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GENERAL METHOD TO GENERATE FUZZY EQUIVALENCE RELATIONS IN MATRIX FORM

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Abstract. In this paper, new method to generate fuzzy equivalence relations in matrix form is considered, it is not easy to check fuzzy relation in matrix form if it is equivalence relation or not, and if it is transitive or not transitive. We start building fuzzy equivalence relation in matrix forms 3×3 and 4×4 matrices, then by using mathematical induction we will build general method that generates fuzzy equivalence relations of the form $n \times n$ matrices.

1. Introduction

Fuzzy relations played a key role in fuzzy control, fuzzy diagnosis, and fuzzy modeling. They also have been utilized in many applications in fields such as Economics, Medicine, Psychology, and Sociology. Relations are an appropriate method for characterizing correspondences between objects. The employ of fuzzy relations started from the observation that real-life objects can be linked to each other to a certain degree. However, in many cases, we may assume that a certain object X is in relation R with another object Y to a certain degree, but it is possible that we are not so sure about it. In other words, we may be uncertainty about the degree that is assigned to the relationship between X and Y [1]. Uncertain information is inherent and pervasive in many applications in the fields such as medical science, engineering,

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business management, and economics. Uncertain information in these fields could be caused by the randomness of information, incomplete data, and delayed information updates. The problem of uncertainty has been investigated for a long time by mathematicians. A great number of research papers have been published to represent these concepts mathematically [2]. The theory of fuzzy relations is a generalization of crisp relations of a set. The notion of fuzzy set introduced by Zadeh (1965) is a very successful approach to vagueness. Zadeh in [3] introduced the concept of fuzzy relations, he also introduced the concept of fuzzy similarity relations in [4]. This provided an impetus for research in this very important area. Many authors like Chakraborty and Das, Murali, Nemitz [5, 6, 7, 8] studied fuzzy equivalence relations [9].

Shakhatreh, M and Hayajenh, M. They gave some results on fuzzy equivalence relations [10], fuzzy equivalence classes and fuzzy partition [11], and defined S-H Partition [12]. To check the fuzzy relation of form 10×10 matrix that is transitive or not, according to the classic definition of transitivity, we need **1000** operations. In general, it is not easy to build fuzzy equivalence relation of form $n \times n$. In our paper, we aim to find a method that generates fuzzy equivalence relation in matrix form of size $n \times n$.

2. Preliminaries

In this section, we give some definitions and results which will be used later.

Definition 2.1 (Fuzzy Set, see [13], [15], [3])**.** if $\mathbf{X}$ is a collection of objects denoted generally by x, then a fuzzy set $\underset{\sim}{\mathbf{A}}$ in $\mathbf{X}$ is a set of orderd pairs :

$$\underset{\sim}{\mathbf{A}} = \left\{(x, \mu_{\underset{\sim}{\mathbf{A}}}(x)) : x \in \mathbf{X}\right\}, where\ \mu_{\underset{\sim}{\mathbf{A}}}(x)\ is\ the\ grade\ of\ membership\ of\ x\ in\ \underset{\sim}{\mathbf{A}}$$

Definition 2.2 (Fuzzy Relation, see [14])**.** Let X and Y be nonempty sets. Then $\underset{\sim}{\mathbf{R}} = \left\{\Big((x,y), \mu_{\underset{\sim}{\mathbf{R}}}(x,y)\Big) : (x,y) \in X \times Y\right\}$ is called a fuzzy relation in $X \times Y$.

Definition 2.3 (Reflexitivity, see [14])**.** Let $\underset{\sim}{\mathbf{R}}$ be a fuzzy relation in $\mathbf{X} \times \mathbf{X}$. Then $\underset{\sim}{\mathbf{R}}$ is called Reflexive if

$$\mu_{\underset{\sim}{\mathbf{R}}}(x, x) = 1, \forall x \in \mathbf{X}.$$

Definition 2.4 (semmetry,see [14])**.** Let $\underset{\sim}{\mathbf{R}}$ be a fuzzy relation in $\mathbf{X}\times Y$**.** Then $\underset{\sim}{\mathbf{R}}$ is called symmetric in $\mathbf{X} \times \mathbf{X}$

$$\text{if} \quad \mu_{\underset{\sim}{\mathbf{R}}}(x, y) = \mu_{\underset{\sim}{\mathbf{R}}}(y, x); \forall x, y \in \mathbf{X}$$

Definition 2.5 (Max-Min Transitive Fuzzy Relation, see [14])**.** A relation $\underset{\sim}{\mathbf{R}}$ is called Max-Min Transitive fuzzy relation

$$\text{if} \quad \mu_{\underset{\sim}{\mathbf{R}}}(x, z) \geq \max_{y} \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x, y), \mu_{\underset{\sim}{\mathbf{R}}}(y, z) \right\} \right\}$$

3. Fuzzy Equivalence Relations represented by a 3×3 Matrix

We begin with fuzzy relation of the form 3×3 matrix, since it is easy to check the fuzzy relation of the form 2×2 matrix whether it is fuzzy equivalence relation or not.

Theorem 3.1. *Let $\mathbf{X} = \{x_1, x_2, x_3\}$. Then a fuzzy equivalence relation $\underset{\sim}{\mathbf{R}}$ in $\mathbf{X} \times \mathbf{X}$ can be represented by a 3×3 matrix.*

Proof. To ensure reflexivity of $\underset{\sim}{\mathbf{R}}$, we must have $\mu_{\underset{\sim}{\mathbf{R}}}(x_i, x_i) = 1$, $\forall i = 1, 2, 3$. And this requires that we must have 1's along the main diagonal of the matrix. For our relation to be symmetric, we must have $\mu_{\underset{\sim}{\mathbf{R}}}(x_i, x_j) = \mu_{\underset{\sim}{\mathbf{R}}}(x_j, x_i), \forall x_i, x_j \in \mathbf{X}$, $i, j \in \{1, 2, 3\}$. Assume that:

$\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_2) = \mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_1) = a$

$\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_3) = \mu_{\underset{\sim}{\mathbf{R}}}(x_3, x_1) = b$

$\mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_3) = \mu_{\underset{\sim}{\mathbf{R}}}(x_3, x_2) = c$, where a, b, and $c \in [0, 1]$.

We can represent our relation as Table 1 below.

Table 1. Reflexive and symmetric fuzzy relation represented as a 3×3 matrix

$\overset{\frown}{\underset{\sim}{\mathbf{R}}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
$\mathbf{x}_1$	$\mathbf{1}$	a	b
$\mathbf{x}_2$	a	$\mathbf{1}$	c
$\mathbf{x}_3$	b	c	$\mathbf{1}$

$a, b, c \in [0, 1]$

Now, to make this fuzzy relation transitive, we need to fulfill the following requirement:

$$\mu_{\underset{\sim}{\mathbf{R}}}(x, z) \geq \max_{y} \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x, y), \mu_{\underset{\sim}{\mathbf{R}}}(y, z) \right\} \right\}$$

It is enough to discuss only the upper triangle (or lower triangle), since the fuzzy relation is reflexive and symmetric. Now notice that:

a. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{a}$

$$\min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_1), \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_2) \right\} = \min \{1, a\} = a \leq a$$

$$\min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_2), \mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_2) \right\} = \min \{a, 1\} = a \leq a$$

$$\min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_3), \mu_{\underset{\sim}{\mathbf{R}}}(x_3, x_2) \right\} = \min \{b, c\} \leq a \tag{3.1}$$

b. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_3) = \mathbf{b}$

$$\min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_1), \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_3) \right\} = \min \{1, b\} = b \leq b$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_3)\right\}=\min\{a,c\}\le b \tag{3.2}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_3)\right\}=\min\{b,1\}=b\le b$$

c. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_2,\mathbf{x}_3)=\mathbf{c}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_3)\right\}=\min\{a,b\}\le c \tag{3.3}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_3)\right\}=\min\{1,c\}=c\le c$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_3)\right\}=\min\{c,1\}=c\le c$$

From (3.1), (3.2), and (3.3) we have:

$\min\{b,c\}\le a$

$\min\{a,c\}\le b$

$\min\{a,b\}\le c$

$$\text{From (3.1) we have: } b\le c\Longrightarrow b\le a \tag{3.4}$$

$$\text{From (3.2) we have: } a\le c\Longrightarrow a\le b \tag{3.5}$$

From (3.4) and (3.5) we get: $a=b$, and hence:

$$\mathbf{0}\le\mathbf{a}=\mathbf{b}\le\mathbf{c}\le\mathbf{1}\ \text{(Type1)} \tag{3.6}$$

Table 2 below shows a fuzzy equivalence relations (Type 1) represented as a 3×3 matrix.

Table 2. Fuzzy Equivalence Relations represented as a 3×3 matrix (Type 1)

$\underset{\sim}{\mathbf{R}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
$\mathbf{x}_1$	$\mathbf{1}$	a	a
$\mathbf{x}_2$	a	$\mathbf{1}$	c
$\mathbf{x}_3$	a	c	$\mathbf{1}$

$0 \le a \le c \le 1$

□

4. Fuzzy Equivalence Relations represented by a 4×4 Matrix

To generate fuzzy equivalence relation of the form 4×4 matrix, we will use the generated fuzzy equivalence relation of the form 3×3 matrix and enlarging this matrix (Table 2) by adding a column and a row to the matrix. The fuzzy relation matrix of the form 4×4 is reflexive and symmetric fuzzy relation as shown in Table 3.

Theorem 4.1. *Let* $\mathbf{X} = \{x_1, x_2, x_3, x_4\}$ *. Then a fuzzy equivalence relation* $\underset{\sim}{\mathbf{R}}$ *in* $\mathbf{X} \times \mathbf{X}$ *can be represented by a* 4×4 *matrix.*

Proof. We can represent our relation (reflexive and symmetric Fuzzy Relation) as shown in Table 3. Now we discuss the transitivity of the $\mathbf{4} \times \mathbf{4}$ matrix given in Table 3. It is enough to discuss only the upper triangle (or lower triangle), since the fuzzy relation is reflexive and symmetric. We have:

a. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{a}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_1), \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_2)\right\} = \min\{1, a\} = a \le a$$

Table 3. Reflexive and Symmetric Fuzzy Relation represented as a $\mathbf{4} \times \mathbf{4}$ Matrix

$\overset{\frown}{\underset{\sim}{\mathbf{R}}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\mathbf{x}_1$	$\mathbf{1}$	a	a	d
$\mathbf{x}_2$	a	$\mathbf{1}$	c	e
$\mathbf{x}_3$	a	c	$\mathbf{1}$	f
$\mathbf{x}_4$	d	e	f	$\mathbf{1}$

$0 \leq a \leq c \leq 1$

$d, e, f \in [0, 1]$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_2), \mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_2)\right\} = \min\{a, 1\} = a \leq a$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_3), \mu_{\underset{\sim}{\mathbf{R}}}(x_3, x_2)\right\} = \min\{a, c\} = a \leq a$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_4), \mu_{\underset{\sim}{\mathbf{R}}}(x_4, x_2)\right\} = \min\{d, e\} \leq a \tag{4.1}$$

b. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_3) = \mathbf{a}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_1), \mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_3)\right\} = \min\{1, a\} = a \leq a$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_2), \mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_3)\right\} = \min\{a, c\} = a \leq a$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_3), \mu_{\underset{\sim}{\mathbf{R}}}(x_3, x_3)\right\} = \min\{a, 1\} = a \leq a$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_4),\mu_{\underset{\sim}{\mathbf{R}}}(x_4,x_3)\right\}=\min\{d,f\}\leq a \tag{4.2}$$

c. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_2,\mathbf{x}_3)=\mathbf{c}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_3)\right\}=\min\{a,a\}=a\leq c$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_3)\right\}=\min\{1,c\}=c\leq c$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_3)\right\}=\min\{c,1\}=c\leq c$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_4),\mu_{\underset{\sim}{\mathbf{R}}}(x_4,x_3)\right\}=\min\{e,f\}\leq c \tag{4.3}$$

d. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1,\mathbf{x}_4)=\mathbf{d}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_4)\right\}=\min\{1,d\}=d\leq d$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_4)\right\}=\min\{a,e\}\leq d \tag{4.4}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_4)\right\}=\min\{a,f\}\leq d \tag{4.5}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_4),\mu_{\underset{\sim}{\mathbf{R}}}(x_4,x_4)\right\}=\min\{d,1\}=d\leq d$$

e. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_2,\mathbf{x}_4)=\mathbf{e}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_4)\right\}=\min\{a,d\}\leq e \tag{4.6}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_4)\right\}=\min\{1,e\}=e\leq e$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_4)\right\}=\min\{c,f\}\leq e \tag{4.7}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_4),\mu_{\underset{\sim}{\mathbf{R}}}(x_4,x_4)\right\}=\min\{e,1\}=e\leq e$$

f. $\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_3,\mathbf{x}_4)=\mathbf{f}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_4)\right\}=\min\{a,d\}\leq f \tag{4.8}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_4)\right\}=\min\{c,e\}\leq f \tag{4.9}$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_4)\right\}=\min\{1,f\}=f\leq f$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_4),\mu_{\underset{\sim}{\mathbf{R}}}(x_4,x_4)\right\}=\min\{f,1\}=f\leq f$$

From equations (4.1)-(4.9), we have:

$$\min\{d,e\}\leq a$$

$$\min\{d,f\}\leq a$$

$$\min\{e,f\}\leq c$$

$$\min\{a,e\} \leq d$$

$$\min\{a,f\} \leq d$$

$$\min\{a,d\} \leq e$$

$$\min\{c,f\} \leq e$$

$$\min\{a,d\} \leq f$$

$$\min\{c,e\} \leq f$$

From (4.4), we get: $a \leq e \Longrightarrow a \leq d$

From (4.1), we get: $e \leq d \Longrightarrow e \leq a$ but $a \leq e \Longrightarrow a = e$

$$\therefore a = e \tag{4.10}$$

From (4.5), we get: $f \leq a \Longrightarrow f \leq d$

$$f \leq a = e \tag{4.11}$$

From (4.3) and (4.11), we get: $\Longrightarrow f \leq c$

From (4.8), we get: $a \leq d \Longrightarrow a \leq f$ but $f \leq a \Longrightarrow a = f$.

$$\therefore a = e = f \tag{4.12}$$

Now, we have:

$0 \leq a = e = f \leq d \leq 1$

$0 \le a = e = f \le c \le 1$

TABLE 4. Fuzzy Equivalence Relations represented as a $\mathbf{4} \times \mathbf{4}$ Matrix

$\underset{\sim}{\mathbf{R}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\mathbf{x}_1$	$\mathbf{1}$	a	a	d
$\mathbf{x}_2$	a	$\mathbf{1}$	c	a
$\mathbf{x}_3$	a	c	$\mathbf{1}$	a
$\mathbf{x}_4$	d	a	a	$\mathbf{1}$

$0 \le a \le c \le 1$

$0 \le a \le d \le 1$

□

5. Fuzzy Equivalence Relations represented by a $n \times n$ Matrix

In Table 4, we built a fuzzy equivalence relation of form $\mathbf{4} \times \mathbf{4}$ matrix. Now, we will show that a fuzzy equivalence relation in $\mathbf{X} \times \mathbf{X}$ with $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$ can be represented by a $n \times n$ matrix which is satisfied a certain condition. We can represent our relation as shown in Table 5 below. Before begin with the proof, we give a lemma and some notes related to our proof.

Lemma 5.1. *Let* $\underset{\sim}{\mathbf{R}}$ *be any fuzzy relation in* $\mathbf{X} \times \mathbf{X}$, *and* $\mu_{\underset{\sim}{\mathbf{R}}}(x, z)$ *be any grade of membership in* $[\mathbf{0}, \mathbf{1}]$ *of the fuzzy relation of the form* $\mathbf{n} \times \mathbf{n}$ *matrix, such that* $\mu_{\underset{\sim}{\mathbf{R}}}(x, z)$ *is the maximum value of its row or the maximum value of its column i.e.*

$$\mu_{\underset{\sim}{\mathbf{R}}}(x, z) = \max_{y} \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x, y) \right\}, \ \textit{for all } y \in \mathbf{X}$$

or

$$\mu_{\underset{\sim}{\mathbf{R}}}(x, z) = \max_{y} \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(y, z) \right\}, \ \textit{for all } y \in \mathbf{X}$$

TABLE 5. Fuzzy Relation represented as a $\mathbf{n} \times \mathbf{n}$ Matrix

$\underset{\sim}{\overset{\frown}{\mathbf{R}}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbf{x}_n$
$\mathbf{x}_1$	$\mathbf{1}$	a	a	a_4	a_5	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_n
$\mathbf{x}_2$	a	$\mathbf{1}$	c	a	a	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a
$\mathbf{x}_3$	a	c	$\mathbf{1}$	a	a	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a
$\mathbf{x}_4$	a_4	a	a	$\mathbf{1}$	a_5	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_n
$\mathbf{x}_5$	a_5	a	a	a_5	$\mathbf{1}$	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_n
$\mathbf{x}_6$	a_6	a	a	a_6	a_6	$\mathbf{1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$	$\mathbf{a}_n$
$\mathbf{x}_n$	a_n	a	a	a_n	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_n	$\mathbf{1}$

$$0 \leq a \leq a_n \leq a_{n-1} \leq \cdots \leq a_5 \leq a_4 \leq 1$$

$$0 \leq a \leq c \leq 1$$

Then: $\mu_{\underset{\sim}{\mathbf{R}}}(x,z)$ *Satisfies the Condition of the Fuzzy Transitive Relation.*

Proof. We want to proof that:

$$\mu_{\underset{\sim}{\mathbf{R}}}(x,z) \geq \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x,y), \mu_{\underset{\sim}{\mathbf{R}}}(y,z) \right\} \right\}, \text{for all } y \in \mathbf{X}$$

Case 1: If $\mu_{\underset{\sim}{\mathbf{R}}}(x,z) = \max_y \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x,y) \right\}$, for all $y \in \mathbf{X}$, and $\mu_{\underset{\sim}{\mathbf{R}}}(x,z) = \max_y \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(y,z) \right\}$, for all $y \in \mathbf{X}$. Then:

$$\mu_{\underset{\sim}{\mathbf{R}}}(x,z) = \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x,y), \mu_{\underset{\sim}{\mathbf{R}}}(y,z) \right\} \right\}, \text{for all } y \in \mathbf{X}$$

Case 2: If $\mu_{\underset{\sim}{\mathbf{R}}}(x,z) = \max\limits_{y}\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x,y)\right\}$, for all $y \in \mathbf{X}$ but $\mu_{\underset{\sim}{\mathbf{R}}}(x,z) \neq \max\limits_{y}\left\{\mu_{\underset{\sim}{\mathbf{R}}}(y,z)\right\}$, for all $y \in \mathbf{X}$. Then:

$$\Longrightarrow \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x,y), \mu_{\underset{\sim}{\mathbf{R}}}(y,z)\right\} \leq \max_{y}\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x,y)\right\}, \text{for all } y \in \mathbf{X}$$

$$\Longrightarrow \mu_{\underset{\sim}{\mathbf{R}}}(x,z) \geq \max_{y}\left\{\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x,y), \mu_{\underset{\sim}{\mathbf{R}}}(y,z)\right\}\right\}, \text{for all } y \in \mathbf{X}$$

Case 3: If $\mu_{\underset{\sim}{\mathbf{R}}}(x,z) = \max\limits_{y}\left\{\mu_{\underset{\sim}{\mathbf{R}}}(y,z)\right\}$, for all $y \in \mathbf{X}$ but $\mu_{\underset{\sim}{\mathbf{R}}}(x,z) \neq \max\limits_{y}\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x,y)\right\}$, for all $y \in \mathbf{X}$. Then: Case 3 is the same as Case 2.

From Case 1, Case 2, and Case 3, we conclude that the fuzzy relation satisfies the condition of fuzzy transitive for the grade $\mu_{\underset{\sim}{\mathbf{R}}}(x,z)$. □

Conclusion 1:

If $\mu_{\underset{\sim}{\mathbf{R}}}(x,x) = 1$, then this grade satisfies the condition of the Fuzzy transitive and fuzzy symmetric relation.

Conclusion 2:

If the fuzzy relation is symmetric i.e. $\mu_{\underset{\sim}{\mathbf{R}}}(x,y) = \mu_{\underset{\sim}{\mathbf{R}}}(y,x), \forall x,y \in \mathbf{X}$, then the upper triangle of the fuzzy matrix has the same cases of the lower triangle. Notice that:

(1) If all the values of the same column in fuzzy relation matrix are equal i.e. $\mu_{\underset{\sim}{\mathbf{R}}}(x,y) = a, \forall y \in \mathbf{X}, \mathbf{0} \leq \mathbf{a} \leq \mathbf{1}$, then all grades of this column satisfy the condition of the Fuzzy transitive relation.

(2) If all the values of the same row in fuzzy relation matrix are equal i.e. $\mu_{\underset{\sim}{\mathbf{R}}}(y,z) = a, \forall y \in \mathbf{X}, \mathbf{0} \leq \mathbf{a} \leq \mathbf{1}$, then all grades of this row satisfy the condition of the Fuzzy transitive relation.

(3) A fuzzy relation of the form $\mathbf{n} \times \mathbf{n}$ matrix that has entries of the same value is transitive fuzzy relation.

Theorem 5.1. *Let* $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$. *Then a fuzzy equivalence relation* $\underset{\sim}{\mathbf{R}}$ *in* $\mathbf{X} \times \mathbf{X}$ *can be represented by a* $n \times n$ *matrix.*

Proof. Using mathematical induction, when n=3, according to our previous results in Table 2, the following matrix represent fuzzy equivalence relation.

Suppose that the statement is true for n=k, then the following $k \times k$ matrix shown in Table 6 generates fuzzy equivalence relation.

TABLE 6. Fuzzy Equivalence Relations represented as a $\mathbf{k} \times \mathbf{k}$ Matrix

$\underset{\sim}{\mathbf{R}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbf{x}_k$
$\mathbf{x}_1$	$\mathbf{1}$	a	a	a_4	a_5	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k
$\mathbf{x}_2$	a	$\mathbf{1}$	c	a	a	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a
$\mathbf{x}_3$	a	c	$\mathbf{1}$	a	a	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a
$\mathbf{x}_4$	a_4	a	a	$\mathbf{1}$	a_5	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k
$\mathbf{x}_5$	a_5	a	a	a_5	$\mathbf{1}$	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k
$\mathbf{x}_6$	a_6	a	a	a_6	a_6	$\mathbf{1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\mathbf{a}_k$
$\mathbf{x}_k$	a_k	a	a	a_k	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k	$\mathbf{1}$

$$0 \leq a \leq a_k \leq a_{k-1} \leq \cdots \leq a_5 \leq a_4 \leq 1$$

$$0 \leq a \leq c \leq 1$$

Now, we want to prove it when $\mathbf{n} = \mathbf{k} + \mathbf{1}$, i.e. we want to prove that the following $\mathbf{k} + \mathbf{1} \times \mathbf{k} + \mathbf{1}$ matrix (see Table 7) generates a fuzzy equivalence relation.

It is obvious that the fuzzy relation of the form $\mathbf{k} + \mathbf{1} \times \mathbf{k} + \mathbf{1}$ matrix (Table 7) is Reflexive and Symmetric. Now, it is enough to prove that the fuzzy relation in (Table

Table 7. Fuzzy Relation represented as a $\mathbf{k+1} \times \mathbf{k+1}$ Matrix

$\overset{\frown}{\underset{\sim}{\mathrm{R}}}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbf{x}_k$	$\mathbf{x}_{k+1}$
$\mathbf{x}_1$	$\mathbf{1}$	a	a	a_4	a_5	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k	a_{k+1}
$\mathbf{x}_2$	a	$\mathbf{1}$	c	a	a	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a	a
$\mathbf{x}_3$	a	c	$\mathbf{1}$	a	a	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a	a
$\mathbf{x}_4$	a_4	a	a	$\mathbf{1}$	a_5	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k	a_{k+1}
$\mathbf{x}_5$	a_5	a	a	a_5	$\mathbf{1}$	a_6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k	$\vdots$
$\mathbf{x}_6$	a_6	a	a	a_6	a_6	$\mathbf{1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\mathbf{a}_k$	a_{k+1}
$\mathbf{x}_k$	a_k	a	a	a_k	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_k	$\mathbf{1}$	a_{k+1}
$\mathbf{x}_{k+1}$	a_{k+1}	a	a	a_{k+1}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_{k+1}	a_{k+1}	$\mathbf{1}$

$$0 \leq a \leq a_{k+1} \leq a_k \leq \cdots \leq a_5 \leq a_4 \leq 1$$

$$0 \leq a \leq c \leq 1$$

7) is transitive. Since the fuzzy relation in (Table 7) is symmetric fuzzy relation; it is enough to prove that the upper triangle in (Table 7) is transitive. To prove that, we will divide our proof into two parts:

Part One : The upper triangle of the fuzzy relation of the form $\mathbf{k} \times \mathbf{k}$ matrix is still transitive in the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix, i.e. If $\mu_{\underset{\sim}{\mathbf{R}}}(x_i, x_j)$,$\forall i, j = 1, 2, ..., k$. be any grade of membership of the fuzzy relation of the form $\mathbf{k} \times \mathbf{k}$ matrix, then it satisfies the condition of the transitive fuzzy relation on the form $\mathbf{k} \times \mathbf{k}$ matrix i.e.

$$\mu_{\underset{\sim}{\mathbf{R}}}(x_i, x_j) \geq \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}}(x_i, y), \mu_{\underset{\sim}{\mathbf{R}}}(y, x_j) \right\} \right\}, \forall y \in \mathbf{X}, \forall i, j = 1, 2, ..., k$$

Now, we have:

$$\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_j) \geq \begin{cases} \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_j)\right\} \\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_j)\right\} \\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_j)\right\} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_k),\mu_{\underset{\sim}{\mathbf{R}}}(x_k,x_j)\right\} \end{cases}$$

In the matrix of the form $\mathbf{k+1\times k+1}$, we want to prove that:

$$\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_j) \geq \begin{pmatrix} \begin{cases} \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_1),\mu_{\underset{\sim}{\mathbf{R}}}(x_1,x_j)\right\} \\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_2),\mu_{\underset{\sim}{\mathbf{R}}}(x_2,x_j)\right\} \\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_3),\mu_{\underset{\sim}{\mathbf{R}}}(x_3,x_j)\right\} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_k),\mu_{\underset{\sim}{\mathbf{R}}}(x_k,x_j)\right\} \end{cases} \end{pmatrix} \text{Already proved when } n=k$$
$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_{k+1}),\mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1},x_j)\right\}$$

So, it is obvious that we just need to prove that:

$$\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_j) \geq \min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_{k+1}),\mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1},x_j)\right\}$$

Part Two: $\mathbf{X}_{k+1}$Column in the form $\mathbf{k+1\times k+1}$ matrix is satisfies the condition of the fuzzy transitive relation i.e. If $\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_{k+1}), i=1,2,.....,k,k+1.$ be any type of the fuzzy relation of the form $\mathbf{k+1\times k+1}$ matrix, then

$$\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_{k+1}) \geq \max_y\left\{\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_i,x_j),\mu_{\underset{\sim}{\mathbf{R}}}(x_j,x_{k+1})\right\}\right\}, ,\forall j=1,2,.....,k,k+1$$

Now, we present the full proof for part one and part two:

Part One: The upper triangle of the fuzzy relation of the form $\mathbf{k\times k}$ matrix is still transitive in the form $\mathbf{k+1\times k+1}$ matrix.

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{a}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_2)\right\} = \min\{a_{k+1}, a\} = a \leq a$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_3) = \mathbf{a}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_3)\right\} = \min\{a_{k+1}, a\} = a \leq a$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_2, \mathbf{x}_3) = \mathbf{c}$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_3)\right\} = \min\{a, a\} = a \leq c$$

Now, we have:

$$0 \leq a \leq a_{k+1} \leq a_k \leq a_{k-1} \leq \cdots\cdots \leq a_5 \leq a_4 \leq 1$$

$$\text{if } \forall i = 4, 5, \cdots, k, \text{ then } \mathbf{a} \leq \mathbf{a}_{k+1} \leq \mathbf{a}_i$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_i) = \mathbf{a}_i, \forall i = 4, 5, \cdots, k.$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_1, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_i)\right\} = \min\{a_{k+1}, a_{k+1}\} = a_{k+1} \leq \mathbf{a}_i$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_2, \mathbf{x}_j) = \mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_3, \mathbf{x}_j) = \mathbf{a}, \forall i = 4, 5, \cdots, k.$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_2, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_j)\right\} = \min\{a, a_{k+1}\} = a \leq a$$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_3, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_j)\right\} = \min\{a, a_{k+1}\} = a \leq a$$

$$\textbf{Now} \text{ if } l = 5, 6 \cdots, k, \text{ then } \mathbf{a} \leq \mathbf{a}_{k+1} \leq \mathbf{a}_l$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_m, \mathbf{x}_l) = \mathbf{a}_l, \forall m = 4, 5, \cdots, k-1, \forall l = m+1, m+2, ..., k-1, k.$

$$\min\left\{\mu_{\underset{\sim}{\mathbf{R}}}(x_m, x_{k+1}), \mu_{\underset{\sim}{\mathbf{R}}}(x_{k+1}, x_l)\right\} = \min\{a_{k+1}, a_{k+1}\} = a_{k+1} \leq \mathbf{a}_l$$

$\therefore$ The upper triangle of the fuzzy relation of the form $\mathbf{k} \times \mathbf{k}$ matrix is still transitive in the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix.

Part Two: X_{k+1} column in the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix is satisfies the condition of the fuzzy transitive relation i.e.$\mu_{\underset{\sim}{\mathbf{R}}}(x_i, x_{k+1}), i = 1, 2, ..., k+1$ satisfies the condition of the fuzzy transitive relation.

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_1, \mathbf{x}_{k+1}) = \mathbf{a}_{k+1}$

$$\max\{a_{k+1}, a, a, a_{k+1}, \cdots\cdots, a_{k+1}\} = a_{k+1} \leq a_{k+1}$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_2, \mathbf{x}_{k+1}) = \mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_3, \mathbf{x}_{k+1}) = \mathbf{a}$

$$\max\{a, a, a, a, \cdots\cdots, a\} = a \leq a.$$

$\mu_{\underset{\sim}{\mathbf{R}}}(\mathbf{x}_i, \mathbf{x}_{k+1}) = \mathbf{a}_{k+1}, \forall i = 4, 5, ..., k$

$$\max\{\mathbf{a}_{k+1}, a, a, \mathbf{a}_{k+1}, ..., \mathbf{a}_{k+1}\} = a_{k+1} \leq a_{k+1}$$

$\therefore X_{k+1}$ Column in the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix is satisfies the condition of the fuzzy transitive relation.

The upper triangle of the fuzzy relation of the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix is transitive and also the generated fuzzy relation of the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix is transitive.

$\therefore$ The generated fuzzy relation of the form $\mathbf{k+1} \times \mathbf{k+1}$ matrix is fuzzy equivalence relation of the form $\mathbf{k+1} \times \mathbf{k+1}$ Matrix. □

6. Conclusion

In this paper, we present a new method to generate fuzzy equivalence relations in matrix form. The new method check if fuzzy relation in matrix form is equivalence relation or not, in particular if it is transitive or not transitive. We start building fuzzy equivalence relation in matrix forms 3×3 and 4×4 matrices, then by using mathematical induction we build general method that generates fuzzy equivalence relations of the form $n \times n$ matrices.

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