Jordan Journal of Mathematics and Statistics (JJMS) 13(1), 2020, pp 17 - 35

ON DOMINATED CLASSES OF HARMONIC CONVEX FUNCTIONS AND ASSOCIATED INTEGRAL INEQUALITIES

ERHAN SET[1], MUHAMMAD UZAIR AWAN[2],*, MUHAMMAD ASLAM NOOR[3], KHALIDA INAYAT NOOR[4] AND NOUSHEEN AKHTAR[5]

Abstract. In this article, we introduce and investigate dominated classes the notion of (h, g)-harmonic convex dominated functions. As particular cases of (h, g)-harmonic convex dominated functions, we also define some other new classes of harmonic convex dominated. We derive some integral inequalities of Hermite-Hadamard type via (h, g)-harmonic convex dominated functions and also give the fractional version of these inequalities. Some new special cases are also discussed which can be deduced from our main results.

1. Introduction

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f(\vartheta x + (1-\vartheta)y) \leq \vartheta f(x) + (1-\vartheta)f(y) \quad \forall x, y \in I, \vartheta \in [0,1].$$

Convexity plays a pivotal role in modern analysis. It has great impact in our daily life through its numerous applications in other fields of pure and applied sciences. In recent years several authors have shown their keen interest in the theory of convexity. As a result many people have generalized the notion of classical convexity in different

2000 *Mathematics Subject Classification.* 26A33, 26D07, 26D10, 26D15.

Key words and phrases. h-convex functions, harmonically convex functions, (h, g)-harmonic convex dominated functions, Hermite-Hadamard, inequalities.

Received: Sept. 4, 2018 Accepted: Feb. 17, 2019 .

directions using novel and innovative ideas, see [1, 2, 3, 5, 7, 9, 10, 15, 11, 16, 17, 18]. Dragomir et al. [5] introduced the notion of g-convex dominated functions as follows:

Definition 1.1. Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on I. The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be g-convex dominated on I, if

$$\begin{aligned} &|\vartheta f(x) + (1-\vartheta) f(y) - f(\vartheta x + (1-\vartheta) y)| \\ &\leq \vartheta g(x) + (1-\vartheta) g(y) - g(\vartheta x + (1-\vartheta) y), \end{aligned} \tag{1.1}$$

for all $x, y \in I$ and $\vartheta \in [0,1]$.

Dragomir et al. [7] obtained the following Hermite-Hadamard like inequalities for g-convex dominated functions

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - g\left(\frac{a+b}{2}\right),$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x.$$

The notions of harmonic convex function and $h-$convex function were introduced as the following:

Definition 1.2. [18] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or f belongs to the class $SX(h; I)$, if f is nonnegative and for all $x, y \in I$ and $\vartheta \in (0,1)$, we have

$$f(\vartheta x + (1-\vartheta) y) \leq h(\vartheta) f(x) + h(1-\vartheta) f(y).$$

Definition 1.3. [10] A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex, if

$$f\left(\frac{xy}{\vartheta x + (1-\vartheta) y}\right) \leq (1-\vartheta) f(x) + \vartheta f(y), \quad \forall x, y \in I, \vartheta \in [0,1].$$

Noor et al. [15] introduced the notion of harmonic h-convex functions, which not only generalizes the class of harmonic convex functions but also includes some other new classes of harmonic convex functions.

Iscan [10] obtained Hermite-Hadamard like inequalities for harmonic convex functions. Noor et al. [15] extended the result of Iscan for the class of harmonic h-convex functions and discussed some other special cases. For some recent investigations on Hermite-Hadamard type inequalities via different classes of convex functions, see [4, 6, 8, 13, 14].

The aim of this article is to introduced the notion of (h, g)-harmonic convex dominated convex functions. We obtain some Hermite-Hadamard like inequalities and their fractional version as well. Several new special cases are also discussed. This is the main motivation of this article. It is expected that the ideas and techniques of this article may inspire interested readers to explore the applications of (h, g)-harmonic convex dominated functions in other fields of pure and applied sciences. The results of this article may be useful in the fields of numerical analysis and statistics where the error analysis and different means of real numbers are required respectively.

2. New Definitions

First of all we introduce the class of (h, g)-harmonic convex dominated functions.

Definition 2.1. Let I, J be two intervals in $\mathbb{R}$, $h : (0, 1) \subseteq J \to \mathbb{R}$ be non-negative function, $h \neq 0$ and $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic h-convex function on I. The function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be (h, g)-harmonic convex dominated on I, if

$$\begin{aligned} &\left| h(\vartheta)f(x) + h(1-\vartheta)f(y) - f\left(\frac{xy}{(1-\vartheta)x + \vartheta y}\right) \right| \\ &\leq h(\vartheta)g(x) + h(1-\vartheta)g(y) - g\left(\frac{xy}{(1-\vartheta)x + \vartheta y}\right) \end{aligned} \tag{2.1}$$

for all $x, y \in I$ and $\vartheta \in (0, 1)$.

Now we discuss some special cases of Definition 2.1.

I. If $h(\vartheta)=\vartheta$ in Definition 2.1, then we have the following new concept.

Definition 2.2. Let $g: I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonic convex function on I. The function $f: I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be g-harmonic convex dominated on I, if

$$\left|\vartheta f(x)+(1-\vartheta)f(y)-f\left(\frac{xy}{(1-\vartheta)x+\vartheta y}\right)\right|$$
$$\leq \vartheta g(x)+(1-\vartheta)g(y)-g\left(\frac{xy}{(1-\vartheta)x+\vartheta y}\right),$$

for all $x,y\in I$ and $\vartheta\in[0,1]$.

II. If $h(\vartheta)=\vartheta^s$ in Definition 2.1, then we have the following new concept.

Definition 2.3. Let $g: I\subseteq\mathbb{R}^+\to\mathbb{R}$ be a Breckner type of harmonic s-convex function on I. The function $f: I\subseteq\mathbb{R}^+\to\mathbb{R}$ is said to be Breckner type of (s,g)-harmonic convex dominated on I, if

$$\left|\vartheta^s f(x)+(1-\vartheta)^s f(y)-f\left(\frac{xy}{(1-\vartheta)x+\vartheta y}\right)\right|$$
$$\leq \vartheta^s g(x)+(1-\vartheta)^s g(y)-g\left(\frac{xy}{(1-\vartheta)x+\vartheta y}\right),$$

for all $x,y\in I$, $\vartheta\in[0,1]$ and $s\in(0,1]$.

III. If $h(\vartheta)=\vartheta^{-s}$ in Definition 2.1, then we have the following new concept.

Definition 2.4. Let $g: I\subseteq\mathbb{R}^+\to\mathbb{R}$ be a Godunova-Levin type of harmonic s-convex function on I. The function $f: I\subseteq\mathbb{R}^+\to\mathbb{R}$ is said to be Godunova-Levin type of (s,g)-harmonic convex dominated on I, if

$$\left|\frac{1}{\vartheta^s}f(x)+\frac{1}{(1-\vartheta)^s}f(y)-f\left(\frac{xy}{(1-\vartheta)x+\vartheta y}\right)\right|$$
$$\leq \frac{1}{\vartheta^s}g(x)+\frac{1}{(1-\vartheta)^s}g(y)-g\left(\frac{xy}{(1-\vartheta)x+\vartheta y}\right)$$

for all $x,y\in I$, $\vartheta\in(0,1)$ and $s\in[0,1]$.

IV. If $h(\vartheta) = \vartheta^{-1}$ in Definition 2.1, then we have the following new concept.

Definition 2.5. Let $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a Godunova-Levin type of harmonic convex function on I. The function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin g-harmonic convex dominated on I, if

$$\begin{aligned} &\left| \frac{1}{\vartheta} f(x) + \frac{1}{1-\vartheta} f(y) - f\left(\frac{xy}{(1-\vartheta)x + \vartheta y}\right) \right| \\ &\leq \frac{1}{\vartheta} g(x) + \frac{1}{1-\vartheta} g(y) - g\left(\frac{xy}{(1-\vartheta)x + \vartheta y}\right) \end{aligned}$$

for all $x, y \in I$, $\vartheta \in (0,1)$.

V. If $h(\vartheta) = 1$ in Definition 2.1, then, we have following new concept.

Definition 2.6. Let $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic P-convex function on I. The function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be a (P, g)-harmonic convex dominated on I, if

$$\left| f(x) + f(y) - f\left(\frac{xy}{(1-\vartheta)x + \vartheta y}\right) \right| \leq g(x) + g(y) - g\left(\frac{xy}{(1-\vartheta)x + \vartheta y}\right)$$

for all $x, y \in I$, $\vartheta \in (0,1)$.

3. Hermite-Hadamard type Inequalities

In this section, we obtain some Hermite-Hadamard like inequalities for (h, g)-harmonic convex dominated functions.

Theorem 3.1. *Let $h : (0,1) \subseteq J \to \mathbb{R}$ be a non-negative function, and $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic h-convex function. Suppose $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a (h, g)-harmonic convex dominated function on I and $f \in L_1[a, b]$, then for $h\left(\frac{1}{2}\right) \neq 0$,* *we have*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - \frac{1}{2h\left(\frac{1}{2}\right)} g\left(\frac{2ab}{a+b}\right).$$

Proof. Using $\vartheta = \frac{1}{2}$, $x = \frac{ab}{\vartheta a+(1-\vartheta)b}$ and $y = \frac{ab}{(1-\vartheta)a+\vartheta b}$ where $\vartheta \in [0,1]$ in the Definition 2.1, we have

$$\left|h\Big(\frac{1}{2}\Big)\left[f\left(\frac{ab}{\vartheta a+(1-\vartheta)b}\right)+f\left(\frac{ab}{(1-\vartheta)a+\vartheta b}\right)\right]-f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq h\Big(\frac{1}{2}\Big)\left[g\left(\frac{ab}{\vartheta a+(1-\vartheta)b}\right)+g\left(\frac{ab}{(1-\vartheta)a+\vartheta b}\right)\right]-g\left(\frac{2ab}{a+b}\right).$$

Integrating above inequality with respect to ϑ on $[0,1]$, we have

$$\left|2h\Big(\frac{1}{2}\Big)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq 2h\Big(\frac{1}{2}\Big)\frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - g\left(\frac{2ab}{a+b}\right).$$

This implies that

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - \frac{1}{2h\left(\frac{1}{2}\right)} g\left(\frac{2ab}{a+b}\right).$$

This completes the proof. □

We now discuss some special cases of Theorem 3.1.

Corollary 3.1. *Let* $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be harmonic convex function. Suppose* $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *is* g*-harmonic convex dominated function on* I *where* $f \in L_1[a,b]$, *then, we have*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - f\left(\frac{2ab}{a+b}\right)\right| \leq \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - g\left(\frac{2ab}{a+b}\right).$$

Corollary 3.2. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Breckner type of harmonic s-convex function. Suppose $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is (s, g)-harmonic convex dominated function on I where $f \in L_1[a, b]$, then for $s \in (0, 1]$, we have*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - 2^{s-1}f\left(\frac{2ab}{a+b}\right)\right| \leq \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - 2^{s-1}g\left(\frac{2ab}{a+b}\right).$$

Corollary 3.3. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Godunova-Levin type of harmonic s-convex function. Suppose $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is Godunova-Levin type of (s, g)-harmonic convex dominated function on I where $f \in L_1[a, b]$, then for $s \in [0, 1]$, we have*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - \frac{1}{2^{s+1}}f\left(\frac{2ab}{a+b}\right)\right| \leq \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - \frac{1}{2^{s+1}}g\left(\frac{2ab}{a+b}\right).$$

Corollary 3.4. *Let $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic P-convex function. Suppose $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is (P, g)-harmonic convex dominated function on I where $f \in L_1[a, b]$, then we have*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x - \frac{1}{2}f\left(\frac{2ab}{a+b}\right)\right| \leq \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - \frac{1}{2}g\left(\frac{2ab}{a+b}\right).$$

Theorem 3.2. *Let $h : (0, 1) \subseteq J \to \mathbb{R}$ be a non-negative function, $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic h-convex function. Suppose $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is (g, h)-harmonic convex dominated function on I where $f \in L_1[a, b]$, then*

$$\left|[f(a) + f(b)]\int_0^1 h(\vartheta)\mathrm{d}\vartheta - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq [g(a) + g(b)]\int_0^1 h(\vartheta)\mathrm{d}\vartheta - \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x.$$

Proof. Let $x = a$ and $y = b$ in the Definition 2.1, we have

$$\left| h(\vartheta)f(a) + h(1-\vartheta)f(b) - f\left(\frac{ab}{(1-\vartheta)a+\vartheta b}\right)\right|$$
$$\leq h(\vartheta)g(a) + h(1-\vartheta)g(b) - g\left(\frac{ab}{(1-\vartheta)a+\vartheta b}\right),$$

Integrating above inequalities with respect to ϑ on $[0,1]$, we have

$$\left| f(a)\int_0^1 h(\vartheta)\mathrm{d}\theta + f(b)\int_0^1 h(1-\vartheta)\mathrm{d}\theta - \int_0^1 f\left(\frac{ab}{(1-\vartheta)a+\vartheta b}\right)\mathrm{d}\vartheta\right|$$
$$\leq g(a)\int_0^1 h(\vartheta)\mathrm{d}\theta + g(b)\int_0^1 h(1-\vartheta)\mathrm{d}\theta - \int_0^1 g\left(\frac{ab}{(1-\vartheta)a+\vartheta b}\right)\mathrm{d}\vartheta,$$

This implies

$$\left| [f(a)+f(b)]\int_0^1 h(\vartheta)\mathrm{d}\vartheta - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq [g(a)+g(b)]\int_0^1 h(\vartheta)\mathrm{d}\vartheta - \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x.$$

This completes the proof. □

We now discuss some special cases of Theorem 3.2.

Corollary 3.5. *Let $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic convex function. Suppose $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is g-harmonic convex dominated function on I where $f \in L_1[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \leq \frac{g(a)+g(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x.$$

Corollary 3.6. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Breckner type of harmonic s-convex function. Suppose $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is Breckner type of (s,g)-harmonic convex dominated* *function on I where $f \in L_1[a,b]$, then for $s \in (0,1]$, we have*

$$\left|\frac{f(a)+f(b)}{s+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \leq \frac{g(a)+g(b)}{s+1} - \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x.$$

Corollary 3.7. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Godunova-Levin type of harmonic s-convex function. Suppose $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is Godunova-Levin type of (s,g)-harmonic convex dominated function on I where $f \in L_1[a,b]$, then for $s \in [0,1]$, we have*

$$\left|\frac{f(a)+f(b)}{1-s} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \leq \frac{g(a)+g(b)}{1-s} - \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x.$$

Corollary 3.8. *Let $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic P-convex function. Suppose $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is (P,g)-harmonic convex dominated function on I where $f \in L_1[a,b]$, then we have*

$$\left|f(a)+f(b) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \leq g(a)+g(b) - \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x.$$

4. Fractional Hermite-Hadamard type inequalities

In this section, we derive the fractional version of the Hermite-Hadamard type inequalities obtained in the previous section. First of all, we recall some preliminaries of the fractional calculus which will be helpful in obtaining the results of this section.

Definition 4.1 ([12])**.** Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} x^{\alpha-1} dx,$$

is the Gamma function.

The well known Beta function is defined as:

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

It is known [12] that

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We now derive some fractional Hermite-Hadamard type inequalities for (h,g)-harmonic convex dominated functions.

Theorem 4.1. *Let $h : (0,1) \subseteq J \to \mathbb{R}$ be a non-negative function and $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonic h-convex function. Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be (h,g)-harmonic convex dominated function such that $f \in L_1[a,b]$, then for $\alpha > 0$ and $h\left(\frac{1}{2}\right) \neq 0$, following inequality holds:*

$$\begin{aligned}&\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right\} - \frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{2ab}{a+b}\right)\right|\\ &\leq \left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(g\circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}^+}(g\circ w)\left(\frac{1}{a}\right)\right\} - \frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{2ab}{a+b}\right)\right|.\end{aligned}$$

Proof. Using $\vartheta = \frac{1}{2}$, $x = \frac{ab}{(1-\mu)a+\mu b}$ and $y = \frac{ab}{\mu a+(1-\mu)b}$, where $\mu \in [0,1]$ in the Def of (h,g)-harmonic convex dominated functions, we have

$$\begin{aligned}&\left|h\left(\frac{1}{2}\right)f\left(\frac{ab}{(1-\mu)a+\mu b}\right) + h\left(\frac{1}{2}\right)f\left(\frac{ab}{\mu a+(1-\mu)b}\right) - f\left(\frac{2ab}{a+b}\right)\right|\\ &\leq h\left(\frac{1}{2}\right)g\left(\frac{ab}{(1-\mu)a+\mu b}\right) + h\left(\frac{1}{2}\right)g\left(\frac{ab}{\mu a+(1-\mu)b}\right) - g\left(\frac{2ab}{a+b}\right).\end{aligned}$$

Multiplying both sides of above inequality with $\mu^{\alpha-1}$ and then integrating with respect to μ on $[0,1]$, we have

$$\left| h\left(\frac{1}{2}\right)\left\{ \int_0^1 \mu^{\alpha-1} f\left(\frac{ab}{(1-\mu)a+\mu b}\right)\mathrm{d}\mu + \int_0^1 \mu^{\alpha-1} f\left(\frac{ab}{\mu a+(1-\mu)b}\right)\mathrm{d}\mu \right\} \right.$$
$$\left. - f\left(\frac{2ab}{a+b}\right)\int_0^1 \mu^{\alpha-1}\mathrm{d}\mu \right|$$
$$\leq \left| h\left(\frac{1}{2}\right)\left\{ \int_0^1 \mu^{\alpha-1} g\left(\frac{ab}{(1-\mu)a+\mu b}\right)\mathrm{d}\mu + \int_0^1 \mu^{\alpha-1} g\left(\frac{ab}{\mu a+(1-\mu)b}\right)\mathrm{d}\mu \right\} \right.$$
$$\left. - g\left(\frac{2ab}{a+b}\right)\int_0^1 \mu^{\alpha-1}\mathrm{d}\mu \right|.$$

This implies that

$$\left| h\left(\frac{1}{2}\right)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{ \int_{\frac{1}{b}}^{\frac{1}{a}} \left(x-\frac{1}{b}\right)^{\alpha-1} f\left(\frac{1}{x}\right)\mathrm{d}x + \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a}-x\right)^{\alpha-1} f\left(\frac{1}{x}\right)\mathrm{d}x \right\} \right.$$
$$\left. -\frac{1}{\alpha} f\left(\frac{2ab}{a+b}\right) \right|$$
$$\leq \left| h\left(\frac{1}{2}\right)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{ \int_{\frac{1}{b}}^{\frac{1}{a}} \left(x-\frac{1}{b}\right)^{\alpha-1} g\left(\frac{1}{x}\right)\mathrm{d}x + \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a}-x\right)^{\alpha-1} g\left(\frac{1}{x}\right)\mathrm{d}x \right\} \right.$$
$$\left. -\frac{1}{\alpha} g\left(\frac{2ab}{a+b}\right) \right|.$$

This further implies

$$\left| \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{ J^{\alpha}_{\frac{1}{a}-}(f\circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}+}(f\circ w)\left(\frac{1}{a}\right)\right\} - \frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) \right|$$
$$\leq \left| \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{ J^{\alpha}_{\frac{1}{a}-}(g\circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}+}(g\circ w)\left(\frac{1}{a}\right)\right\} - \frac{1}{h\left(\frac{1}{2}\right)} g\left(\frac{2ab}{a+b}\right) \right|.$$

This completes the proof. □

Now we discuss some special cases of Theorem 4.1.

Corollary 4.1. *Let $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic convex function. Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be g-harmonic convex dominated function such that $f \in L_1[a, b]$, then for $\alpha > 0$, following inequality holds:*

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right\}-2f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(g\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(g\circ w)\left(\frac{1}{a}\right)\right\}-2g\left(\frac{2ab}{a+b}\right)\right|.$$

Corollary 4.2. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a Breckner type of harmonic s-convex function. Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Breckner type of (s, g)-harmonic convex dominated function such that $f \in L_1[a, b]$, then for $\alpha > 0$ and $s \in (0, 1]$, following inequality holds:*

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right\}-2^sf\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(g\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(g\circ w)\left(\frac{1}{a}\right)\right\}-2^sg\left(\frac{2ab}{a+b}\right)\right|.$$

Corollary 4.3. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a Godunova-Levin type of harmonic s-convex function. Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Godunova-Levin type of (s, g)-harmonic convex dominated function such that $f \in L_1[a, b]$, then for $\alpha > 0$ and $s \in [0, 1]$, following inequality holds:*

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right\}-\frac{1}{2^s}f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(g\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(g\circ w)\left(\frac{1}{a}\right)\right\}-\frac{1}{2^s}g\left(\frac{2ab}{a+b}\right)\right|.$$

Corollary 4.4. *Let $g : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic P-convex function. Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}\precsim$ be (P, g)-harmonic convex dominated function such that*

$f \in L_1[a,b]$, then for $\alpha > 0$, following inequality holds:

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right\}-f\left(\frac{2ab}{a+b}\right)\right|$$
$$\leq\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}^-}(g\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(g\circ w)\left(\frac{1}{a}\right)\right\}-g\left(\frac{2ab}{a+b}\right)\right|.$$

Theorem 4.2. *Let $h:(0,1)\subseteq J\to\mathbb{R}$ be a non-negative function and $g:I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonic h-convex function. Let $f:I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be (h,g)-harmonic convex dominated function such that $f\in L_1[a,b]$, then for $\alpha>0$, following inequality holds:*

$$\left|\psi_1 f(a)+\psi_2 f(b)-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)\right|$$
$$+\left|\psi_2 f(a)+\psi_1 f(b)-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right|$$
$$\leq\psi_3[g(a)+g(b)]-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right],$$

where

$$\psi_1=\int_0^1\vartheta^{\alpha-1}h(\vartheta)\mathrm{d}\vartheta,$$

$$\psi_2=\int_0^1\vartheta^{\alpha-1}h(1-\vartheta)\mathrm{d}\vartheta,$$

and

$$\psi_3=\int_0^1\vartheta^{\alpha-1}[h(\vartheta)+h(1-\vartheta)]\mathrm{d}\vartheta.$$

Proof. Let $x = a$ and $y = b$ in the Def of (h, g)-harmonic convex dominated functions, we have

$$\begin{aligned}
&\left|h(\vartheta)f(a) + h(1-\vartheta)f(b) - f\left(\frac{ab}{(1-\vartheta)a + \vartheta b}\right)\right| \\
&+ \left|h(1-\vartheta)f(a) + h(\vartheta)f(b) - f\left(\frac{ab}{\vartheta a + (1-\vartheta)b}\right)\right| \\
&\leq [h(\vartheta) + h(1-\vartheta)][g(a) + g(b)] - \left[g\left(\frac{ab}{(1-\vartheta)a + \vartheta b}\right) + g\left(\frac{ab}{\vartheta a + (1-\vartheta)b}\right)\right].
\end{aligned}$$

Multiplying both sides of above inequality with respect to $\vartheta^{\alpha-1}$ and then integrating it with respect to ϑ on $[0, 1]$, we have

$$\begin{aligned}
&\left|f(a)\int_0^1 \vartheta^{\alpha-1}h(\vartheta)\mathrm{d}\vartheta + f(b)\int_0^1 \vartheta^{\alpha-1}h(1-\vartheta)\mathrm{d}\vartheta - \int_0^1 \vartheta^{\alpha-1}f\left(\frac{ab}{(1-\vartheta)a + \vartheta b}\right)\mathrm{d}\vartheta\right| \\
&+ \left|f(a)\int_0^1 \vartheta^{\alpha-1}h(1-\vartheta)\mathrm{d}\vartheta + f(b)\int_0^1 \vartheta^{\alpha-1}h(\vartheta)\mathrm{d}\vartheta - \int_0^1 \vartheta^{\alpha-1}f\left(\frac{ab}{\vartheta a + (1-\vartheta)b}\right)\mathrm{d}\vartheta\right| \\
&\leq [g(a) + g(b)]\int_0^1 \vartheta^{\alpha-1}[h(\vartheta) + h(1-\vartheta)]\mathrm{d}\vartheta \\
&\quad - \left[\int_0^1 \vartheta^{\alpha-1}g\left(\frac{ab}{(1-\vartheta)a + \vartheta b}\right)\mathrm{d}\vartheta + \int_0^1 \vartheta^{\alpha-1}g\left(\frac{ab}{\vartheta a + (1-\vartheta)b}\right)\mathrm{d}\vartheta\right].
\end{aligned}$$

This implies

$$\left|f(a)\int_0^1 \vartheta^{\alpha-1}h(\vartheta)\mathrm{d}\vartheta + f(b)\int_0^1 \vartheta^{\alpha-1}h(1-\vartheta)\mathrm{d}\vartheta - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)\right|$$
$$+\left|f(a)\int_0^1 \vartheta^{\alpha-1}h(1-\vartheta)\mathrm{d}\vartheta + f(b)\int_0^1 \vartheta^{\alpha-1}h(\vartheta)\mathrm{d}\vartheta - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right|$$
$$\leq [g(a)+g(b)]\int_0^1 \vartheta^{\alpha-1}[h(\vartheta)+h(1-\vartheta)]\mathrm{d}\vartheta$$
$$-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right].$$

□

Now we discuss some special cases of Theorem 4.2.

Corollary 4.5. *Let $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonic convex function. Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be g-harmonic convex dominated function such that $f \in L_1[a,b]$, then for $\alpha > 0$, following inequality holds:*

$$\left|\psi_1^* f(a)+\psi_2^* f(b)-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)\right|$$
$$+\left|\psi_2^* f(a)+\psi_1^* f(b)-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right|$$
$$\leq \psi_3^*[g(a)+g(b)]-\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right],$$

where

$$\psi_1^* = \frac{1}{\alpha+1},$$
$$\psi_2^* = \frac{1}{\alpha(\alpha+1)},$$

and

$$\psi_3^* = \frac{1}{\alpha}.$$

Corollary 4.6. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a Breckner type of harmonic s-convex function. Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Breckner type of (s, g)-harmonic convex dominated function such that $f \in L_1[a, b]$, then for $\alpha > 0$ and $s \in (0, 1]$, following inequality holds:*

$$\begin{aligned}&\left|\psi_1^{**} f(a) + \psi_2^{**} f(b) - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}^-}(f \circ w)\left(\frac{1}{b}\right)\right| \\ &+ \left|\psi_2^{**} f(a) + \psi_1^{**} f(b) - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}^+}(f \circ w)\left(\frac{1}{a}\right)\right| \\ &\leq \psi_3^{**}[g(a) + g(b)] - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}^-}(f \circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}^+}(f \circ w)\left(\frac{1}{a}\right)\right],\end{aligned}$$

where

$$\psi_1^{**} = \frac{1}{\alpha + s},$$
$$\psi_2^{**} = B(\alpha, s + 1),$$

and

$$\psi_3^{**} = \frac{1}{\alpha + s} + B(\alpha, s + 1).$$

Corollary 4.7. *Let $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a Godunova-Levin type of harmonic s-convex function. Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be Godunova-Levin type of (s, g)-harmonic convex dominated function such that $f \in L_1[a, b]$, then for $\alpha > 0$ and $s \in [0, 1]$, following inequality holds:*

$$\begin{aligned}&\left|\psi_1^{***} f(a) + \psi_2^{***} f(b) - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}^-}(f \circ w)\left(\frac{1}{b}\right)\right| \\ &+ \left|\psi_2^{***} f(a) + \psi_1^{***} f(b) - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}^+}(f \circ w)\left(\frac{1}{a}\right)\right| \\ &\leq \psi_3^{***}[g(a) + g(b)] - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}^-}(f \circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}^+}(f \circ w)\left(\frac{1}{a}\right)\right],\end{aligned}$$

where

$$\psi_1^{***} = \frac{1}{\alpha - s},$$
$$\psi_2^{**} = B(\alpha, 1 - s),$$

and

$$\psi_3^{**} = \frac{1}{\alpha - s} + B(\alpha, 1-s).$$

Corollary 4.8. *Let $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a P-function. Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be (P, g)-harmonic convex dominated function such that $f \in L_1[a, b]$, then, for $\alpha > 0$, following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{\alpha} - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right)\right| \\
&+\left|\frac{f(a)+f(b)}{\alpha} - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right| \\
&\leq \frac{2[g(a)+g(b)]}{\alpha} - \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}^-}(f\circ w)\left(\frac{1}{b}\right) + J^{\alpha}_{\frac{1}{b}^+}(f\circ w)\left(\frac{1}{a}\right)\right].
\end{aligned}$$

Acknowledgement

Authors would like to thank the editor and the referees for their valuable comments and suggestions.
This research is supported by HEC NRPU project no: 8081/Punjab/NRPU/R&D/HEC/2017.

References

[1] W. W. Breckner, Stetigkeitsaussagen fiir eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen. Pupl. Inst. Math. 23, (1978), 13-20.

[2] G. Cristescu, L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, (2002).

[3] G. Cristescu, M. A. Noor, M. U. Awan, Bounds of the second degree cumulative frontier gaps of functions with generalized convexity, Carpath. J. Math. 31(2), (2015).

[4] S. S. Dragomir, Inequalities of Hermite–Hadamard type for h-convex functions on linear spaces, RGMIA Res. Rep. Coll., Article 72 (2013).

[5] S. S. Dragomir, N. M. Ionescu, On some inequalities for convex-dominated functions. Anal. Numér. Théor. Approx. 19, 21-28 (1990).

[6] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications. Victoria University, Australia, (2000).

[7] S. S. Dragomir, C. E. M. Pearce, J. E. Pecaric, Means, g-convex dominated & Hadamard-type inequalities. Tamsui Oxford Univ. J. Math. Sci. 18(2), 161-173 (2002).

[8] S. S. Dragomir, J. Pečarić, L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21, 335-341, (1995).

[9] E. K. Godunova, V. I. Levin, Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii. Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva. (1985) 138-142, (in Russian).

[10] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe J. Math. Stat., 43(6), 935-942, (2014).

[11] H. Kavurmacı, M. E. Özdemir, M. Z. Sarıkaya, New definitions and theorems via different kinds of convex dominated functions. RGMIA Research Report Collection (Online) 15, Article ID 9 (2012).

[12] A. Kilbas , H. M. Srivastava, J. J. Trujillo: Theory and applications of fractional differential equations, Elsevier B.V., Amsterdam, Netherlands, (2006).

[13] M. V. Mihai, M. A. Noor, K. I. Noor, M. U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions, Appl. Math. Comput. 252, 257-262, (2015).

[14] M. A. Noor, G. Cristescu, M. U. Awan, Generalized fractional Hermite-Hadamard inequalities for twice differentiable s-convex functions, Filomat, 29(4), (2015), 807-815.

[15] M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, Some integral inequalities for harmonically h-convex functions U. P. B. Sci. Bull., Series A. 77(1), 5-16, (2015).

[16] M. E. Ozdemir, M. Gürbüz, H. Kavurmacı, Hermite-Hadamard-type inequalities for (g, φ_h)-convex dominated functions, J. Inequal. Appl., 2013, 2013:184.

[17] M. E. Ozdemir, M. Tunç, H. Kavurmacı, Two new different kinds of convex dominated functions and inequalities via Hermite-Hadamard type. arXiv:1202.2054v1 [math.CA] 9 Feb 2012.

[18] S. Varošanec, On h-convexity, J. Math. Anal. Appl., 326, 303-311, (2007).

(1)Department of Mathematics, Faculty of Science and Letters, Ordu University, Turkey.

E-mail address: erhanset@yahoo.com

(2) Department of Mathematics, GC University, Faisalabad, Pakistan.

E-mail address: awan.uzair@gmail.com

*Corresponding Author

(3) Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.

E-mail address: noormaslam@gmail.com

(4) Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.

E-mail address: khalidan@gmail.com

(5) Department of Mathematics, GC University, Faisalabad, Pakistan.

E-mail address: nousheenakhtar86@gmail.com