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PARTITION DIMENSION AND STRONG METRIC DIMENSION OF CHAIN CYCLE

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Abstract. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For an ordered k-partition $\Pi = \{Q_1, \ldots, Q_k\}$ of $V(G)$, the representation of a vertex $v \in V(G)$ with respect to Π is the k-vectors $r(v|\Pi) = (d(v, Q_1), \ldots, d(v, Q_k))$, where $d(v, Q_i)$ is the distance between v and Q_i. The partition Π is a resolving partition if $r(u|\Pi) \neq r(v|\Pi)$, for each pair of distinct vertices $u, v \in V(G)$. The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension of G. A vertex $w \in V(G)$ strongly resolves two distinct vertices $u, v \in V(G)$ if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. An ordered set $W = \{w_1, \ldots, w_t\} \subseteq V(G)$ is a strong resolving set for G if for every two distinct vertices u and v of G there exists a vertex $w \in W$ which strongly resolves u and v. A strong metric basis of G is a strong resolving set of minimal cardinality. The cardinality of a strong metric basis is called strong metric dimension of G. In this paper, we determine the partition dimension and strong metric dimension of a chain cycle constructed by even cycles and a chain cycle constructed by odd cycles.

1. Introduction

Let G be a finite, simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u and v of G is the length of the shortest path from u to v in G and is denoted by $d(u, v)$. Two distinct vertices u and v are

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called adjacent if there is an edge between them and denoted by uv. The degree of a vertex u is the number of vertices adjacent to it and is denoted by $d_G(v)$ or simply $d(v)$. The set of neighborhood of a vertex $u \in V(G)$, denoted by $N(u)$, is the set of all vertices of G that are adjacent to u.The diameter of a graph G, denoted by $d(G)$, is defined as $d(G) = max\{d(u,v) \mid u,v \in V(G)\}$. A cycle of lenght n is denoted by C_n.

A vertex $w \in V(G)$ resolves two vertices u and v of G if $d(w,u) \neq d(w,v)$. An ordered set $W = \{w_1,\ldots,w_t\} \subseteq V(G)$ is a resolving set for G if for every two distinct vertices u and v of G there exists a vertex $w \in W$ which resolves u and v. The representation of a vertex $u \in V(G)$ with respect to W, denoted by $r_G(u|W)$, is defined by the t-vectors $r_G(u|W) = (d(u,w_1), d(u,w_2), \ldots, d(u,w_t))$. The metric basis of G is a resolving set of minimal cardinality. The cardinality of the metric basis is called metric dimension of G and is denoted by $dim(G)$. The metric dimension of graphs was introduced independently by Harary and Melter in [4]. For more detail see [1, 3, 4, 5, 8, 17].

Later on the concept of partition dimension was given by Chartrand et al. [2] in 2000. Given an ordered partition $\Pi = \{Q_1,\ldots,Q_t\}$ of the vertices of G, the partition representation of a vertex $u \in V(G)$ with respect to Π is the vector $r(u|\Pi) = (d(u,Q_1),\ldots,d(u,Q_t))$, where $d(u,Q_j) = \min\{d(u,q) \mid q \in Q_j\}$, for each $j = 1,2,\ldots,t$. The partition Π is a resolving partition of G if for every pair of distinct vertices $u,v \in V(G)$, $r(u|\Pi) \neq r(v|\Pi)$. The partition dimension of G is the cardinality of a minimum resolving partition of G and is denoted by $pd(G)$. See [2, 6, 11, 15, 18] for more results.

Sebő and Tannier [16], in 2004, gave more strict version of metric dimension of a graph called the strong metric dimension of a graph. A vertex $w \in V(G)$ strongly resolves two distinct vertices $u,v \in V(G)$ if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. An ordered set $W = \{w_1,\ldots,w_t\} \subseteq V(G)$ is a strong resolving set for G if for every two distinct vertices u and v of G there exists a vertex $w \in W$ which strongly resolves u and v. A strong metric basis of G is a strong resolving set of minimal cardinality. The cardinality of a strong metric basis is called strong metric dimension of G and is denoted by $sdim(G)$. For more detail, see [7, 9, 13, 14].

A set S of vertices of G is a vertex cover of G if every edge of G is incident with at least one vertex of S. The vertex cover number of G, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of G. The largest cardinality of a set of vertices of G, no two of which are adjacent, is called the independence number of G and is denoted by $\beta(G)$. Since for any graph G of order n the complement of an independent set $S \subseteq V(G)$ is a vertex cover of G and therefore $\alpha(G) + \beta(G) = n$.

A vertex $u \in V(G)$ is maximally distant from $v \in V(G)$, denoted by $u\text{MD}v$, if for every vertex w in the neighborhood of u, $d_G(v, w) \leq d_G(u, v)$. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are mutually maximally distant and we denote it as $u\text{MMD}v$. The strong resolving graph of G is a graph G_{SR} whose vertex set is $V(G)$ and two vertices $u, v \in V(G)$ are adjacent in G_{SR} if and only if $u\text{MMD}v$. Oellermann and Peters-Fransen [13] showed that finding the strong metric dimension of a connected graph G is equivalent to finding the vertex cover number of G_{SR}.

Theorem 1.1 (Oellermann and Peters-Fransen [13]). *For any connected graph G, $sdim(G) = \alpha(G_{SR})$.*

Let $\{G_i\}_{i=1}^{m}$ be a set of finite pairwise disjoint simple connected graphs. The chain graph

$$\mathcal{C}(G_1, G_2, \ldots, G_m) = \mathcal{C}(G_1, G_2, ..., G_m; x_1, w_1, x_2, w_2, \ldots, x_m, w_m)$$

of $\{G_i\}_{i=1}^m$ with respect to the vertices $\{x_i, w_i \in V(G_i) \mid i = 1, 2, \ldots, m\}$ is the graph obtained from the graphs $G_1, \ldots, G_m$ by identifying the vertex w_i and the vertex x_{i+1}, as shown in Figure 1, for all $i \in \{1, 2, \ldots, m-1\}$. For more results and detail about chain graph, see [10, 12].

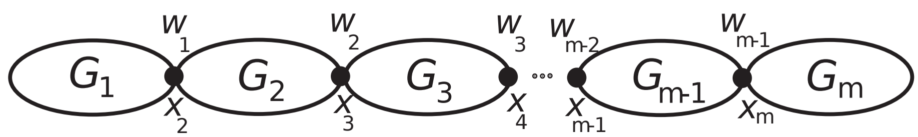

FIGURE 1. A chain graph

Let $\{C_{n_i}\}_{i=1}^m$ be a set of finite pairwise disjoint simple cycles. Let $V(C_{n_i}) = \{v_j^i \mid j = 1, 2, \ldots, n_i\}$, where $i \in \{1, 2, \ldots, m\}$. Assume that n_i is even for each $i = 1, 2, \ldots, m$. We consider a chain cycle of $\{C_{n_i}\}_{i=1}^m$ given by

$$\begin{aligned}&\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})\\&= \mathcal{C}\left(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1+1}{2}+1}, v_1^3, v^2_{\frac{n_2+1}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}+1}{2}+1}, v^m_{\frac{n_m+1}{2}+1}\right)\end{aligned}$$

with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$. A chain cycle of $\{C_8, C_{10}, C_8\}$ with respect to vertices $\{v_5^1, v_1^2, v_6^2 v_1^3\}$ is shown in Figure 2.

Now, assume that n_i is odd for each $i = 1, 2, \ldots, m$. We consider a chain cycle of $\{C_{n_i}\}_{i=1}^m$ given by

$$\begin{aligned}&C(C_{n_1}, C_{n_2}, \ldots, C_{n_m})\\&= C\left(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1}{2}+1}, v_1^3, v^2_{\frac{n_2}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}}{2}+1}, v^m_{\frac{n_m}{2}+1}\right)\end{aligned}$$

with respect to the vertices $\{v^i_{\frac{n_i+1}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$. A chain cycle of $\{C_5, C_7, C_5\}$ with respect to vertices $\{v_4^1, v_1^2, v_5^2, v_1^3\}$ is shown in Figure 3.

Through out the paper, we denote the vertex set and the edge set of chain cycle by $V(\mathcal{C})$ and $E(\mathcal{C})$ instead of $V(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$ and $E(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$, respectively.

2. Partition dimension of chain cycles

In this section, we find the partition dimension of chain cycle constructed by even cycles and chain cycle constructed by odd cycles. Following two results are important tools for proving our results.

Theorem 2.1 (Chartrand et al. [2]). *If G is a nontrivial connected graph, then $pd(G) \leq dim(G) + 1$.*

Proposition 2.1 (Chartrand et al. [2]). *Let G be a connected graph of order $n \geq 2$. Then $pd(G) = 2$ if and only if $G \cong P_n$.*

In the following theorem, we compute the partition dimension of chain cycle constructed by even cycles.

Theorem 2.2. *The partition dimension of chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ $= \mathcal{C}\Big(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1}{2}+1}, v_1^3, v^2_{\frac{n_2}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}}{2}+1}, v^m_{\frac{n_m}{2}+1}\Big)$ is 3, where n_i is even for each $i = 1, 2, \ldots, m$.*

Proof. Let $\Pi = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v_1^1, \ldots, v^1_{\frac{n_1}{2}-1}, v^1_{\frac{n_1}{2}+2}, \ldots, v^1_{n_1}\}$, $Q_2 = \{v^2_{\frac{n_2}{2}+3}, v^2_{\frac{n_2}{2}+4}, \ldots, v^2_{n_2}, v^3_{\frac{n_3}{2}+3}, v^3_{\frac{n_3}{2}+4}, \ldots, v^3_{n_3}, \ldots, v^{m-1}_{\frac{n_{m-1}}{2}+3}, v^{m-1}_{\frac{n_{m-1}}{2}+4}, \ldots, v^{m-1}_{n_{m-1}}\} \cup \{v^m_{n_m}\}$ and $Q_3 = V(\mathcal{C}) \setminus \{Q_1 \cup Q_2\}$ be a partition of $V(\mathcal{C})$. We show that Π is a resolving partition of $V(\mathcal{C})$ with minimum cardinality. The representation of each vertex of $V(\mathcal{C})$ with respect to Π is given as:

$$r(v^1_{\frac{n_1}{2}}|\Pi) = (1, 2, 0), \quad r(v^1_{\frac{n_1}{2}+1}|\Pi) = (1, 1, 0), \quad r(v^m_{n_m}|\Pi) = \left(\sum_{k=2}^{m} \frac{n_k}{2} + 2, 0, 1\right).$$

$$r(v_j^1|\Pi) = \begin{cases} (0, n_1 - j - 1, n_1 - j - 3) & \text{if } 1 \leq j \leq \frac{n_1}{2} - 1 \\ (0, j - \frac{n_1}{2}, j - \frac{n_1}{2} - 1) & \text{if } \frac{n_1}{2} + 2 \leq j \leq n_1, \end{cases}$$

$$r(v_j^i|\Pi) = \begin{cases} (j, j, 0) & \text{if } 1 \leq j \leq \lceil \frac{n_2}{4} \rceil \\ (j, \frac{n_2}{2} - j + 2, 0) & \text{if } \lceil \frac{n_2}{4} \rceil + 1 \leq j \leq \frac{n_2}{2}, \end{cases}$$

$$r(v_j^i|\Pi) = \begin{cases} \left(\sum\limits_{k=3}^{m} \frac{n_k}{2} + j, j, 0\right) & \text{if } 1 \le j \le \lceil \frac{n_i}{4} \rceil, 3 \le i \le m \\ \left(\sum\limits_{k=3}^{m} \frac{n_k}{2} + j, \frac{n_i}{2} - j + 2, 0\right) & \text{if } \lceil \frac{n_i}{4} \rceil + 1 \le j \le \frac{n_i}{2}, 3 \le i \le m, \end{cases}$$

$$r(v_{\frac{n_i}{2}+j}^i|\Pi) = \begin{cases} \left(\sum\limits_{k=2}^{m-1} \frac{n_k}{2}, 1, 0\right) & \text{if } 2 \le i \le m-1, j = 2 \\ \left(\sum\limits_{k=2}^{m-1} \frac{n_k}{2} + n_m - j + 2, n_i - j, 0\right) & \text{if } i = m, \frac{n_i}{2} + 1 \le j \le n_i - 1, \end{cases}$$

$$r(v_j^i|\Pi) = \begin{cases} (n_i + 2 - j, 0, j - \frac{n_i}{2} - 2) & \text{if } \frac{n_2}{2} + 3 \le j \le \lceil \frac{3n_2}{4} \rceil + 1 \\ (n_i + 2 - j, 0, n_i + 1 - j) & \text{if } \lceil \frac{3n_2}{4} \rceil + 2 \le j \le n_2, \end{cases}$$

$$r(v_j^i|\Pi) = \begin{cases} \left(\sum\limits_{k=3}^{m} \frac{n_k}{2} + n_i - j, 0, j - \frac{n_i}{2} - 2\right) & \text{if } \frac{n_i}{2} + 3 \le j \le \lceil \frac{3n_i}{4} \rceil + 1, 3 \le i \le m \\ \left(\sum\limits_{k=3}^{m} \frac{n_k}{2} + n_i - j, 0, n_i + 1 - j\right) & \text{if } \lceil \frac{3n_i}{4} \rceil + 2 \le j \le n_i, 3 \le i \le m. \end{cases}$$

It is easily seen that the representation of each vertex with respect to Π is distinct. This shows that Π is a resolving partition of $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$. Thus $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \le 3$.

On the other hand, by Proposition 2.1 it follows that $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \ge 3$. Hence $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) = 3$. □

In the following example, we find the partition dimension of a chain cycle constructed by C_8, C_{10} and C_8.

Example 2.1. *Let $m = 3$ and $n_1 = 8$, $n_2 = 10$ and $n_3 = 8$. The chain cycle constructed by C_8, C_{10} and C_8 with respect to the vertices $\{v_5^1, v_1^2, v_6^2, v_1^3\}$ is denoted by $\mathcal{C}(C_8, C_{10}, C_8) = \mathcal{C}(C_8, C_{10}, C_8; v_1^1, v_1^2, v_5^1, v_1^3, v_6^2, v_5^3)$ and is given in Figure 2.*

Using Theorem 2.2, we construct a resolving partition of $\mathcal{C}(C_8, C_{10}, C_8)$ as $\Pi = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v_1^1, v_2^1, v_3^1, v_6^1, v_7^1, v_8^1\}$, $Q_2 = \{v_8^2, v_9^2, v_{10}^2, v_8^3\}$ and $Q_3 = \{v_4^1, v_5^1, v_2^2, v_3^2, v_4^2, v_5^2, v_6^2, v_7^2, v_2^3, v_3^3, v_4^3, v_4^3, v_5^3, v_6^3, v_7^3\}$. Again by Theorem 2.2, we note that each vertex of $\mathcal{C}(C_8, C_{10}, C_8)$ has distinct representation with respect to Π, as shown in Table 1. Hence $pd(\mathcal{C}(C_8, C_{10}, C_8)) = 3$.

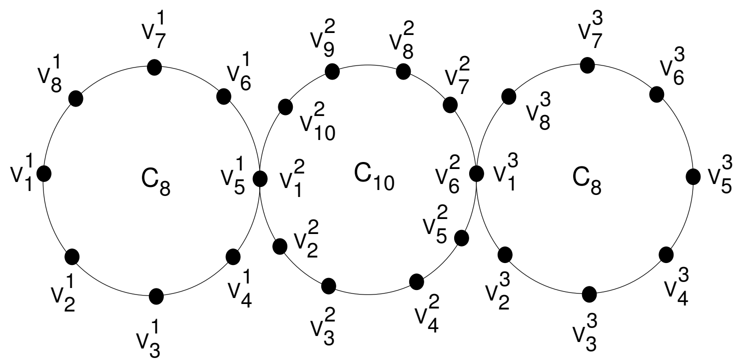

FIGURE 2. Chain cycle of C_8, C_{10} and C_8

TABLE 1. Representation of v_j^i with respect to Π

$r(v_1^1\|\Pi) = (0,5,3)$	$r(v_2^2\|\Pi) = (2,2,0)$	$r(v_{10}^2\|\Pi) = (2,0,1)$
$r(v_2^1\|\Pi) = (0,4,2)$	$r(v_3^2\|\Pi) = (3,3,0)$	$r(v_2^3\|\Pi) = (7,2,0)$
$r(v_3^1\|\Pi) = (0,3,1)$	$r(v_4^2\|\Pi) = (4,3,0)$	$r(v_3^3\|\Pi) = (8,3,0)$
$r(v_4^1\|\Pi) = (1,2,0)$	$r(v_5^2\|\Pi) = (5,2,0)$	$r(v_4^3\|\Pi) = (9,4,0)$
$r(v_5^1\|\Pi) = (1,1,0)$	$r(v_6^2\|\Pi) = (6,1,0)$	$r(v_5^3\|\Pi) = (10,3,0)$
$r(v_6^1\|\Pi) = (0,2,1)$	$r(v_7^2\|\Pi) = (5,1,0)$	$r(v_6^3\|\Pi) = (9,2,0)$
$r(v_7^1\|\Pi) = (0,3,2)$	$r(v_8^2\|\Pi) = (4,0,1)$	$r(v_7^3\|\Pi) = (8,1,0)$
$r(v_8^1\|\Pi) = (0,4,3)$	$r(v_9^2\|\Pi) = (3,0,2)$	$r(v_8^3\|\Pi) = (7,0,1)$

Theorem 2.3. *The partition dimension of chain cycle* $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ $= \mathcal{C}\Big(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1+1}{2}+1}, v_1^3, v^2_{\frac{n_2+1}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}+1}{2}+1}, v^m_{\frac{n_m+1}{2}+1}\Big)$ *is* 3, *where* n_i *is odd for each* $i = 1, 2, \ldots, m$.

Proof. Let $\Pi = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v_1^1, \ldots, v^1_{\lceil\frac{n_1}{2}\rceil-1}, v^1_{\lceil\frac{n_1}{2}\rceil+2}, \ldots, v^1_{n_1}\}$, $Q_2 = \{v^2_{\lceil\frac{n_2}{2}\rceil+3}, v^2_{\lceil\frac{n_2}{2}\rceil+4}, \ldots, v^2_{n_2}, v^3_{\lceil\frac{n_3}{2}\rceil+3}, v^3_{\lceil\frac{n_3}{2}\rceil+4}, \ldots, v^3_{n_3}, \ldots, v^{m-1}_{\lceil\frac{n_{m-1}}{2}\rceil+3}, v^{m-1}_{\lceil\frac{n_{m-1}}{2}\rceil+4}, \ldots, v^{m-1}_{n_{m-1}}\} \cup \{v^m_{n_m}\}$ and $Q_3 = V(\mathcal{C}) \setminus \{Q_1 \cup Q_2\}$ be a partition of $V(\mathcal{C})$. We show that Π is a resolving partition of $V(\mathcal{C})$ with minimum cardinality. The representation of each vertex of $V(\mathcal{C})$ with respect to Π is given as:

$$r(v^1_{\lceil \frac{n_1}{2} \rceil}|\Pi) = (1,2,0), \quad r(v^1_{\lceil \frac{n_1}{2} \rceil+1}|\Pi) = (1,1,0), \quad r(v^m_{n_m}|\Pi) = \left(\sum_{k=2}^{m} \lfloor \frac{n_k}{2} \rfloor + 2, 0, 1\right).$$

$$r(v^1_j|\Pi) = \begin{cases} (0, n_1 - \lfloor \frac{n_1}{2} \rfloor, n_1 - \lfloor \frac{n_1}{2} \rfloor - 1) & \text{if j=1} \\ (0, n_1 - j - 1, n_1 - j - 3) & \text{if } 2 \leq j \leq \lceil \frac{n_1}{2} \rceil - 1 \\ (0, j - \lceil \frac{n_1}{2} \rceil, j - \lceil \frac{n_1}{2} \rceil - 1) & \text{if } \lceil \frac{n_1}{2} \rceil + 2 \leq j \leq n_1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} (j, j, 0) & \text{if } 1 \leq j \leq \lceil \frac{n_2}{4} \rceil + 1 \\ (j, \lceil \frac{n_2}{2} \rceil - j + 3, 0) & \text{if } \lceil \frac{n_2}{4} \rceil + 2 \leq j \leq \lceil \frac{n_2}{2} \rceil + 1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} \left(\sum_{k=3}^{m} \lfloor \frac{n_k}{2} \rfloor + j, j, 0\right) & \text{if } 1 \leq j \leq \lceil \frac{n_i}{4} \rceil + 1, 3 \leq i \leq m \\ \left(\sum_{k=3}^{m} \lfloor \frac{n_k}{2} \rfloor + j, \lceil \frac{n_i}{2} \rceil - j + 3, 0\right) & \text{if } \lceil \frac{n_i}{4} \rceil + 2 \leq j \leq \lceil \frac{n_i}{2} \rceil + 1, 3 \leq i \leq m, \end{cases}$$

$$r(v^i_{\frac{n_i}{2}+j}|\Pi) = \begin{cases} \left(\sum_{k=2}^{m-1} \lfloor \frac{n_k}{2} \rfloor, 1, 0\right) & \text{if } 2 \leq i \leq m-1, j = 2 \\ \left(\sum_{k=2}^{m-1} \lfloor \frac{n_k}{2} \rfloor + n_m - j + 2, n_i - j, 0\right) & \text{if } i = m, \lceil \frac{n_i}{2} \rceil + 2 \leq j \leq n_i - 1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} (n_i + 2 - j, 0, j - \lceil \frac{n_i}{2} \rceil - 2) & \text{if } \lceil \frac{n_2}{2} \rceil + 3 \leq j \leq \lceil \frac{3n_2}{4} \rceil + 1 \\ (n_i + 2 - j, 0, n_i + 1 - j) & \text{if } \lceil \frac{3n_2}{4} \rceil + 2 \leq j \leq n_2, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} \left(\sum_{k=3}^{m} \lceil \frac{n_k}{2} \rceil + n_i - j, 0, j - \lceil \frac{n_i}{2} \rceil - 2\right) & \text{if } \lceil \frac{n_i}{2} \rceil + 3 \leq j \leq \lceil \frac{3n_i}{4} \rceil + 1, 3 \leq i \leq m \\ \left(\sum_{k=3}^{m} \lceil \frac{n_k}{2} \rceil + n_i - j, 0, n_i + 1 - j\right) & \text{if } \lceil \frac{3n_i}{4} \rceil + 2 \leq j \leq n_i, 3 \leq i \leq m. \end{cases}$$

It is easily seen that the representation of each vertex with respect to Π is distinct. This shows that Π is a resolving partition of $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$. Thus $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \leq 3$.

On the other hand, by Proposition 2.1 it follows that $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \geq 3$. Hence $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) = 3$. □

In the following example, we compute the partition dimension of the chain cycle constructed by C_5, C_7 and C_5.

Example 2.2. *Let* $m = 3$ *and* $n_1 = 5$, $n_2 = 7$ *and* $n_3 = 5$. *The chain cycle constructed by* C_5, C_7 *and* C_5 *with respect to vertices* $\{v_4^1, v_1^2, v_5^2, v_1^3\}$ *is denoted by* $\mathcal{C}(C_5, C_7, C_5) = \mathcal{C}(C_5, C_7, C_5; v_1^1, v_1^2, v_4^1,\ v_1^3, v_5^2, v_4^3)$ *and is given in Figure 3.*

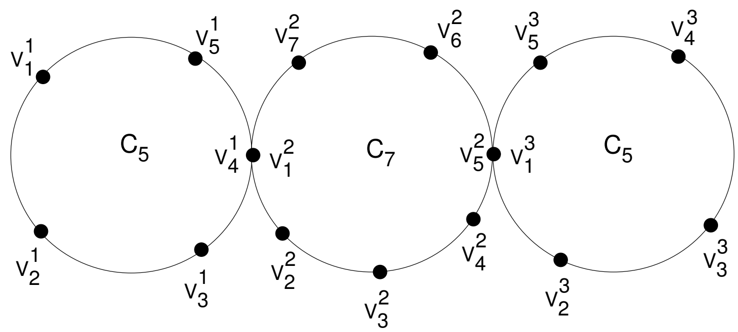

FIGURE 3. Chain cycle of C_5, C_7 and C_5

Using Theorem 2.3, we construct a resolving partition of $\mathcal{C}(C_5, C_7, C_5)$ *as* $\Pi = \{Q_1, Q_2, Q_3\}$, *where* $Q_1 = \{v_1^1, v_2^1, v_5^1\}$, $Q_2 = \{v_7^2, v_5^3\}$ *and* $Q_3 = \{v_3^1, v_4^1, v_2^2, v_3^2, v_4^2, v_5^2, v_6^2, v_2^3, v_3^3, v_4^3\}$. *Again by Theorem 2.3, we note that each vertex of* $\mathcal{C}(C_5, C_7, C_5)$ *has distinct representation with respect to* Π, *as shown in Table* 2. *Hence* $pd(\mathcal{C}(C_5, C_7, C_5)) = 3$.

3. Strong metric dimension of chain cycle

In this section, we find the strong metric dimension of the chain cycle constructed by even cycles and the chain cycle constructed by odd cycles. Let $V_1 = \{v_1^2, v_1^3, \ldots, v_1^m\}$ and $V_2 = V(\mathcal{C}) \backslash V_1$. Through out the section, we denote the strong resolving graph of a chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ by $\mathcal{C}_{SR}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$. Furthermore, we denote the vertex set and the edge set of the strong resolving graph of $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ by $V(\mathcal{C}_{SR})$ and $E(\mathcal{C}_{SR})$, respectively.

TABLE 2. Representation of v_j^i with respect to Π

$r(v_1^1\|\Pi) = (0,3,2)$	$r(v_5^2\|\Pi) = (4,1,0)$
$r(v_2^1\|\Pi) = (0,3,1)$	$r(v_6^2\|\Pi) = (3,1,0)$
$r(v_3^1\|\Pi) = (1,2,0)$	$r(v_7^2\|\Pi) = (2,0,1)$
$r(v_4^1\|\Pi) = (1,1,0)$	$r(v_2^3\|\Pi) = (5,2,0)$
$r(v_5^1\|\Pi) = (0,2,1)$	$r(v_3^3\|\Pi) = (6,2,0)$
$r(v_2^2\|\Pi) = (2,2,0)$	$r(v_4^3\|\Pi) = (6,1,0)$
$r(v_3^2\|\Pi) = (3,3,0)$	$r(v_5^3\|\Pi) = (5,0,1)$
$r(v_4^2\|\Pi) = (4,2,0)$	

Following two lemmas are easy observations from the structure of a cycle C_n and a chain cycle constructed by even cycles as well as a chain cycle constructed by even cycles, respectively.

Lemma 3.1. *Let C_n be a cycle. Then for two distinct vertices $u_i, u_j \in V(C_n)$ we have $u_i MMD u_j$ if and only if $d(u_i, u_j) = d(C_n)$.*

Lemma 3.2. *Let $x \in V_1$ and $y \in V(\mathcal{C})$. Then x and y are not mutually maximally distant.*

In the next theorem, we find the mutually maximally distant vertices in chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$, with each n_i is even, with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$. Here, we denote $U_1(C_{n_i}) = \{v_1^i, v_2^i, \ldots, v^i_{\frac{n_i}{2}}\}$ and $U_2(C_{n_i}) = \{v^i_{\frac{n_i}{2}+1}, v^i_{\frac{n_i}{2}+2}, \ldots, v^i_{n_i}\}, i \in \{1, 2, \ldots, m\}$.

Theorem 3.1. *Let $v_j^i, v_l^k \in V_2$, where $i, k \in \{1, 2, \ldots, m\}$, in a chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ constructed by even cycles with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$.*

a: Let $i = k$. Then $v_j^i MMD v_l^i$ if and only if $d(v_j^i, v_l^i) = d(C_{n_i})$.

b: *Let $i \neq k$. Then $v_j^i MMD v_l^k$ if and only if $d(v_j^i, v_l^k) = d(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$.*

Proof. (a). Let $d(v_j^i, v_l^i) = d(C_{n_i})$. Then from Lemma 3.1, we have $v_j^i\text{MMD}v_l^i$.

Conversely, let $v_j^i\text{MMD}v_l^i$ and $j < l$. On the contrary, assume that $d(v_j^i, v_l^i) < d(C_{n_i})$. Since $N(v_j^i) = \{v_{j-1}^i, v_{j+1}^i\}$ and $N(v_l^i) = \{v_{l-1}^i, v_{l+1}^i\}$. Note that either $v_j^i v_{j+1}^i \ldots v_l^i$ or $v_j^i v_{j-1}^i \ldots v_1^i v_{n_i}^i \ldots v_l^i$ is a shortest path from v_j^i to v_l^i. This shows that v_j^i and v_l^i are not mutually maximally distant which contradicts our supposition that $v_j^i\text{MMD}v_l^i$.

(b). Let $d(v_j^i, v_l^k) = d(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$. Then clearly $v_j^i\text{MMD}v_l^k$.

Conversely, let $u_j^i\text{MMD}u_l^k$ and let $d(v_j^i, v_l^k) < d(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$. Since $N(v_j^i) = \{v_{j-1}^i,\ v_{j+1}^i\}$ and $N(v_l^k) = \{v_{l-1}^k, u_{l+1}^k\}$. If $v_j^i \in U_1(C_{n_i})$ and $v_l^k \in U_1(C_{n_k})$, then $P_1 = v_j^i v_{j+1}^i \ldots v_1^{i+1} v_2^{i+1} \ldots\ v_1^k v_2^k \ldots v_l^k$ is a shortest path from v_j^i to v_l^k. This clearly shows that v_j^i is not mutually maximally distant to v_l^k. Similarly, if $v_j^i \in U_2(C_{n_i})$ and $v_l^k \in U_2(C_{n_k})$, then $P_2 = v_j^i v_{j-1}^i \ldots v_1^{i+1} v_{n_{i+1}}^{i+1} \ldots v_1^k\ v_{n_k}^k \ldots v_l^k$ is a shortest path from v_j^i to v_l^k. This shows that v_j^i is not mutually maximally distant to v_l^k. Moreover, If $v_j^i \in U_1(C_{n_i})$ and $v_l^k \in U_2(C_{n_k})$, then $R_1 = v_j^i v_{j+1}^i \ldots v_1^{i+1} v_2^{i+1} \ldots v_1^k,\ v_{n_k}^k \ldots v_l^k$ is a shortest path from v_j^i to v_l^k. This clearly shows that v_j^i is not mutually maximally distant to v_l^k. Similarly, if $v_j^i \in U_2(C_{n_i})$ and $v_l^k \in U_1(C_{n_k})$, then $R_2 = v_j^i v_{j-1}^i \ldots v_1^{i+1} v_2^{i+1} \ldots v_1^k$, $v_2^k \ldots v_l^k$ is a shortest path from v_j^i to v_l^k, which shows that v_j^i is not mutually maximally distant to v_l^k. □

For each $i \in \{1, 2, \ldots, m\}$, Theorem 3.1 (a) implies

$$(3.1) \qquad A = \{v_j^i v_{j+\frac{n_i}{2}}^i \mid j = 2, 3, \ldots, \frac{n_i}{2}, \frac{n_i}{2} + 2, \frac{n_i}{2} + 3, \ldots, n_i\} \subseteq E(\mathcal{C}_{SR}),$$

where $j + \frac{n_i}{2}$ are integers modulo n_i. Similarly, Theorem 3.1 (b) implies $v_1^1 v_{\frac{n_m}{2}+1}^m \in E(\mathcal{C}_{SR})$. Thus $E(\mathcal{C}_{SR}) = A \cup \{v_1^1 v_{\frac{n_m}{2}+1}^m\}$.

Lemma 3.3. *Let $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$ ba a chain cycle constructed by even cycles with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v^{i+1}_1 \mid i = 1, 2, \dots, m-1\}$ and each $n_i \geq 4$. Then $\alpha(\mathcal{C}_{SR}(C_{n_1}, C_{n_2}\ , \dots, C_{n_m}) = 1 + \sum_{i=1}^{m} \frac{n_i-2}{2}$.*

Proof. We construct a vertex cover of strong resolving graph of $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$ with minimum cardinality. From (3.1), we note that the vertices $\{v^i_j, v^i_{j+\frac{n_i}{2}} \mid j = 2, 3, \dots, \frac{n_i}{2}, \frac{n_i}{2}+2, \frac{n_i}{2}+3, \dots, n_i\}$, for each $i \in \{1, 2, \dots, m\}$, form $\sum_{i=1}^{m} \frac{n_i-2}{2}$ copies of K_2. Thus, the $\sum_{i=1}^{m} \frac{n_i-2}{2}$ vertices $\{v^i_j \mid j = 2, 3, \dots, \frac{n_i}{2}, \frac{n_i}{2}+2, \frac{n_i}{2}+3, \dots, n_i\}$, for each $i \in \{1, 2, \dots, m\}$, are minimum number of vertices to cover the edges of A. Let $S = \{v^i_j \mid j = 2, 3, \dots, \frac{n_i}{2}, \frac{n_i}{2}+2, \frac{n_i}{2}+3, \dots, n_i\}$. Furthermore, since $v^1_1 v^m_{\frac{n_m}{2}+1} \in E(\mathcal{C}_{SR})$. Thus, the vertex cover of the strong resolving graph of chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$ with minimum cardinality is $S := S \cup \{v^1_1\}$. Hence $\alpha(\mathcal{C}_{SR}(C_{n_1}, C_{n_2}, \dots, C_{n_m})) = 1 + \sum_{i=1}^{m} \frac{n_i-2}{2}$. □

Theorem 3.2. *Let $\{C_{n_i}\}_{i=1}^{m}$ be m disjoint cycles with each n_i is even and $n_i \geq 4$, then $sdim(\mathcal{C}(C_{n_1},\ C_{n_2}, \dots, C_{n_m})) = 1 + \sum_{i=1}^{m} \frac{n_i-2}{2}$.*

Proof. The proof follows from Lemma 3.3 and Theorem 1.1. □

In the next theorem, we find the mutually maximally distant vertices in chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$, with each n_i is odd, with respect to the vertices $\{v^i_{\frac{n_i+1}{2}+1}, v^{i+1}_1 \mid i = 1, 2, \dots, m-1\}$.

Theorem 3.3. *Let $v^i_j, v^k_l \in V_2$, where $i, k \in \{1, 2, \dots, m\}$, in a chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$ constructed by odd cycles with respect to the vertices $\{v^i_{\frac{n_i+1}{2}+1}, v^{i+1}_1 \mid i = 1, 2, \dots, m-1\}$.*

(a): *Let $i = k$. Then $v^i_j MMD v^i_l$ if and only if $d(v^i_j, v^i_l) = d(C_{n_i})$.*

(b): *Let $i \neq k$.*

(1): *For $i = 1$ and $k = m$, then $v^1_j MMD v^m_l$ if and only if $j \in \{1, 2\}$ and $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$,* (2): *For* $i = 1$ *and* $k \in \{2, 3, \ldots, m-1\}$, *then* $v_j^1 MMD v_l^k$ *if and only if* $j \in \{1, 2\}$ *and* $l = \frac{n_k+1}{2}$

(3): *For* $i \in \{2, 3, \ldots, m-1\}$ *and* $k = m$, *then* $v_l^i MMD v_j^m$ *if and only if* $j = 2$ *and* $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$,

(4): *For* $i \in \{2, 3, \ldots, m-1\}$ *and* $k \in \{2, 3, \ldots, m-1\}$, *then* $v_j^i MMD v_l^k$ *if and only if* $j = 2$ *and* $l = \frac{n_k+1}{2}$.

Proof. (a). Proof is similar to the proof of Theorem 3.1 part (a).

(b). Let $i \neq k$. We prove the cases (1) and (2) and the proof of the cases (3) and (4) are similar.

(1). Suppose that $j \in \{1, 2\}$ and $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$. We shall show that v_j^1MMDv_l^m. Note that $N(v_1^1) = \{v_2^1, v_{n_1}^1\}$ and $N(v_2^1) = \{v_3^1, v_1^1\}$. Then $P_1 := v_1^1 v_{n_1}^1 v_{n_1-1}^1 \ldots v_1^2 v_{n_2}^2 \ldots v_1^m v_{n_m}^m \ldots\ v^m_{\frac{n_m+1}{2}+2} v^m_{\frac{n_m+1}{2}+1}$ and $P_2 := v_2^1 v_3^1 v_4^1 \ldots v_1^2 v_{n_2}^2 \ldots v_1^m v_{n_m}^m \ldots v^m_{\frac{n_m+1}{2}+2} v^m_{\frac{n_m+1}{2}+1}$ are shortest path from v_1^1 to $v^m_{\frac{n_m+1}{2}+1}$ and from v_2^1 to $v^m_{\frac{n_m+1}{2}+1}$, respectively, of length $\sum_{i=1}^m \frac{n_i-1}{2} = s$. Thus from P_1 and P_2, we have

$$d(v_{n_1}^1, v^m_{\frac{n_m+1}{2}+1}) = d(v_3^1, v^m_{\frac{n_m+1}{2}+1}) = s - 1.$$

That is, v_j^1MD$v^m_{\frac{n_m+1}{2}+1}$ for $j \in \{1, 2\}$.
Again note that $N(v^m_{\frac{n_m+1}{2}+1}) = \{v^m_{\frac{n_m+1}{2}+2}, v^m_{\frac{n_m+1}{2}}\}$. Then
$Q_1 := v^m_{\frac{n_m+1}{2}+1}\ v^m_{\frac{n_m+1}{2}+2} \ldots v_1^m v^{m-1}_{\frac{n_{m-1}+1}{2}+2}\ v^{m-1}_{\frac{n_{m-1}+1}{2}+3} \ldots v_1^2 v^1_{\frac{n_1+1}{2}+2} \ldots v^1_{n_1} v_1^1$ and
$Q_2 := v^m_{\frac{n_m+1}{2}+1} v^m_{\frac{n_m+1}{2}+2} \ldots v_1^m v^{m-1}_{\frac{n_{m-1}+1}{2}+2} v^{m-1}_{\frac{n_{m-1}+1}{2}+3} \ldots v_1^2 v^1_{\frac{n_1+1}{2}} \ldots\ v_3^1 v_2^1$ are shortest path from $v^m_{\frac{n_m+1}{2}+1}$ to v_1^1 and from $v^m_{\frac{n_m+1}{2}+1}$ to v_2^1 of lenght s. Thus from Q_1 and Q_2, we have

$$d(v_1^1, v^m_{\frac{n_m+1}{2}+2}) = d(v_2^1, v^m_{\frac{n_m+1}{2}+2}) = s - 1. \tag{3.2}$$

Also, $Q_3 := v^m_{\frac{n_m+1}{2}} v^m_{\frac{n_m+1}{2}-1} \ldots v_1^m v^{m-1}_{\frac{n_{m-1}+1}{2}+2} v^{m-1}_{\frac{n_{m-1}+1}{2}+3} \ldots v_1^2 v^1_{\frac{n_1+1}{2}+2} \ldots v^1_{n_1} v_1^1$ and $Q_4 := v^m_{\frac{n_m+1}{2}} v^m_{\frac{n_m+1}{2}-1} \ldots v_1^m v^{m-1}_{\frac{n_{m-1}+1}{2}+2} v^{m-1}_{\frac{n_{m-1}+1}{2}+3} \ldots v_1^2 v^1_{\frac{n_1+1}{2}} \ldots v_3^1 v_2^1$ are shortest path from $v^m_{\frac{n_m+1}{2}}$ to v_1^1 and from $v^m_{\frac{n_m+1}{2}}$ to v_2^1 of lenght s. Thus from Q_3, Q_4 and equation (3.2), we have $v^m_{\frac{n_m+1}{2}+1}$MDv^1_j for $j \in \{1,2\}$. Hence v^1_jMMD$v^m_{\frac{n_m+1}{2}+1}$ for $j \in \{1,2\}$. Similarly, we can prove that v^1_jMMD$v^m_{\frac{n_m+1}{2}}$ for $j \in \{1,2\}$. Summing up, we have v^1_jMMDv^m_l for $j \in \{1,2\}$ and $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$.

Conversely, Suppose v^1_jMMDv^m_l then we show that $j \in \{1,2\}$ and $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$. On the contrary, we shall prove the following cases:

Case 1: $j \notin \{1,2\}$ but $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$,

Case 2: $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$ but $j \in \{1,2\}$,

Case 3: $j \notin \{1,2\}$ and $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$.

Case 1: First suppose that $j \notin \{1,2\}$ and $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$. Let $j \in \{3,4,\dots,\frac{n_1+1}{2}\}$. Note that $N(v^1_j) = \{v^1_{j-1}, v^1_{j+1}\}$. Then $v^1_j v^1_{j+1} v^1_{j+2} \dots v^2_1 v^2_{n_2} \dots v^m_1 v^m_{n_m} \dots v^m_{\frac{n_m+1}{2}+2} v^m_{\frac{n_m+1}{2}+1}$ is a shortest path from v^1_j to $v^m_{\frac{n_m+1}{2}+1}$ of length say r. But $v^1_{j-1} v^1_j v^1_{j+1} v^1_{j+2}$ $\dots v^2_1 v^2_{n_2} \dots v^m_1 v^m_{n_m} \dots v^m_{\frac{n_m+1}{2}+2} v^m_{\frac{n_m+1}{2}+1}$ is a shortest path from v^1_{j-1} to $v^m_{\frac{n_m+1}{2}+1}$ of length $r+1$. That is,

$$d(v^1_{j-1}, v^m_{\frac{n_m+1}{2}+1}) = r+1.$$

Thus v^1_j and $v^m_{\frac{n_m+1}{2}+1}$ are not MMD.

Now let $j \in \{\frac{n_1+1}{2}+2, \frac{n_1+1}{2}+3, \dots, n_1\}$. Note that $N(v^1_j) = \{v^1_{j-1}, v^1_{j+1}\}$. Then $v^1_j v^1_{j-1} v^1_{j-2} \dots v^2_1 v^2_{n_2} \dots v^m_1 v^m_{n_m} \dots v^m_{\frac{n_m+1}{2}+2} v^m_{\frac{n_m+1}{2}+1}$ is a shortest path from v^1_j to $v^m_{\frac{n_m+1}{2}+1}$ of length say r'. But $v^1_{j+1} v^1_j v^1_{j-1} v^1_{j-2} \dots v^2_1 v^2_{n_2} \dots v^m_1 v^m_{n_m} \dots v^m_{\frac{n_m+1}{2}+2} v^m_{\frac{n_m+1}{2}+1}$ is a shortest path from v^1_j to $v^m_{\frac{n_m+1}{2}+1}$ of length $r'+1$. That is,

$$d(v^1_{j+1}, v^m_{\frac{n_m+1}{2}+1}) = r'+1.$$

Thus v^1_j and $v^m_{\frac{n_m+1}{2}+1}$ are not MMD for $j \notin \{1,2\}$. Similarly, we can prove that v^1_j and $v^m_{\frac{n_m+1}{2}}$ are not MMD for $j \notin \{1,2\}$. Summing up, we have v^1_j and v^m_l are not MMD for $j \notin \{1,2\}$ and $l \in \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$.

Case 2: Secondly, suppose that $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$ and $j \in \{1,2\}$. Let $l \in \{2,3,\dots \frac{n_m+1}{2}-1\}$. Note that $N(v^m_l) = \{v^m_{l-1}, v^m_{l+1}\}$. Then $v^m_l v^m_{l-1} \dots v^m_1 v^{m-1}_{\frac{n_m+1}{2}+2} \dots v^2_1$

$v^1_{\frac{n_m+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^m_l to v^1_1 of length say t. But $v^m_{l+1}v^m_l v^m_{l-1}\dots v^m_1 v^{m-1}_{\frac{n_m+1}{2}+2}\dots v^2_1 v^1_{\frac{n_m+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^m_{l+1} to v^1_1 of length $t+1$. That is,

$$d(v^1_1, v^m_{l+1}) = t+1.$$

Thus v^m_l and v^1_1 are not MMD.

Now let $l \in \{\frac{n_m+1}{2}+2, \frac{n_m+1}{2}+3, \dots n_m\}$. Note that $N(v^m_l) = \{v^m_{l-1}, v^m_{l+1}\}$. Then $v^m_l v^m_{l+1}\dots v^m_1 v^{m-1}_{\frac{n_m+1}{2}+2}\dots v^2_1 v^1_{\frac{n_m+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^m_l to v^1_1 of length say t'. But $v^m_{l-1}v^m_l v^m_{l+1}\dots v^m_1 v^{m-1}_{\frac{n_m+1}{2}+2}\dots v^2_1 v^1_{\frac{n_m+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^m_{l-1} to v^1_1 of length $t'+1$. That is,

$$d(v^1_1, v^m_{l-1}) = t'+1.$$

This implies, v^m_l and v^1_1 are not MMD. Hence v^m_l and v^1_1 are not MMD for $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$. Similarly, we can prove that v^m_l and v^1_2 are not MMD for $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$. Summing up, we have v^m_l and v^1_2 are not MMD for $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$ and $j \in \{1,2\}$.

Case 3: Thirdly, suppose that $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$ and $j \notin \{1,2\}$. Then the proof is straigt forward from the cases $l \notin \{\frac{n_m+1}{2}, \frac{n_m+1}{2}+1\}$ or $j \notin \{1,2\}$. This completes the proof of (1).

(2). Suppose that $j \in \{1,2\}$ and $l = \frac{n_k+1}{2}$. We shall show that $v^1_j\text{MMD}v^k_l$. Note that $N(v^1_1) = \{v^1_2, v^1_{n_1}\}$ and $N(v^1_2) = \{v^1_3, v^1_1\}$. Then

$P_3 := v^1_1 v^1_{n_1} v^1_{n_1-1}\dots v^2_1 v^2_{n_2}\dots v^k_1 v^k_2\dots v^k_{\frac{n_k+1}{2}-1}v^k_{\frac{n_k+1}{2}}$ and

$P_4 := v^1_2 v^1_3 v^1_4\dots\ v^2_1 v^2_{n_2}\dots v^k_1 v^k_2\dots v^k_{\frac{n_k+1}{2}-1}v^k_{\frac{n_k+1}{2}}$ are shortest path from v^1_1 to $v^k_{\frac{n_k+1}{2}}$ and from v^1_2 to $v^k_{\frac{n_k+1}{2}}$, respectively, of length $\sum_{i=1}^k \frac{n_i-1}{2} = s$. Thus from P_3 and P_4, we have

$$d(v^1_{n_1}, v^k_{\frac{n_k+1}{2}}) = d(v^1_3, v^k_{\frac{n_k+1}{2}+1}) = s-1.$$

That is, $v^1_j \text{MD} v^k_{\frac{n_k+1}{2}}$ for $j \in \{1,2\}$ and $l = \frac{n_k+1}{2}$.
Again note that $N(v^k_{\frac{n_k+1}{2}}) = \{v^k_{\frac{n_k+1}{2}+1}, v^k_{\frac{n_k+1}{2}-1}\}$. Then
$R_1 := v^k_{\frac{n_k+1}{2}} v^k_{\frac{n_k+1}{2}-1} \dots v^m_1 v^{m-1}_{\frac{n_{m-1}+1}{2}+2} v^{m-1}_{\frac{n_{m-1}+1}{2}+3} \dots v^2_1 v^1_{\frac{n_1+1}{2}+2} \dots v^1_{n_1} v^1_1$ and
$R_2 := v^k_{\frac{n_k+1}{2}} v^k_{\frac{n_k+1}{2}-1} \dots v^k_1 v^{k-1}_{\frac{n_{k-1}+1}{2}+2} v^{k-1}_{\frac{n_{k-1}+1}{2}+3} \dots v^2_1 v^1_{\frac{n_1+1}{2}} \dots v^1_3 v^1_2$ are shortest path from $v^k_{\frac{n_k+1}{2}}$ to v^1_1 and from $v^k_{\frac{n_k+1}{2}}$ to v^1_2 of lenght s. Thus from R_1 and R_2, we have

$$d(v^1_1, v^k_{\frac{n_k+1}{2}-1}) = d(v^1_2, v^k_{\frac{n_k+1}{2}-1}) = s - 1. \tag{3.3}$$

Also, $R_3 := v^k_{\frac{n_k+1}{2}+1} v^k_{\frac{n_k+1}{2}+2} \dots v^k_1 v^{k-1}_{\frac{n_{k-1}+1}{2}+2} v^{k-1}_{\frac{n_{k-1}+1}{2}+3} \dots v^2_1 v^1_{\frac{n_1+1}{2}+2} \dots v^1_{n_1} v^1_1$ and $R_4 := v^k_{\frac{n_k+1}{2}+1} v^k_{\frac{n_k+1}{2}+2} \dots v^k_1 v^{k-1}_{\frac{n_{k-1}+1}{2}+2} v^{k-1}_{\frac{n_{k-1}+1}{2}+3} \dots v^2_1 v^1_{\frac{n_1+1}{2}} \dots v^1_3 v^1_2$ are shortest path from $v^k_{\frac{n_k+1}{2}+1}$ to v^1_1 and from $v^k_{\frac{n_k+1}{2}+1}$ to v^1_2 of lenght s. Thus from R_3, R_4 and equation (3.3), we have $v^k_{\frac{n_k+1}{2}} \text{MD} v^1_j$ for $j \in \{1,2\}$. Hence $v^1_j \text{MMD} v^k_{\frac{n_k+1}{2}}$ for $j \in \{1,2\}$. Thus we have $v^1_j \text{MMD} v^m_l$ for $j \in \{1,2\}$ and $l = \frac{n_m+1}{2}$.

Conversely, Suppose $v^1_j \text{MMD} v^k_l$ and we show that $j \in \{1,2\}$ and $l = \frac{n_k+1}{2}$. On the contrary, we shall prove the following cases:

Case 1: $j \notin \{1,2\}$ but $l = \frac{n_k+1}{2}$,

Case 2: $l \neq \frac{n_k+1}{2}$ but $j \in \{1,2\}$,

Case 3: $j \notin \{1,2\}$ and $l \neq \frac{n_k+1}{2}$.

Case 1: First suppose that $j \notin \{1,2\}$ and $l = \frac{n_k+1}{2}$. Let $j \in \{3,4,\dots,\frac{n_1+1}{2}\}$. Note that $N(v^1_j) = \{v^1_{j-1}, v^1_{j+1}\}$. Then $v^1_j v^1_{j+1} v^1_{j+2} \dots v^2_1\ v^2_{n_2} \dots v^k_1 v^k_2 \dots v^k_{\frac{n_k+1}{2}-1} v^k_{\frac{n_k+1}{2}}$ is a shortest path from v^1_j to $v^k_{\frac{n_k+1}{2}}$ of length say x. But
$v^1_{j-1} v^1_j v^1_{j+1} v^1_{j+2} \dots v^2_1 v^2_{n_2} \dots v^k_1 v^k_2 \dots v^k_{\frac{n_k+1}{2}-1} v^{k\,\frac{n_k+1}{2}}$ is a shortest path from v^1_{j-1} to $v^k_{\frac{n_k+1}{2}}$ of length $x+1$. That is,

$$d(v^1_{j-1}, v^k_{\frac{n_k+1}{2}}) = x + 1.$$

Thus v^1_j and $v^k_{\frac{n_k+1}{2}}$ are not MMD.
Now let $j \in \{\frac{n_1+1}{2}+2, \frac{n_1+1}{2}+3 \dots, n_1\}$. Note that $N(v^1_j) = \{v^1_{j-1}, v^1_{j+1}\}$. Then $v^1_j v^1_{j-1} v^1_{j-2} \dots v^2_1 v^2_{n_2}\ \dots v^k_1 v^k_2 \dots v^k_{\frac{n_k+1}{2}-1} v^k_{\frac{n_k+1}{2}}$ is a shortest path from v^1_j to $v^k_{\frac{n_k+1}{2}}$ of

length say x'. But $v^1_{j+1}v^1_jv^1_{j-1}v^1_{j-2}\dots v^2_1v^2_{n_2}\dots v^k_1v^k_2\dots\ v^k_{\frac{n_k+1}{2}-1}v^k_{\frac{n_k+1}{2}}$ is a shortest path from v^1_j to $v^k_{\frac{n_k+1}{2}}$ of length $x'+1$. That is,

$$d(v^1_{j+1}, v^k_{\frac{n_k+1}{2}}) = x' + 1.$$

Thus v^1_j and $v^k_{\frac{n_k+1}{2}}$ are not MMD for $j \notin \{1,2\}$. Summing up, we have v^1_j and v^m_l are not MMD for $j \notin \{1,2\}$ and $l = \frac{n_k+1}{2}$.

Case 2: Secondly, suppose that $l \neq \frac{n_k+1}{2}$ and $j \in \{1,2\}$. Let $l \in \{2,3,\dots \frac{n_k+1}{2}-1\}$. Note that $N(v^k_l) = \{v^k_{l-1}, v^k_{l+1}\}$. Then $v^k_lv^k_{l-1}\dots v^k_1\ v^{k-1}_{\frac{n_k+1}{2}+2}\dots v^2_1v^1_{\frac{n_k+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^m_l to v^1_1 of length say y. But $v^k_{l+1}v^k_lv^k_{l-1}\dots v^k_1v^{k-1}_{\frac{n_k+1}{2}+2}\dots v^2_1v^1_{\frac{n_k+1}{2}+2}$ $\dots v^1_{n_1}v^1_1$ is a shortest path from v^k_{l+1} to v^1_1 of length $y+1$. That is,

$$d(v^1_1, v^k_{l+1}) = y + 1.$$

Thus v^k_l and v^1_1 are not MMD.

Now let $l \in \{\frac{n_m+1}{2}+2, \frac{n_m+1}{2}+3, \dots n_k\}$. Note that $N(v^k_l) = \{v^k_{l-1}, v^k_{l+1}\}$. Then $v^k_lv^k_{l+1}\dots v^k_1v^{k-1}_{\frac{n_k+1}{2}+2}\ \dots v^2_1v^1_{\frac{n_k+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^k_l to v^1_1 of length say y'. But $v^k_{l-1}v^k_lv^k_{l+1}\dots v^k_1v^{k-1}_{\frac{n_k+1}{2}+2}\ \dots v^2_1v^1_{\frac{n_k+1}{2}+2}\dots v^1_{n_1}v^1_1$ is a shortest path from v^k_{l-1} to v^1_1 of length $y'+1$. That is,

$$d(v^1_1, v^k_{l-1}) = y' + 1.$$

This implies, v^k_l and v^1_1 are not MMD. Hence v^k_l and v^1_1 are not MMD for $l \neq \frac{n_k+1}{2}$. Summing up, we have v^m_l and v^1_j are not MMD for $l \neq \frac{n_k+1}{2}$ and $j \in \{1,2\}$.

Case 3: Thirdly, let $j \notin \{1,2\}$ and $l \neq \frac{n_k+1}{2}$. Then the proof is straigt forward from the cases $j \notin \{1,2\}$ or $l \neq \frac{n_k+1}{2}$. This completes the proof of (2). □

From Theorem 3.3 (a), for all $i \in \{2,3,\dots,m-1\}$, we have

$$A_1 = \{v^1_jv^1_{j+\frac{n_1-1}{2}} \mid j = 1,2,\dots,\tfrac{n_1+1}{2}, \tfrac{n_1+1}{2}+2, \dots n_1\} \subseteq E(C_{SR}), \tag{3.4}$$

$$A_2 = \{v^m_jv^m_{j+\frac{n_m-1}{2}} \mid j = 2,3,\dots,n_m\} \subseteq E(C_{SR}), \tag{3.5}$$

$$A_3 = \{v^i_jv^i_{j+\frac{n_i-1}{2}} \mid j = 2,3,\dots,\tfrac{n_i+1}{2}, \tfrac{n_i+1}{2}+2, \dots n_i\} \subseteq E(C_{SR}). \tag{3.6}$$

From Theorem 3.3 (b), for all $i, k \in \{2, 3, \ldots, m-1\}$, we have

$$B_1 = \{v_j^1 v_l^m \mid j = 1, 2. \text{ and } l = \tfrac{n_m+1}{2}, \tfrac{n_m+1}{2} + 1.\} \subseteq E(C_{SR}), \tag{3.7}$$

$$B_2 = \{v_j^1 v_{\frac{n_k+1}{2}}^k \mid j = 1, 2.\} \subseteq E(C_{SR}), \tag{3.8}$$

$$B_3 = \{v_2^i v_l^m \mid l = \tfrac{n_m+1}{2}, \tfrac{n_m+1}{2} + 1.\} \subseteq E(C_{SR}), \tag{3.9}$$

$$B_4 = \{v_2^i v_{\frac{n_k+1}{2}}^k\} \subseteq E(C_{SR}). \tag{3.10}$$

Thus from (3.4)∼(3.10), we have $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup B_4 = E(C_{SR})$.

Note that the set of edges A_1 form a path P_{n_1-1} with initial vertex v_1^1 and final vertex v_2^1, that is,

$$P_{n_1-1} := v_1^1 v_{1+\lfloor\frac{n_1}{2}\rfloor}^1 v_{1+2\lfloor\frac{n_1}{2}\rfloor}^1 \ldots v_{1+(n_1-2)\lfloor\frac{n_1}{2}\rfloor}^1,$$

where $1 + (n_1 - 2)\lfloor\frac{n_1}{2}\rfloor \equiv 2 \ (mod\ n_1)$.

Similarly, the set of edges A_2 form a path P_{n_m-1} with initial vertex $v_{\frac{n_m+1}{2}}^m$ and final vertex $v_{\frac{n_m+1}{2}+1}^m$, that is,

$$P_{n_m-1} := v_{\frac{n_m+1}{2}}^m v_{\frac{n_m+1}{2}+\lfloor\frac{n_m}{2}\rfloor}^m v_{\frac{n_m+1}{2}+2\lfloor\frac{n_m}{2}\rfloor}^m \ldots v_{\frac{n_m+1}{2}+(n_m-2)\lfloor\frac{n_m}{2}\rfloor}^m,$$

where $\frac{n_m+1}{2} + (n_m - 2)\lfloor\frac{n_m}{2}\rfloor \equiv \frac{n_m+1}{2} + 1 \ (mod\ n_m)$.

Also, the set of edges A_3 form $m-2$ paths P_{n_i-2}, $i \in \{2, 3, \ldots, m-1\}$, with initial vertex v_2^i and final vertex $v_{\frac{n_i+1}{2}}^i$, that is,

$$P_{n_i-2} := v_2^i v_{2+\lceil\frac{n_i}{2}\rceil}^i v_{2+2\lceil\frac{n_i}{2}\rceil}^i \ldots v_{2+(n_i-3)\lceil\frac{n_i}{2}\rceil}^i,$$

where $2 + (n_i - 3)\lceil\frac{n_i}{2}\rceil \equiv \frac{n_i+1}{2} \ (mod\ n_i)$.

Lemma 3.4. *Let $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ ba a chain cycle constructed by odd cycles with respect to the vertices $\{v_{\frac{n_i+1}{2}+1}^i, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$ and each $n_i \geq 5$. Then $\alpha(\mathcal{C}_{SR}(C_{n_1}, C_{n_2}\ , \ldots, C_{n_m}) = m - 1 + \lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_m}{2}\rfloor + \sum_{i=2}^{m-1}\lfloor\frac{n_i-2}{2}\rfloor$.*

Proof. We construct a vertex cover of $\mathcal{C}_{SR}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ with minimum cardinality. Note that the m vertices v_1^1 and v_2^i, $i \in \{2, 3, \ldots, m-1\}$, are nonadjacent vertices in $\mathcal{C}_{SR}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ and cover all edges of sets B_1, B_2, B_3 and B_4. Let $S = \{v_1^1, v_2^i \mid i = 2, 3, \ldots, m-1\}$. In order to cover the edges of path P_{n_1-1} we need $\frac{n_1-1}{2}$ vertices. Since $v_1^1, v_2^1 \in S$, therefore we must take $\frac{n_1-1}{2}-1$ more vertices of P_{n_1-1}. Thus we take the vertices $v^1_{1+2\lfloor\frac{n_1}{2}\rfloor}, v^1_{1+4\lfloor\frac{n_1}{2}\rfloor}, \ldots, v^1_{1+(n_1-3)\lfloor\frac{n_1}{2}\rfloor}$ in S. That is, we augment the set S by taking $S := S \cup \{v^1, v^1_{1+2\lfloor\frac{n_1}{2}\rfloor}, v^1_{1+4\lfloor\frac{n_1}{2}\rfloor}, \ldots, v^1_{1+(n_1-3)\lfloor\frac{n_1}{2}\rfloor}\}$. Similarly, to cover the edges of the path P_{n_m-1}, we must take $\frac{n_m-1}{2}$ vertices of P_{n_m-1}. Thus we take the vertices $v^m_{\frac{n_m+1}{2}}, v^m_{\frac{n_m+1}{2}+2\lfloor\frac{n_m}{2}\rfloor}, v^m_{\frac{n_m+1}{2}+4\lfloor\frac{n_m}{2}\rfloor}, \ldots, v^m_{\frac{n_m+1}{2}+(n_m-3)\lfloor\frac{n_m}{2}\rfloor}$ in S. That is, we again augment the set S by taking $S := S \cup \{v^m_{\frac{n_m+1}{2}}, v^m_{\frac{n_m+1}{2}+2\lfloor\frac{n_m}{2}\rfloor}, v^m_{\frac{n_m+1}{2}+4\lfloor\frac{n_m}{2}\rfloor}, \ldots, v^m_{\frac{n_m+1}{2}+(n_m-3)\lfloor\frac{n_m}{2}\rfloor}\}$. Finally, to cover the edges of each path P_{n_i-2} we must take $\lceil\frac{n_i-2}{2}\rceil$ vertices of P_{n_i-1}, $i \in \{2, 3, \ldots, m-1\}$. Since $v_2^i \in S$, therefore we take $\lfloor\frac{n_i-2}{2}\rfloor$ more vertices of P_{n_i-1}, $i \in \{2, 3, \ldots, m-1\}$. Thus we take the vertices $v^i_{2+2\lceil\frac{n_i}{2}\rceil}, v^i_{2+4\lceil\frac{n_i}{2}\rceil}, \ldots, v^i_{2+(n_i-3)\lceil\frac{n_i}{2}\rceil}$, $i \in \{2, 3, \ldots, m-2\}$ in S. That is, we again augment the set S by taking $S := S \cup \{v^i_{2+2\lceil\frac{n_i}{2}\rceil}, v^i_{2+4\lceil\frac{n_i}{2}\rceil}, \ldots, v^i_{2+(n_i-3)\lceil\frac{n_i}{2}\rceil} \mid i = 2, 3, \ldots, m-2\}$. Thus S is the vertex cover of the strong resolving graph of the chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ with minimum cardinality $|S| = m - 1 + \lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_m}{2}\rfloor + \sum_{i=2}^{m-1}\lfloor\frac{n_i-2}{2}\rfloor$. □

Theorem 3.4. *Let* $\{C_{n_i}\}_{i=1}^m$ *be* m *disjoint cycles with each* n_i *is odd and* $n_i \geq 5$, *then* $sdim(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) = m - 1 + \lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_m}{2}\rfloor + \sum_{i=2}^{m-1}\lfloor\frac{n_i-2}{2}\rfloor$.

Proof. The proof follows from Lemma 3.4 and Theorem 1.1. □

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References

[1] S. Akhter, R. Farooq, Metric dimension of fullerene graphs, *Electronic Journal of Graph Theory and Applications*, **7(1)** (2019), 87–99.

[2] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Math.*, **59** (2000), 45–54.

[3] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99–113.

[4] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, **2** (1976), 191–195.

[5] H. Iswadi, E. T. Baskoro, A. N. M. Salman, R. Simanjuntak, The metric dimension of amalgamation of cycles, *Far East J. Math. Sci.*, **41(1)** (2010), 19–31.

[6] J. A. Rodríguez-Velázquez, I. G. Yero, M. Lemanska, On the partition dimension of trees, *Discrete Appl. Math.*, **166** (2014), 204–209.

[7] D. Kuziak, Strong resolvability in product graphs, Doctoral Thesis, Universitat Rovira i Virgili: The public university of Tarragona, Spain, 2014.

[8] D. Kuziak, J. A. Rodríguez-Velázquez, I. G. Yero, Ismael G, Computing the metric dimension of a graph from primary subgraphs, *Discuss. Math. Graph Theory*, **37** (2017), 273–293.

[9] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Closed formulae for the strong metric dimension of lexicographic product graphs, *Discuss. Math. Graph Theory*, **36** (2016), 1051–1064.

[10] T. Mansour, M. Schork , The PI index of bridge and chain graphs, *Match*, **61** (2009), 723.

[11] N. Mehreen, R. Farooq, S. Akhter, On partition dimension of fullerene graphs, *AIMS Mathematics*, **3** (2018), 343–352.

[12] Nilanjan De, Hyper Zagreb Index of Bridge and Chain Graphs, *arXiv: 1703.08325*, 2017.

[13] O. R. Oellermann, J. Peters-Fransen, The strong metric dimension of graphs and digraphs, *Discrete Appl. Math.*, **155** (2007), 356–364.

[14] J.A. Rodríguez-Velázquez, I. G. Yero, D. Kuziak, O. R. Oellermann, On the strong metric dimension of Cartesian and direct products of graphs, *Discrete Math.*, **335** (2014), 8–19.

[15] J. A. Rodríguez-Velázquez, I. G. Yero, H. Fernau, On the partition dimension of unicyclic graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, **57(4)** (2014), 381–391.

[16] A. Sebő, E. Tannier, On metric generators of graphs, *Math. Oper. Res.* **29** (2004), 383–393.

[17] I. Tomescu, Discrepancies between metric dimension and partition dimension of a connected graph, *Discrete Math.*, **308** (2008), 5026–5031.

[18] I. Tomescu, I. Javaid, Slamin, On the partition dimension and connected partition dimension of wheels, *Ars Combin.*, **84** (2007), 311–317.

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