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ON PROJECTIVELY FLAT SPECIAL (α, β)-METRIC

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Abstract. In this paper, we discussed the projectively flat exponential (α, β)-metric of type $L = \alpha\, e^{\beta/\alpha}$ We obtained a necessary and sufficient condition for this metric to be locally projectively flat and we established the conditions for this metric to be Berwald type and Douglas type.

1. Introduction

M. Matsumoto [9] introduced the concept of (α, β)-metric on a differentiable manifold M^n, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. The Matsumoto metric is an interesting (α, β)-metric introduced by using gradient of slope, speed and gravity [5]. This metric formulates the model of a Finsler space. Many authors [5, 1, 11] studied this metric by different perspectives. In projective Finsler geometry, we have remarkable theorem called Rapcsak theorem, which plays an important role in the projective geometry of Finsler spaces.

In fact this theorem gives the necessary and sufficient condition that a Finsler space is projective to another Finsler space. An extensive study of projectively flat Finsler metrics was taken up by authors [6, 7, 10] and [12, 8]. An interesting and important class of Finsler spaces is the class of Berwald spaces. As a generalization of a Berwald

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space S. Bacso and M. Matsumoto [2] introduced the notion of a Douglas space. A Douglas space is a Finsler space whose Douglas tensor vanishes.

2. Preliminaries

Definition 2.1. A smooth manifold is a differentiable manifold for which all transition maps are smooth i.e. derivatives of all orders exist.

Definition 2.2. A Finsler metric on M^n is a function $L : TM^n \to [0, \infty)$ with the following properties:

- L is C^∞ on TM_0^n,
- L is positively 1-homogeneous on the fiber of tangent bundle TM^n,
- the Hessian of F^2 with element $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is regular on TM_0^n, i.e., $det(g_{ij}) \neq 0$.

The pair (M^n, L) is called a Finsler space. L is called fundamental function and g_{ij} is called fundamental tensor.

Let M^n be a real smooth manifold of dimension n and let $F_n = (M^n, L)$ be a Finsler space on the differentiable manifold M^n endowed with a fundamental function $L(x,y)$, where

$$L = \alpha e^{\beta/\alpha}. \tag{2.1}$$

Differentiating (2.1) partially with respect to α and β, we get

$$\begin{cases} a) \ \ L_\alpha = e^{\beta/\alpha} - \frac{\beta}{\alpha} e^{\beta/\alpha}, \\ b) \ \ L_\beta = e^{\beta/\alpha}, \\ c) \ \ L_{\alpha\alpha} = \frac{\beta^2}{\alpha^3} e^{\beta/\alpha}, \\ d) \ \ L_{\beta\beta} = \frac{e^{\beta/\alpha}}{\alpha}, \end{cases} \tag{2.2}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$, $L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha$, $L_{\beta\beta} = \partial L_\beta/\partial\beta$.

Definition 2.3. A special class of Finsler metrics called (α,β)-metrics, which is defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta = b_i(x)y^i$.

Definition 2.4. The (α,β)-metric of type $L = \alpha e^{\beta/\alpha}$ is locally projectively flat if and only if

(1) β is parallel with respect to α,

(2) α is locally projectively flat, i.e. of constant curvature.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, $\beta = b_iy^i$ is a 1-form and let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where ϕ is a positive C^∞ function defined in a neighborhood of the origin $s = 0$. It is well known that $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ is a Finsler metric for any α and β with $b = \|\beta\|_\alpha < b_0$ if and only if

$$\phi(s) > 0,\ \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,\ (|s| \le b < b_0).$$

By taking $b = s$, we obtain

$$\phi(s) - s\phi'(s) > 0,\ (|s| < b_0).$$

Let G^i and G^i_α denote the spray coefficients of L and α respectively, given by

$$G^i = \frac{g^{il}}{4}\left\{[L^2]_{x^ky^l}y^k - [L^2]_{x^k}\right\},$$
$$G^i_\alpha = \frac{a^{il}}{4}\left\{[\alpha^2]_{x^ky^l}y^k - [\alpha^2]_{x^k}\right\}, \tag{2.3}$$

where $g_{ij} = \frac{1}{2}[L^2]_{y^iy^j}$, $(a^{ij}) = (a_{ij})^{-1}$, $L_{x^k} = \frac{\partial L}{\partial x^k}$ and $L_{y^k} = \frac{\partial L}{\partial y^k}$.

For (α,β)-metric $L(\alpha,\beta)$ the space $R^n = (M^n,\alpha)$ is called associated Riemannian space to the Finsler space $F_n = (M^n, L)$. The covariant differentiation with respect to the Levi-Civita connection $\gamma^i_{jk}(x)$ of R^n is denoted by $(;)$.

Lemma 2.1. [3] *The spray coefficient G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J(r_{00} - 2\alpha Q s_0)\frac{y^i}{\alpha} + H(r_{00} - 2\alpha Q s_0)\{b^i - \frac{y^i}{\alpha}\}, \tag{2.4}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$J = \frac{(\phi - s\phi')\phi'}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$
$$H = \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

$s_{l0} = s_{li}y^i$, $s_0 = s_{l0}b^l$, $r_{00} = r_{ij}y^iy^j$, $r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i})$, $s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i})$, $r^i_j = a^{ir}r_{rj}$, $s^i_j = a^{ir}s_{rj}$, $r_j = b_r r^r_j$, $s_j = b_r s^r_j$, $b^i = a^{ir}b_r$ *and* $b^2 = a^{rs}b_rb_s$.

It is well-known that a Finsler metric $L = L(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if [4]

$$L_{x^k y^l} y^k - L_{x^l} = 0. \tag{2.5}$$

By (2.5), we have the following lemma:

Lemma 2.2. [14] *An* (α, β)*-metric* $L = \alpha\phi(s)$*, where* $s = \frac{\beta}{\alpha}$*, is projectively flat on an open subset* $U \subset \mathbb{R}^n$ *if and only if*

$$(a_{ml}\alpha^2 - y_my_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(r_{00} - 2\alpha Q s_0)(b_l\alpha - sy_l) = 0. \tag{2.6}$$

The functions G^i of F_n with an (α, β)-metric are written in the form [7]

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.7}$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}, \tag{2.8}$$

provided $\beta^2 + L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$. The subscript 0 means contraction by y^i. We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity.

From (2.7) the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F_n with an (α, β)-metric is given by [7]

$$G^i_j = \dot{\partial}_j G^i = \gamma^i_{0j} + B^i_j, \tag{2.9}$$

(2.10) $$G^i_{jk} = \dot{\partial}_k G^i_j = \gamma^i_{jk} + B^i_{jk},$$

where $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$. On account of [7], B^i_{jk} is determined by

(2.11) $$L_\alpha B^k_{ji} y^j y_t + \alpha L_\beta (B^k_{ji} b_t - b_{j;i}) y^j = 0,$$

where $y_k = a_{ik} y^i$.

A Finsler space F_n with an (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ is $hp(3)$ [2].

From (2.8) B^{ij} is written as follows

(2.12) $$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i).$$

Lemma 2.3. [8] *If $(a_{ml}\alpha^2 - y_m y_l) G^m_\alpha = 0$, then α is projectively flat.*

3. Projectively Flat Special (α, β)-metric

Let L be an exponential (α, β)-metric, i.e.

(3.1) $$L = \alpha\phi(s), \ \ \phi(s) = e^s, \ \ s = \frac{\beta}{\alpha},$$

Let $b_0 > 0$ be the largest number such that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ \ (|s| \leq b < b_0),$$

i.e.,

$$e^s(1 - s + b^2 - s^2) > 0, \quad (|s| \leq b < b_0).$$

Lemma 3.1. *$L = \alpha e^{\beta/\alpha}$ is a regular Finsler metric if and only if $\|\beta\|_\alpha < 1$.*

Proof. If $L = \alpha e^{\beta/\alpha}$ is a regular Finsler metric, then

$$e^s(1 - s + b^2 - s^2) > 0, \quad |s| \leq b < b_0.$$

Let $s = b$, then we get $b < 1$, $\forall\ b < b_0$. Let $b \to b_0$, then $b_0 < 1$. So $\|\beta\|_\alpha < 1$. Now if

$$|s| \leq b < 1,$$

then

$$e^s(1 - s + b^2 - s^2) \geq e^s(1 - s) > 0 \ \ (\textit{because } b^2 - s^2 \geq 0).$$

Thus we see that $L = \alpha e^{\beta/\alpha}$ is a regular Finsler metric. □

By Lemma 2.1, the spray coefficients G^i of L are given by (2.1) with

$$Q = \frac{\alpha}{\alpha - \beta},$$

$$J = \frac{\alpha(\alpha - \beta)}{2(\alpha^2 - \beta^2 - \alpha\beta + b^2\alpha^2)},$$

$$H = \frac{\alpha^2}{2(\alpha^2 - \beta^2 - \alpha\beta + b^2\alpha^2)}.$$

Equation (2.7) is reduced to the following form:

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \frac{\alpha^4}{\alpha - \beta} s_{l0} + \frac{\alpha^3}{2(\alpha^2 - \beta^2 - \alpha\beta + b^2\alpha^2} \times \left[r_{00} - \frac{2\alpha^2 s_0}{\alpha - \beta} \right] (b_l\alpha - \frac{\beta}{\alpha} y_l) = 0. \tag{3.2}$$

Theorem 3.1. *The (α, β)-metric of type $L = \alpha e^{\beta/\alpha}$ is locally projectively flat if and only if*

(1) *β is parallel with respect to α,*

(2) *α is locally projectively flat, i.e. of constant curvature.*

Proof. Suppose that L is locally projectively flat. First, we rewrite (3.2) as a polynomial in y^i and α. We have

$$\Big[(-4\alpha^2\beta+2\beta^3-2b^2\alpha^2\beta)(a_{ml}\alpha^2-y_my_l)G^m_\alpha$$
$$+(2\alpha^6+2b^2\alpha^6-2\alpha^4\beta^2)s_{l0}$$
$$-2\alpha^4s_0(b_l\alpha^2-\beta y_l)-\alpha^2\beta r_{00}(b_l\alpha^2-\beta y_l)\Big]$$
$$+\alpha\Big[(2\alpha^2+2\alpha^2b^2)(a_{ml}\alpha^2-y_my_l)G^m_\alpha$$
$$-2\alpha^4\beta s_{l0}+\alpha^2r_{00}(b_l\alpha^2-\beta y_l)\Big]=0,$$

or

$$U+\alpha V=0, \tag{3.3}$$

where

$$\begin{aligned}U&=(-4\alpha^2\beta+2\beta^3-2b^2\alpha^2\beta)(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\&\quad+(2\alpha^6+2b^2\alpha^6-2\alpha^4\beta^2)s_{l0}-2\alpha^4s_0(b_l\alpha^2-\beta y_l)\\&\quad-\alpha^2\beta r_{00}(b_l\alpha^2-\beta y_l),\end{aligned}$$

and

$$\begin{aligned}V&=(2\alpha^2+2\alpha^2b^2)(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\&\quad-2\alpha^4\beta s_{l0}+\alpha^2r_{00}(b_l\alpha^2-\beta y_l).\end{aligned}$$

The left hand side of (3.3) is a polynomial in y^i, such that U and V are rational in y^i and α is irrational. Therefore we must have

$$U=0 \ and \ V=0,$$

which implies that

$$\begin{aligned}(3.4)\qquad & [-2\alpha^2\beta(2+b^2)+2\beta^3](a_{ml}\alpha^2-y_my_l)G^m_\alpha \\ & = -2\alpha^4[\alpha^2(1+b^2)-\beta^2]s_{l0}+2\alpha^4 s_0(b_l\alpha^2-\beta y_l) \\ & + \alpha^2\beta r_{00}(b_l\alpha^2-\beta y_l)\end{aligned}$$

and

$$\begin{aligned}(3.5)\qquad & 2(1+b^2)(a_{ml}\alpha^2-y_my_l)G^m_\alpha \\ & = 2\alpha^2\beta s_{l0}-r_{00}(b_l\alpha^2-\beta y_l).\end{aligned}$$

Transvecting (3.4) and (3.5) with b^l, we get

$$\begin{aligned}(3.6)\qquad & [-2\alpha^2\beta(2+b^2)+2\beta^3](b_m\alpha^2-y_m\beta)G^m_\alpha \\ & = -2\alpha^4[\alpha^2(1+b^2)-\beta^2]s_0+2\alpha^4 s_0(b^2\alpha^2-\beta^2) \\ & + \alpha^2\beta r_{00}(b^2\alpha^2-\beta^2)\end{aligned}$$

and

$$\begin{aligned}(3.7)\qquad & 2(1+b^2)(b_m\alpha^2-y_m\beta)G^m_\alpha \\ & = 2\alpha^2\beta s_0-r_{00}(b^2\alpha^2-\beta^2).\end{aligned}$$

Multiplying (3.7) by $\alpha^2\beta$ and adding in (3.6), we have

$$(3.8)\qquad \beta(b_m\alpha^2-y_m\beta)G^m_\alpha = \alpha^4 s_0.$$

In (3.8), α^4 is not divisible by β and β is not divisible by α^4. Thus s_0 is divisible by β and $(b_m\alpha^2-y_m\beta)$ is divisible by α^4. Therefore there exist scalar functions $\tau=\tau(x)$ and $\eta=\eta(x)$ such that

$$(3.9)\qquad s_0=\tau\beta,$$

$$(3.10)\qquad (b_m\alpha^2-y_m\beta)G^m_\alpha=\eta\alpha^4.$$

From (3.8), (3.9) and (3.10), we have

$$\tau = \eta. \tag{3.11}$$

In view (3.9), (3.10) and (3.11) equation (3.6), becomes

$$\eta[2\alpha^4(1+b^2) - 2\alpha^2\beta^2] = -r_{00}(b^2\alpha^2 - \beta^2). \tag{3.12}$$

Since $(b^2\alpha^2 - \beta^2)$ is not divisible by $[2\alpha^4(1+b^2) - 2\alpha^2\beta^2]$, it follows from (3.12) that $\eta = 0$.
From (3.9), (3.10), (3.11) and (3.12), we get

$$s_0 = 0, \tag{3.13}$$

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha = 0, \tag{3.14}$$

$$r_{00} = 0. \tag{3.15}$$

In view of (3.14) and (3.15), equation (3.5), gives

$$s_{l0} = 0. \tag{3.16}$$

In view of Lemma (2.3), equation (3.14) implies that α is projectively flat. From (3.15) and (3.16), $b_{i;j} = 0$ i.e., β is parallel with respect to α.
Conversely, if β is parallel with respect to α and α is locally projectively flat, then by Lemma (2.2), we see that L is locally projectively flat. □

Example: The special form of (α, β)-metric

$$L = \alpha + \epsilon\beta + k\left(\frac{\beta^2}{\alpha}\right),$$

where ϵ and k are non-zero constant to illustrate Theorem 3.1

4. Berwald and Douglas Spaces

In this section, we obtained a necessary and sufficient condition for a Finsler space F_n with (α,β)-metric to be a Berwald space.
In view of (2.2) equation (2.11), becomes

$$\alpha e^{\beta/\alpha}B^t_{ji}y^jy_t-\beta e^{\beta/\alpha}B^t_{ji}y^jy_t+\alpha^2(B^t_{ji}b_t-b_{j;i})y^j=0. \tag{4.1}$$

Assume that F_n is a Berwald space i.e., $G^i_{jk}=G^i_{jk}(x)$. Then we have $B^t_{ji}=B^t_{ji}(x)$. Since α is irrational in y^i, from (4.1), we have

$$e^{\beta/\alpha}B^t_{ji}y^jy_t=0, \tag{4.2}$$

$$-\beta e^{\beta/\alpha}B^t_{ji}y^jy_t+\alpha^2(B^t_{ji}b_t-b_{j;i})y^j=0. \tag{4.3}$$

From (4.2) and (4.3), we get

$$B^t_{ji}y^jy_t=0 \ \ and \ \ (B^t_{ji}b_t-b_{j;i})y^j=0,$$

which yield

$$B^t_{ji}a_{th}+B^t_{hi}a_{tj}=0, \ \ (B^t_{ji}b_t-b_{j;i})y^j=0.$$

Thus, by the well known Christoffel process, we get $B^t_{ji}=0$.
Therefore we have the following Theorem

Theorem 4.1. *An exponential (α,β)-metric of type $L=\alpha e^{\beta/\alpha}$ provides a Berwald metric if and only if $b_{j;i}=0$, then Berwald connection is Riemannian.*

Now we consider the condition for a Finsler space F^n with an (α,β)-metric to be Douglas space. Using (2.2) in (2.12), we obtain

$$\begin{aligned}&2B^{ij}(\alpha-\beta)(\alpha^2-\alpha\beta+b^2\alpha^2-\beta^2)-2\alpha^2(\alpha^2-\alpha\beta+b^2\alpha^2-\beta^2)\\&\times(s^i_0y^j-s^j_0y^i)-\alpha^2(b^iy^j-b^jy^i)\{(\alpha-\beta)r_{00}-2\alpha^2s_0\}\end{aligned} \tag{4.4}$$

Suppose that F^n is a Douglas space, i.e. B^{ij} are $hp(3)$. Separating (4.4) in rational and irrational terms of y^i, we have

$$\begin{aligned}[] &[(-2\alpha^4-2b^2\alpha^4+2\alpha^2\beta^2)(s^i_0y^j-s^j_0y^i)+\alpha^2\beta r_{00}(b^iy^j-b^jy^i)\\ &+2\alpha^4s_0(b^iy^j-b^jy^i)+(-4\alpha^2\beta+2\beta^3-2b^2\alpha^2\beta)B^{ij}]\\ &+\alpha[B^{ij}(2\alpha^2+2b^2\alpha^2)+2\alpha^2\beta(s^i_0y^j-s^j_0y^i)-\alpha^2r_{00}(b^iy^j-b^jy^i)] \end{aligned} \tag{4.5}$$

The equation (4.5) is divided into two equations as follows:

$$\begin{aligned}[] &[(-2\alpha^4-2b^2\alpha^4+2\alpha^2\beta^2)(s^i_0y^j-s^j_0y^i)+\alpha^2\beta r_{00}(b^iy^j-b^jy^i)\\ &+2\alpha^4s_0(b^iy^j-b^jy^i)+(-4\alpha^2\beta+2\beta^3-2b^2\alpha^2\beta)B^{ij}], \end{aligned} \tag{4.6}$$

$$[B^{ij}(2\alpha^2+2b^2\alpha^2)+2\alpha^2\beta(s^i_0y^j-s^j_0y^i)-\alpha^2r_{00}(b^iy^j-b^jy^i)]. \tag{4.7}$$

Eliminating B^{ij} from (4.6) and (4.7), we get

$$A(s^i_0y^j-s^j_0y^i)+B(b^iy^j-b^jy^i)=0, \tag{4.8}$$

where

$$A=-2\alpha^4-4b^2\alpha^4+4\alpha^2\beta^2-2b^4\alpha^4+4b^2\alpha^2\beta^2-2\beta^4, \tag{4.9}$$

and

$$B=r_{00}(\beta^3-\alpha^2\beta^2)+2\alpha^4(1+b^2)s_0. \tag{4.10}$$

Transvecting (4.8) by b_iy_j, we get

$$As_0\alpha^2+B(b^2\alpha^2-\beta^2)=0. \tag{4.11}$$

The term of (4.11) which do not contain α^2 is $-\beta^5r_{00}$. Hence there exits $hp(5)$: v_5 such that

$$-\beta^5r_{00}=\alpha^2v_5 \tag{4.12}$$

Here we consider two cases:

case (i): When $v_5 = 0$, this leads to $r_{00} = 0$. Therefore substituting $r_{00} = 0$ into (4.11), we get

$$s_0(A + \gamma^2 B_1) = 0, \tag{4.13}$$

where $B_1 = 2\alpha^2(1 + b^2)$.

If $A + \gamma^2 B_1 = 0$, then term of $A + \gamma^2 B_1 = 0$ which do not contain α^2 is $-2\beta^4$.Thus there exits $hp(2) : v_2$ such that

$$-2\beta^4 = \alpha^2 v_2, \tag{4.14}$$

hence we have $v_2 = 0$, which leads to a contradiction. Therefore we must have $a + \gamma^2 B_1 \neq 0$. Therefore we have $s_0 = 0$, from (4.13). Substituting $s_0 = 0$ and $r_{00} = 0$ into (4.8), we get

$$A(s^i_0 y^j - s^j_0 y^i) = 0. \tag{4.15}$$

If $A = 0$, then from (4.9), we have

$$-2\alpha^4 - 4b^2\alpha^4 + 4\alpha^2\beta^2 - 2b^4\alpha^4 + 4b^2\alpha^2\beta^2 - 2\beta^4 = 0. \tag{4.16}$$

The term of (4.15) which does not contain α^2 is $-2\beta^4$. Thus there exist $hp(2)$: v_2 such that

$$-2\beta^4 = \alpha^2 v_2, \tag{4.17}$$

from which we have $v_2 = 0$. It is a contradiction, therefore we must have $A \neq 0$. From (4.15), we get

$$s^i_0 y^j - s^j_0 y^i = 0.$$

Transvecting above equation y_j gives $s^i_0 = 0$, which imply $s_{ij} = 0$. Consequently, we have $r_{ij} = s_{ij} = 0$, i.e. $b_{i;j} = 0$.

case (ii): The equation (4.12) shows that there exists a function $k = k(x)$ satisfying

$$r_{00} = k(x)\alpha^2.$$

Thus we have the term of (4.11) does not contain α^2 is included in the term $-\beta^5 r_{00}$. Hence we get $r_{00} = 0$. From (4.14), we have $A(s^i_0 y^j - s^j_0 y^i) = 0$. If $A = 0$, then it is a contradiction. Hence $A \neq 0$. Therefore we obtain $s^i_0 y^j - s^j_0 y^i = 0$. Transvecting this equation by y_j we get $s^i_0 = 0$. Hence both the cases (i) and (ii) lead to $r_{00} = 0$ and $s_{ij} = 0$, i.e. $b_{i;j} = 0$.
Conversely if $b_{i;j} = 0$, then F^n is a Berwald space, so F^n is a Douglas space.
Thus we have the following Theorem

Theorem 4.2. *An exponential (α, β)-metric of type $L = \alpha\, e^{\beta/\alpha}$ is of Douglas type if and only if $\alpha^2 \not\equiv 0 (\mod \beta)$ and $b_{i;j} = 0$.*

From Theorem 4.1 and Theorem 4.2, we have

Theorem 4.3. *If the exponential (α, β)-metric of type $L = \alpha\, e^{\beta/\alpha}$ is of Douglas type, then it is Barwaldian.*

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