Jordan Journal of Mathematics and Statistics (JJMS) 13(3), 2020, pp 389 - 399

FUNDAMENTAL RESULTS ON SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we analyze the solutions of a linear system of fractional differential equations involving the Caputo-Fabrizio fractional derivative. We first transform the system to a equivalent system of integro-differential equations with integer derivative. We then establish a uniqueness result for the system of fractional differential equations and present a necessary condition to guarantee the existence of a solution. Moreover, if the solution exists, the unique solution of the fractional system is obtained explicitly and is given in a closed form. Two examples are presented to illustrate the validity of the obtained results.

1. Introduction

In this paper, we consider the following linear system of fractional differential equations

$$\frac{d^\alpha}{dt^\alpha}\mathbf{Y}(t) = A\mathbf{Y}(t) + \mathbf{G}(t),\ t > a,\ \mathbf{Y}(a) = \mathbf{Y}_0, \tag{1.1}$$

where $\mathbf{Y}, \mathbf{G} \in \mathbb{R}^n, A \in \mathbb{R}^n \times \mathbb{R}^n, 0 < \alpha < 1$, and $\frac{d^\alpha}{dt^\alpha} = {}^{CFC}D_a^\alpha$ is the Caputo-Fabrizio fractional derivative of Caputo sense.

Recently, several types of non-local fractional derivatives with non-singular kernel have been introduced [7, 16]. The role of applications of these types of fractional

Key words and phrases. Systems of fractional differential equations, Caputo-Fabrizio fractional derivative.

Received: Feb. 11, 2019 Accepted: July 28, 2019 .

derivatives is indicated by several authors, see [14, 17]. Since the theory of fractional models is effected by the type of the fractional derivative, several papers have been devoted to study the new types of fractional models, see [3, 8, 9, 15]. Also, it is been noted that certain classes of fractional models involving the Caputo-Fabrizio fractional derivative can be transformed to integro-differential models of especial types, see [10, 11, 12, 13].

This paper is devoted to study linear systems of fractional differential equations involving the Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$. While there are some studies on systems of fractional differential equations, see [1, 18, 20, 23], to the best of our knowledge, this is the first study on fractional systems involving the Caputo-Fabrizio fractional derivative. We start with the definition and main properties of the Caputo-Fabrizio fractional derivative.

Definition 1.1. *For $p \in [1, \infty]$ and Ω an open subset of $\mathbb{R}$, the Sobolev space $H^p(\Omega)$ is defined by*

$$H^p(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq p\}.$$

Definition 1.2. [16] *Let $f \in H^1(a, b)$, $a < b$, $a \in (-\infty, t)$, $0 < \alpha < 1$, the left Caputo-Fabrizio fractional derivative in the Caputo sense is defined by*

$$({}^{CFC}D_a^\alpha f)(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t f'(s) e^{-\frac{\alpha}{1-\alpha}(t-s)} ds, \tag{1.2}$$

where $N(\alpha) > 0$ is a normalization function satisfying $N(0) = N(1) = 1$.

The corresponding fractional integral is defined by, see [22]

$$({}^{CFC}I_a^\alpha f)(t) = \frac{1-\alpha}{N(\alpha)} f(t) + \frac{\alpha}{N(\alpha)} \int_a^t f(s) ds, \quad 0 < \alpha < 1. \tag{1.3}$$

The relation between the Caputo-Fabrizio fractional derivative and the corresponding integral is given by

$$({}^{CFC}I_a^\alpha)({}^{CFC}D_a^\alpha) f(t) = f(t) - f(a). \tag{1.4}$$

For more about the Caputo-Fabrizio fractional derivatives we refer the readers to [2, 4, 16, 19, 22]. Let $B(\alpha) = \frac{N(\alpha)}{1-\alpha}$, and $\mu_\alpha = \frac{\alpha}{1-\alpha}$, then

(1.5)
$$({}^{CFC}D_a^\alpha y)(t) = B(\alpha)\int_a^t y'(s)e^{-\mu_\alpha(t-s)}ds = B(\alpha)e^{-\mu_\alpha t}\int_a^t y'(s)e^{\mu_\alpha s}ds,\ 0<\alpha<1.$$

2. Main Results

In this section, we present a necessary condition to guarantee the existence of a solution to the system (1.1) and obtain a closed formula of the solution, if it exists. We start with the following result in which we transform the fractional system to a equivalent system of integro-differential equations. We have

Theorem 2.1. *Consider the linear system in (1.1) with $y_i \in H^1(a,b)$, and let*

$$v_i(t) = \int_a^t e^{\mu_\alpha s} y_i(s)ds,\ i = 1,2,\cdots,n. \tag{2.1}$$

Then the system (1.1) is equivalent to the system of first order differential equations

$$D\ \mathbf{V}' = -\mu_\alpha B(\alpha)\mathbf{V} - \mathbf{H}(\mathbf{t}),\ \mathbf{V}(a) = 0, \tag{2.2}$$

where

$$D = A - B(\alpha)I,\quad \mathbf{H}(\mathbf{t}) = B(\alpha)e^{\mu_\alpha a}\mathbf{Y}(\mathbf{a}) + e^{\mu_\alpha t}\mathbf{G}(\mathbf{t}) \text{ and } V(t) = (v_1(t),\cdots,v_n(t)).$$

Proof. Integration by parts of $({}^{CFC}D_a^\alpha y_i)(t)$ yields

$$\begin{aligned} ({}^{CFC}D_a^\alpha y_i)(t) &= B(\alpha)e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s} y_i'(s)ds \\ &= B(\alpha)e^{-\mu_\alpha t}\left(e^{\mu_\alpha s}y_i(s)|_a^t - \mu_\alpha\int_a^t e^{\mu_\alpha s}y_i(s)ds\right) \\ &= B(\alpha)\left(y_i(t) - y_i(a)e^{-\mu_\alpha(t-a)} - \mu_\alpha e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s}y_i(s)ds\right). \end{aligned}$$

Since $v_i(t) = \int_a^t e^{\mu_\alpha s}y_i(s)ds$, and y_i is continuous, we have

$$v_i'(t) = e^{\mu_\alpha t}y_i(t), \text{or } y_i(t) = e^{-\mu_\alpha t}v_i'(t). \tag{2.3}$$

Thus,

$$({}^{CFC}D_a^{\alpha}y_i)(t) \quad = \quad B(\alpha)e^{-\mu_\alpha t}\Big(v_i'(t) - e^{\mu_\alpha a}y_i(a) - \mu_\alpha v_i(t)\Big). \tag{2.4}$$

We have the system (1.1)

$$\begin{aligned}
{}^{CFC}D_a^{\alpha}y_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + g_1(t),\\
{}^{CFC}D_a^{\alpha}y_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + g_2(t),\\
&\vdots\\
{}^{CFC}D_a^{\alpha}y_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + g_n(t).
\end{aligned}$$

Using the results in Eq.'s (2.3)-(2.4), we have

$$\begin{aligned}
B(\alpha)e^{-\mu_\alpha t}\left(v_1' - e^{\mu_\alpha a}y_1(a) - \mu_\alpha v_1\right) &= e^{-\mu_\alpha t}\left(a_{11}v_1' + a_{12}v_2' + \cdots + a_{1n}v_n'\right) + g_1(t),\\
B(\alpha)e^{-\mu_\alpha t}\left(v_2' - e^{\mu_\alpha a}y_2(a) - \mu_\alpha v_2\right) &= e^{-\mu_\alpha t}\left(a_{21}v_1' + a_{22}v_2' + \cdots + a_{2n}v_n'\right) + g_2(t),\\
&\vdots\\
B(\alpha)e^{-\mu_\alpha t}\left(v_n' - e^{\mu_\alpha a}y_n(a) - \mu_\alpha v_n\right) &= e^{-\mu_\alpha t}\left(a_{n1}v_1' + a_{n2}v_2' + \cdots + a_{nn}v_n'\right) + g_n(t).
\end{aligned}$$

The above system is reduced to

$$\begin{aligned}
\big(a_{11} - B(\alpha)\big)v_1' + a_{12}v_2' + \cdots + a_{1n}v_n' &= -\mu_\alpha B(\alpha)v_1 - B(\alpha)e^{\mu_\alpha a}y_1(a) - e^{\mu_\alpha t}g_1(t),\\
a_{21}v_1' + \big(a_{22} - B(\alpha)\big)v_2' + \cdots + a_{2n}v_n' &= -\mu_\alpha B(\alpha)v_2 - B(\alpha)e^{\mu_\alpha a}y_2(a) - e^{\mu_\alpha t}g_2(t),\\
&\vdots\\
a_{n1}v_1' + a_{n2}v_2' + \cdots + \big(a_{nn} - B(\alpha)\big)v_n' &= -\mu_\alpha B(\alpha)v_n - B(\alpha)e^{\mu_\alpha a}y_n(a) - e^{\mu_\alpha t}g_n(t),
\end{aligned}$$

which proves the result. □

Proposition 2.1. *A necessary condition for the existence of a solution to the system (1.1) is that*

$$AY(a) + G(a) = 0, \tag{2.5}$$

where $\mathbf{Y} = (y_i)_{i=1}^n$, *and* $y_i \in H^1(a,b) \cap C[a,b]$.

Proof. Since $({}^{CFC}D_a^{\alpha} y_i)(a_i) = 0$, see [8], then $\left(\frac{d^\alpha}{dt^\alpha}\mathbf{Y}\right)(a) = \underline{0}$, and the result follows by the continuity of a solution to (1.1). □

Remark 2.1. *Analogous conditions were discussed in the case of Atangana-Baleanu fractional derivative, see* [5, 6, 21]. *The author in* [6] *generalized the Mittag-Leffler kernel to remove the necessary condition (2.5), and such that the homogenous equation has a nontrivial solution.*

Proposition 2.2. *Consider the homogeneous system (1.1) with* $\mathbf{G}(t) = \mathbf{0}$. *If* A *is invertible then it has only the zero solution.*

Proof. Since $\mathbf{G}(a) = \mathbf{0}$, and A is invertible by the necessary condition in Eq. (2.5) we have $\mathbf{Y}(a) = \mathbf{0}$. Then by Eq. (2.2) the problem is equivalent to the homogeneous system

$$D\mathbf{v}' = -\mu_\alpha B(\alpha)\mathbf{V},\ \ \mathbf{V}(a) = \mathbf{0}. \tag{2.6}$$

Applying the Laplace transform to the above system and using the fact that $\mathbf{V}(a) = \mathbf{0}$, we have

$$\Big(sD + \mu_\alpha B(\alpha) I\Big) L(\mathbf{V}) = \mathbf{0}.$$

Since the matrix $sD + \mu_\alpha B(\alpha)I$, is nonsingular, then the above system has only the zero solution $L(\mathbf{V}) = \mathbf{0}$. By the continuity of $\mathbf{V}$, we have $\mathbf{V} = \mathbf{0}$. Thus the system (2.6) possesses only the zero solution and hence the result. □

Proposition 2.3. *Consider the system of first order differential equations*

$$\mathbf{Z} = A\mathbf{Z}' + \mathbf{R}(t), \tag{2.7}$$

where $\mathbf{Z}, \mathbf{G} \in \mathbb{R}^n$, *and* $A \in \mathbb{R}^n \times \mathbb{R}^n$. *Let* $\lambda_1, \cdots, \lambda_n$, *be the eigenvalues of* A *and* $\mathbf{X}_1, \cdots, \mathbf{X}_n$ *be the corresponding n-linearly independent eigenvectors of* A. *Let* $\mathbf{Z}_p$ *be a particular solution to (2.7). If* $\lambda_i \neq 0, i = 1, \cdots, n$, *then the general solution to (2.7) is given by*

$$\mathbf{Z} = c_1 e^{\frac{1}{\lambda_1}t}\mathbf{X}_1 + c_2 e^{\frac{1}{\lambda_2}t}\mathbf{X}_2 \cdots + c_n e^{\frac{1}{\lambda_n}t}\mathbf{X}_n + \mathbf{Z}_p. \tag{2.8}$$

Proof. We have

$$\mathbf{Z}' = \frac{c_1}{\lambda_1} e^{\frac{1}{\lambda_1}t}\mathbf{X}_1 + \frac{c_2}{\lambda_2} e^{\frac{1}{\lambda_2}t}\mathbf{X}_2 \cdots + \frac{c_n}{\lambda_n} e^{\frac{1}{\lambda_n}t}\mathbf{X}_n + \mathbf{Z}_p,$$

and

$$\begin{aligned} A\mathbf{Z}' &= \frac{c_1}{\lambda_1} e^{\frac{1}{\lambda_1}t}A\mathbf{X}_1 + \frac{c_2}{\lambda_2} e^{\frac{1}{\lambda_2}t}A\mathbf{X}_2 \cdots + \frac{c_n}{\lambda_n} e^{\frac{1}{\lambda_n}t}A\mathbf{X}_n + A\mathbf{Z}'_p, \\ &= c_1 e^{\frac{1}{\lambda_1}t}\mathbf{X}_1 + c_2 e^{\frac{1}{\lambda_2}t}\mathbf{X}_2 \cdots + c_n e^{\frac{1}{\lambda_n}t}\mathbf{X}_n + A\mathbf{Z}'_p. \end{aligned}$$

Thus,

$$\begin{aligned} A\mathbf{Z}' + \mathbf{R}(t) &= c_1 e^{\frac{1}{\lambda_1}t}\mathbf{X}_1 + c_2 e^{\frac{1}{\lambda_2}t}\mathbf{X}_2 \cdots + c_n e^{\frac{1}{\lambda_n}t}\mathbf{X}_n + A\mathbf{Z}'_p + \mathbf{R}(t), \\ &= c_1 e^{\frac{1}{\lambda_1}t}\mathbf{X}_1 + c_2 e^{\frac{1}{\lambda_2}t}\mathbf{X}_2 \cdots + c_n e^{\frac{1}{\lambda_n}t}\mathbf{X}_n + \mathbf{Z}_p = \mathbf{Z}, \end{aligned}$$

which proves the result. □

As a consequence of the above result, we have the following existence and uniqueness result for the system (1.1).

Theorem 2.2. *Consider the system (1.1) with* $det(A) \neq 0$, *then the system has a unique solution given by Eq. (2.8).*

3. Illustrated Examples

In this section, we present two examples to illustrate the efficiency of the obtained results.

Example 3.1. *Consider the homogenous system (1.1) with*

$$A = \begin{bmatrix} B(\alpha) & B(\alpha)\mu_\alpha \\ B(\alpha)\mu_\alpha & B(\alpha) \end{bmatrix}, \ t > 0, \mathbf{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

We have $|A| = B(\alpha)^2\big(1-\mu_\alpha^2\big)$. *If* $\mu_\alpha \neq \mp 1$ *then* A *is invertible, and by Proposition 2.2 the system possesses only the zero solution. We now discuss the solution for* $\mu_\alpha = 1$, *or* $\alpha = \frac{1}{2}$. *The case* $\mu_\alpha = -1$ *is not valid as* $\mu_\alpha = \frac{\alpha}{1-\alpha}$. *The system is equivalent to the system of ordinary differential equations*

$$D\mathbf{V}' = -B(\alpha)\mathbf{V} - B(\alpha)\mathbf{Y}(0), \tag{3.1}$$

$$\mathbf{V}(0) = 0, \tag{3.2}$$

where $D = A - B(\alpha)I$. *Let*

$$E = -\frac{1}{B(\alpha)}D = -\frac{1}{B(\alpha)}(A - B(\alpha)I) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of E *are* 1 *and* -1 *and the corresponding eigenvectors are* $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *respectively. Thus, the solution of Eq. (3.1) is given by*

$$V(t) = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

The condition $\mathbf{V}(0) = 0$, *yields* $c_1 = \frac{y_1(0)-y_2(0)}{2}$, *and* $c_2 = \frac{y_1(0)+y_2(0)}{2}$. *The necessary condition in (2.5) yields* $y_2(0) = -y_1(0)$, *and thus* $c_1 = y_1(0)$, $c_2 = 0$, *and the general solution of* $\mathbf{V}(t)$ *is given by*

$$\mathbf{V}(t) = y_1(0)e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} y_1(0) \\ -y_1(0) \end{pmatrix} = y_1(0)(e^t - 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$\mathbf{Y} = e^{-\mu_\alpha t}\mathbf{V}' = e^{-t}\mathbf{V}' = y_1(0) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example 3.2. *Consider the system (1.1) with*

$$A = \begin{bmatrix} 1 & -2B(\alpha) \\ 0 & \frac{1}{2}B(\alpha) \end{bmatrix}, \mathbf{G}(t) = e^{-2t} \begin{pmatrix} -1 \\ B(\alpha)t(\frac{1}{2} - t) \end{pmatrix}, \mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and $\alpha = \frac{2}{3}$. *The above system is equivalent to the system of of first order differential equations*

$$D\ \mathbf{V}' = -2B(\alpha)\mathbf{V} - \mathbf{H}(\mathbf{t}),\ \ \mathbf{V}(0) = 0,$$

where

$$D = \begin{bmatrix} 1 - B(\alpha) & -2B(\alpha) \\ 0 & \frac{-B(\alpha)}{2} \end{bmatrix} \text{ and } \mathbf{H}(t) = \begin{pmatrix} B(\alpha) - 1 \\ B(\alpha)t(\frac{1}{2} - t) \end{pmatrix}.$$

The eigenvalues of the matrix $E = -\frac{1}{2B(\alpha)}D$, *are* $\frac{1}{4}$ *and* r, *and the corresponding eigenvectors are* $(1 \quad 1 - r)^t$, *and* $(1 \ \ 0)^t$, *where* $r = \frac{B(\alpha)-1}{2B(\alpha)}$. *The particular solution is given by*

$$\mathbf{V}_p = \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix}.$$

Thus the general solution is

$$\mathbf{V} = c_1 e^{\frac{1}{r}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ \frac{1}{4} - r \end{pmatrix} + \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix}.$$

The condition $\mathbf{V}(0) = 0$ *yields* $c_1 = c_2 = 0$, *and thus*

$$\mathbf{V}(t) = \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix}.$$

The solution $\mathbf{Y}$ *is given by*

$$\mathbf{Y}(t) = e^{-2t}\mathbf{V}'(t) = e^{-2t} \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

4. Concluding Remarks

We have established existence and uniqueness results for a system of fractional differential equations of order $0 < \alpha < 1$. The unique solution of the system is given in a closed form. The obtained results are of interests for many researchers as the Caputo-Fabrizio derivative is connected with many applications.

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