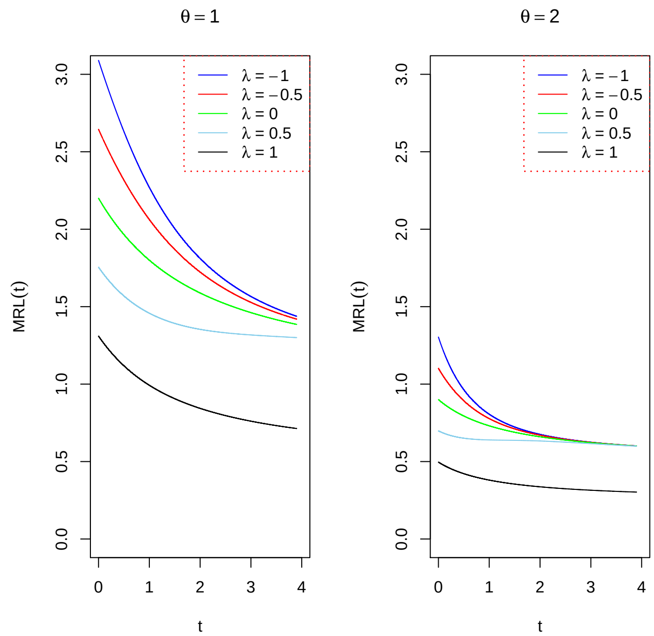


(B) Reversed hazard



(C) Odds function

FIGURE 4. The mean residual life function of the TAD with $\theta = 1, 2$, and variant values of λ



5. ORDER STATISTICS

Let X_1, X_2, … X_n be a random sample from TAD with *pdf* $g(x)$ in (2.6) and *cdf* $G(x)$ in (2.5). If $X_{(1)}$, $X_{(2)}$, ... $X_{(n)}$ is the order statistic of this sample, where $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Then the *pdf* of the j^{th} order statistics, $X_{(j)}$ (see [8]) is given by

$$\begin{aligned} g_{(j)}(x) &= j\binom{n}{j} g(x)[G(x)]^{j-1}[1-G(x)]^{n-j} \\ g_{(j)}(x) &= j\binom{n}{j}\left[\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2 e^{-\theta x}\left((1-\lambda)+2\lambda e^{-\theta x}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)\right)\right] \\ &\quad \times\left[1+\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\left[(\lambda-1)-\lambda\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right]\right]^{j-1} \\ &\quad \times\left[\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\left[(1-\lambda)+\lambda\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right]\right]^{n-j}. \end{aligned}$$

6. Estimation of Parameters

6.1. **Maximum Likelihood Estimates.** Let $X_1, X_2, \ldots, X_n$ be a random sample from the TAD with a *pdf* $g(x)$ in (2.6) and parameters θ and λ . The likelihood function is defined as

$$\begin{aligned} L(\theta,\lambda|x_1,x_2,\ldots,x_n) &= \prod_{i=1}^{n} g(x_i|\theta,\lambda) \\ &= \left(\frac{\theta^3}{\theta^2+2\theta+2}\right)^n \left[\prod_{i=1}^{n}(1+x_i)\right]^2 e^{-n\theta\bar{x}} \prod_{i=1}^{n}\left((1-\lambda)+2\lambda e^{-\theta x_i}\left(1+\frac{\theta x_i(\theta x_i+2\theta+2)}{\theta^2+2\theta+2}\right)\right), \end{aligned}$$

where $\bar{x}$ is the sample mean. Hence, the log-likelihood function is obtained as

$$\begin{aligned} L^* &= lnL(\theta,\lambda|x_1,x_2,\ldots,x_n) \\ &= nln\left(\frac{\theta^3}{\theta^2+2\theta+2}\right)+2\left[\sum_{i=1}^{n} ln(1+x_i)\right] - n\theta\bar{x} + \sum_{i=1}^{n} ln\left((1-\lambda)+2\lambda e^{-\theta x_i}\left(1+\frac{\theta x_i(\theta x_i+2\theta+2)}{\theta^2+2\theta+2}\right)\right). \end{aligned}$$

Thus, the derivatives of the log-likelihood function with respect to the parameters θ and λ are:

$$\frac{\partial L^*}{\partial\theta} = \frac{n(\theta^2+4\theta+6)}{\theta(\theta^2+2\theta+2)} - n\bar{x} + \sum_{i=1}^{n} \frac{2\lambda x_i e^{-\theta x_i}\left[\frac{2(\theta^2 x_i+\theta^2+2\theta x_i+4\theta+2)}{\theta^2+2\theta+2} - \left(1+\frac{\theta x_i(\theta x_i+2\theta+2)}{\theta^2+2\theta+2}\right)\right]}{(1-\lambda)+2\lambda e^{-\theta x_i}\left(1+\frac{\theta x_i(\theta x_i+2\theta+2)}{\theta^2+2\theta+2}\right)} \tag{6.1}$$

$$\frac{\partial L^*}{\partial\lambda} = \sum_{i=1}^{n} \frac{-1+2e^{-\theta x_i}\left(1+\frac{\theta x_i(\theta x_i+2\theta+2)}{\theta^2+2\theta+2}\right)}{(1-\lambda)+2\lambda e^{-\theta x_i}\left(1+\frac{\theta x_i(\theta x_i+2\theta+2)}{\theta^2+2\theta+2}\right)} \tag{6.2}$$

The maximum likelihood estimators (MLEs) of θ and λ can be obtained by equating the derivatives in (6.1) and (6.2) to zero and solve the resulting nonlinear system of equations by using numerical methods.

6.2. **Method of Moments Estimates.** In order to estimate the parameters (θ, λ) of TAD, we use the first two moments of TAD in (3.3) and (3.4) to get

$$\lambda = \frac{4\theta^4+24\theta^3+64\theta^2+80\theta+48-4\theta(\theta^2+2\theta+2)^2 E(X)}{2\theta^4+12\theta^3+30\theta^2+30\theta+15}, \tag{6.3}$$

and

$$\lambda = \frac{8\theta^4+64\theta^3+208\theta^2+288\theta+192-4\theta^2(\theta^2+2\theta+2)^2 E(X^2)}{6\theta^4+48\theta^3+150\theta^2+180\theta+105}. \tag{6.4}$$

By replacing the first and second population moments in the above equations by their corresponding sample moments and subtracting (6.4) from (6.3), we have

$$\begin{aligned} &(6\theta^4+48\theta^3+150\theta^2+180\theta+105)\left(4\theta^4+24\theta^3+64\theta^2+80\theta+48-4\theta(\theta^2+2\theta+2)^2\bar{x}\right) \\ &-(2\theta^4+12\theta^3+30\theta^2+30\theta+15)\left(8\theta^4+64\theta^3+208\theta^2+288\theta+192-4\theta^2(\theta^2+2\theta+2)^2\sum_{i=1}^{n}\frac{x_i^2}{n}\right)=0. \end{aligned}$$

The method of moment estimate (MOME) $\tilde{\theta}$ of the parameter θ can be obtained by solving the above polynomial equation for $\theta > 0$ using numerical methods. By substituting this estimate of θ and replacing $E(X)$ by $\bar{x}$ in equation (6.3), the MOME $\tilde{\lambda}$ of the parameter λ can be obtained as

$$\tilde{\lambda} = \frac{4\tilde{\theta}^4 + 24\tilde{\theta}^3 + 64\tilde{\theta}^2 + 80\tilde{\theta} + 48 - 4\tilde{\theta}(\tilde{\theta}^2 + 2\tilde{\theta} + 2)^2\bar{x}}{2\tilde{\theta}^4 + 12\tilde{\theta}^3 + 30\tilde{\theta}^2 + 30\tilde{\theta} + 15}. \tag{6.5}$$

7. Renyi Entropy

Entropy is the degree of disorder or randomness in a system, it has many applications in various fields of science, including information theory, psychodynamics, engineering, finance, statistical mechanics, thermodynamics, biomedical and economics. Entropy of a random variable X with *pdf* $g(x)$ is a measure of variation of the uncertainty. A popular entropy measure is the Renyi entropy [16] which is defined as

$$E_R(\beta) = \frac{1}{1-\beta} log \int_0^\infty (g(x))^\beta dx \, ; \beta > 0, \beta \neq 1. \tag{7.1}$$

Theorem 7.1. *The Renyi entropy for the TAD is defined as*

$$E_R(\beta) = \frac{1}{1-\beta} log \Bigg[\sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \binom{2\beta}{m} \binom{\beta}{i} \binom{\beta - i}{l} \binom{i}{j} \binom{j}{k} (-1)^l (\lambda)^{i+l} (\theta+1)^{j-k} \, 2^{i+j-k} \\ \frac{\theta^{3\beta - m - 1}}{(\theta^2 + 2\theta + 2)^{\beta + j}} \, \frac{\Gamma(m+j+k+1)}{(i+\beta)^{m+j+k+1}} \Bigg],$$

where $\Gamma(\eta) = (\eta - 1)!$.

Proof. Using the *pdf* of the TAD in (2.6) and plug it in (7.1), we have

$$\begin{aligned} E_R(\beta) &= \frac{1}{1-\beta} log \int_0^\infty \left[\frac{\theta^3}{\theta^2 + 2\theta + 2} (1+x)^2 e^{-\theta x} \left((1-\lambda) + 2\lambda e^{-\theta x} \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) \right) \right]^\beta dx \\ &= \frac{1}{1-\beta} log \int_0^\infty \left(\frac{\theta^3}{\theta^2 + 2\theta + 2} \right)^\beta (1+x)^{2\beta} e^{-\beta\theta x} \left((1-\lambda) + 2\lambda e^{-\theta x} \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) \right)^\beta dx \end{aligned}$$

By using the Binomial theorem $((x+y)^\beta = \sum_{i=0}^\infty \binom{\beta}{i} x^{\beta - i} y^i \; ; \, |y| < |x|)$, we have

$$(1+x)^{2\beta} = \sum_{m=0}^{\infty} \binom{2\beta}{m} x^m ; |x| < 1,$$

and for $|2\lambda e^{-\theta x}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)| < |1-\lambda|$, we have

$$
\begin{aligned}
&\left((1-\lambda)+2\lambda e^{-\theta x}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)\right)^{\beta} \\
= &\sum_{i=0}^{\infty}\binom{\beta}{i}(1-\lambda)^{\beta-i}(2\lambda)^{i}e^{-i\theta x}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)^{i} \\
= &\sum_{i=0}^{\infty}\binom{\beta}{i}\sum_{l=0}^{\infty}\binom{\beta-i}{l}(-1)^{l}\lambda^{l}(2\lambda)^{i}e^{-i\theta x}\sum_{j=0}^{i}\binom{i}{j}\frac{\theta^{j}x^{j}(\theta x+2\theta+2)^{j}}{(\theta^2+2\theta+2)^{j}} \\
= &\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\binom{\beta}{i}\binom{\beta-i}{l}(-1)^{l}\lambda^{l}(2\lambda)^{i}e^{-i\theta x}\sum_{j=0}^{i}\binom{i}{j}\frac{\theta^{j}x^{j}}{(\theta^2+2\theta+2)^{j}}\sum_{k=0}^{j}\binom{j}{k}\theta^{k}x^{k}(2\theta+2)^{j-k}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_R(\beta) = &\frac{1}{1-\beta}log\Bigg[\left(\frac{\theta^3}{\theta^2+2\theta+2}\right)^{\beta}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{2\beta}{m}\binom{\beta}{i}\binom{\beta-i}{l}\binom{i}{j}\binom{j}{k}(-1)^{l}\lambda^{l}(2\lambda)^{i} \\
&\frac{\theta^{k+j}}{(\theta^2+2\theta+2)^{j}}(2\theta+2)^{j-k}\int_0^{\infty}x^{m+j+k}e^{-(i+\beta)\theta x}dx\Bigg] \\
= &\frac{1}{1-\beta}log\Bigg[\left(\frac{\theta^3}{\theta^2+2\theta+2}\right)^{\beta}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{2\beta}{m}\binom{\beta}{i}\binom{\beta-i}{l}\binom{i}{j}\binom{j}{k}(-1)^{l}\lambda^{l}(2\lambda)^{i} \\
&\frac{\theta^{k+j}}{(\theta^2+2\theta+2)^{j}}(2\theta+2)^{j-k}\frac{\Gamma(m+j+k+1)}{((i+\beta)\theta)^{m+j+k+1}}\Bigg] \\
= &\frac{1}{1-\beta}log\Bigg[\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{2\beta}{m}\binom{\beta}{i}\binom{\beta-i}{l}\binom{i}{j}\binom{j}{k}(-1)^{l}(\lambda)^{i+l}(\theta+1)^{j-k}\,2^{i+j-k} \\
&\frac{\theta^{3\beta-m-1}}{(\theta^2+2\theta+2)^{\beta+j}}\frac{\Gamma(m+j+k+1)}{(i+\beta)^{m+j+k+1}}\Bigg].
\end{aligned}
$$

□

8. Application

In this section, we illustrate the goodness of fit of the proposed distribution using a real lifetime data. This data was studied by [11] which represents the waiting times (in minutes) before service of 100 bank customers. The data is given in Table 3.

For this data, the goodness of fit of the TAD is compared with Aradhana distribution and the following distributions:

- Exponential distribution: $f(x)=\theta e^{-\theta x}\quad ;x>0.$
- Akash distribution [17]: $f(x)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}\quad ;x>0,\theta>0.$
- Ishita distribution [19]: $f(x)=\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x};x>0,\theta>0.$

TABLE 3. Waiting times (in minutes) of 100 bank customers.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1
4.2	4.2	4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9	5.0	5.3	5.5	5.7	5.7	6.1
6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0	8.2	8.6	8.6	8.6
8.8	8.8	8.9	8.9	9.5	9.6	9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4
12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5								

For all fitted distributions, the values of the Cramer-von Mises Criterion (C-M), Anderson-Darling Criterion (A-D), -2 log likelihood ($-2logL$), Kolmogorov Smirnov (KS) statistic and its p-value, are calculated and the results are presented in Table 4. It can be seen that the TAD has the lowest values of the $-2lnL$, C-M, A-D, KS statistic and highest p-value among all fitted distributions. Therefore, the TAD fits this survival times data better than Exponential, Akash, Ishita, and Aradhana distributions.

TABLE 4. $-2logL$, C-M, A-D, KS statistic and its p-value for fitted distributions.

Distribution	*-2 log L*	*C-M*	*A-D*	*KS Statistic*	*p-value*
Exponential	658.0418	0.0270	0.1790	0.173	0.005
Ishita	643.6996	0.0361	0.2349	0.109	0.1861
Akash	641.9292	0.0525	0.3414	0.1003	0.2672
Aradhana	638.343	0.0488	0.3112	0.0801	0.542
Transmuted Aradhana	636.6322	0.0344	0.2222	0.0704	0.7045

The maximum likelihood estimates (MLEs), standard errors, and confidence intervals (CI) of the parameters of the fitted distributions are obtained and given in Table 5.

TABLE 5. The MLEs of the parameters of the fitted distributions and their confidence intervals

Distribution	Parameter	MLE	Standard error	95% Confidence Interval	
				Lower Limit	Upper Limit
Exponential	θ	0.1012487	0.01012388	0.081406	0.121092
Ishita	θ	0.3015717	0.01717704	0.267905	0.335239
Akash	θ	0.2952784	0.01683424	0.262283	0.328274
Aradhana	θ	0.276552	0.01600372	0.245185	0.307919
Transmuted Aradhana	θ	0.2431686	0.02901144	0.186306	0.300031
	λ	0.4098040	0.30538536	-0.18875	1.008359

9. Conclusion

In this article, we propose a new distribution called transmuted Aradhana distribution. We study several mathematical and statistical features of this distribution such as moments, mean, variance, skewness, kurtosis, coefficient of variation, moment generating function and quantile function. We obtain Renyi entropy, order statistics and maximum likelihood estimates of the distribution parameters. Also, we investigate the reliability, hazard rate, cumulative hazard, reversed hazard, odds and mean residual life functions. An application to a real lifetime data illustrates the usefulness of the proposed distribution and shows that TAD fits this data better than other considered distributions. Therefore, the TAD can be considered as a competitive distribution for modeling such a real lifetime data.

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