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QUASI-MULTIPLICATION AND QUASI-COMULTIPLICATION MODULES

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Abstract. In this paper, we will introduce the notion of quasi-multiplication (resp. quasi-comultiplication) modules over a commutative ring as a generalization of multiplication (resp. comultiplication) modules and explore some basic properties of these classes of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Multiplication rings are introduced by W. Krull in 1925 as a generalization of Dedekind domains [9]. In 1981, Barnard [6] has given the concept of multiplication modules. An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [6]. Equivalently, M is a multiplication module if and only if for each submodule N of M, we have $N = (N :_R M)M$. There is a large body of research concerning multiplication modules. H. Ansari-Toroghy and F. Farshadifer introduced [2] the notion of comultiplication module as a dual notion of multiplication module and investigated some properties of this class of modules. An R-module M is said to be a *comultiplication module* if

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for every submodule N of M there exists an ideal I of R such that

$$N = (0 :_M I) = \{m \in M \mid Im = 0\},$$

equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

The purpose of this paper is to introduce the notions of quasi-multiplication and quasi-comultiplication R-modules as generalizations of multiplication and comultiplication R-modules, respectively and investigate some properties of these classes of modules.

2. Quasi-multiplication modules

Definition 2.1. We say that an R-module M is a *quasi-multiplication module* if whenever $Ann_R(rM) = Ann_R(M)$ for each $r \in R$, then $(0 :_M r) = 0$.

Lemma 2.2. Every multiplication R-module is a quasi-multiplication R-module.

Proof. Let M be a multiplication R-module and $Ann_R(rM) = Ann_R(M)$ for some $r \in R$. Then there exists an ideal I of R such that $(0 :_M r) = IM$. It follows that $I \subseteq Ann_R(rM)$. Therefore, $I \subseteq Ann_R(M)$ and so $(0 :_M r) = 0$. □

The converse of Lemma 2.2 need not be true in general as explained in Example 2.3 below.

Example 2.3. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since

$$(\mathbb{Z}_2 \oplus 0 :_{\mathbb{Z}} \mathbb{Z}_2 \oplus \mathbb{Z}_2)(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = 0 \neq \mathbb{Z}_2 \oplus 0,$$

M is not a multiplication $\mathbb{Z}$-module. If r is an even integer, then we have

$$2\mathbb{Z} = Ann_{\mathbb{Z}}(M) \neq Ann_{\mathbb{Z}}(rM) = Ann_{\mathbb{Z}}(0) = \mathbb{Z}.$$

If r is an odd integer, then $Ann_{\mathbb{Z}}(rM) = Ann_{\mathbb{Z}}(M) = 2\mathbb{Z}$ and $(0 :_M r) = 0$. Therefore, M is a quasi-multiplication $\mathbb{Z}$-module.

Proposition 2.4. Let M be an R-module. In either of the following cases, M is a quasi-multiplication R-module.

(a) $(0 :_M r) = ((0 :_M r) :_R M)M$ for each $r \in R$.

(b) If $(Rm :_R M) = Ann_R(M)$ for each $m \in M$, then $Rm = 0$.

Proof. First assume that part (a) holds. Let $r \in R$ and $Ann_R(rM) = Ann_R(M)$. Then by part (a) we have

$$(0 :_M r) = ((0 :_M r) :_R M)M = Ann_R(rM)M = Ann_R(M)M = 0.$$

Therefore, M is a quasi-multiplication R-module. Now assume that part (b) holds. Let $r \in R$ and $Ann_R(rM) = Ann_R(M)$. Suppose that $m \in (0 :_M r)$. Then

$$(Rm :_R M) \subseteq ((0 :_M r) :_R M) = Ann_R(rM) = Ann_R(M).$$

Thus $(Rm :_R M) = Ann_R(M)$ because the reverse inclusion is clear. Now by part (b), $Rm = 0$. It follows that M is a quasi-multiplication R-module. □

Let M be an R-module. The dual notion of $Z_R(M)$, the set of zero divisors of M, is denoted by $W_R(M)$ and defined by

$$W_R(M) = \{a \in R : aM \neq M\}.$$

M is said to be *Hopfian* (resp. *co-Hopfian*) if every surjective (resp. injective) endomorphism f of M is an isomorphism.

A submodule N of M is said to be *pure submodule* if $IN = N \cap IM$ for every ideal I of R [1].

A submodule N of of M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [5].

Theorem 2.5. *Let M be an R-module. Then we have the following.*

(a) *If $W_R(M) = Z_R(R/Ann_R(M))$ and M is a co-Hopfian R-module, then M is a quasi-multiplication module.*

(b) *If M is a Hopfian quasi-multiplication module, then $W_R(M) = Z_R(R/Ann_R(M))$.*

(c) *If N is a pure submodule of an R-module M such that $Ann_R(N) \not\subseteq W_R(M/N)$, then N is a direct summand of M.*

(d) *If N is a copure submodule of an R-module M such that $Ann_R(M/N) \not\subseteq Z_R(N)$, then N is a direct summand of M.*

Proof. (a) Let $r \in R$ such that $Ann_R(rM) = Ann_R(M)$. If $rM = M$, then $(0 :_M r) = 0$ because M is co-Hopfian. So suppose that $rM \neq M$. Hence $r \in W_R(M)$. Thus by assumption, there exists $t \in R \setminus Ann_R(M)$ such that $rt \in Ann_R(M)$. Therefore, $t \in Ann_R(rM) = Ann_R(M)$. This is a desired contradiction.

(b) Clearly, $Z_R(R/Ann_R(M)) \subseteq W_R(M)$. Let $r \in W_R(M)$. Then $rM \neq M$. Now as M is co-Hopfian, $(0 :_M r) \neq 0$. So by assumption, we can choose $t \in Ann_R(rM) \setminus Ann_R(M)$. It follows that $r \in Z_R(R/Ann_R(M))$, as required.

(c) Let $r \in Ann_R(N) \setminus W_R(M/N)$. Then $rN = 0$ and $r(M/N) = M/N$. Thus $M = rM + N$ and $0 = rN = rM \cap N$ because N is pure.

(d) Let $r \in Ann_R(M/N) \setminus Z_R(N)$. Then $rM \subseteq N$. Thus $M \subseteq (N :_M r) = N + (0 :_M r)$. As $r \notin Z_R(N)$, $N \cap (0 :_M r) = 0$ as needed. □

A proper submodule P of an R-module M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [7]. M is said to be a *prime module* if the zero submodule of M is a prime submodule of M.

An R-module M is said to be a *second module* if $M \neq 0$ and for each $a \in R$, the endomorphism $M \xrightarrow{a} M$ is either surjective or zero [11].

Proposition 2.6. Let M be an R-module. Then we have the following.

(a) If M is a prime module, then M is a quasi-multiplication module.

(b) If $t \in R \setminus W_R(M)$ and tM is a quasi-multiplication R-module, then M is a quasi-multiplication R-module.

(c) If R is an integral domain and M is a faithful quasi-multiplication R-module, then M is a prime module.

(d) If M is a second quasi-multiplication R-module, then M is a prime R-module.

Proof. (a) This is clear.

(b) Let $r \in R$ such that $Ann_R(M) = Ann_R(rM)$. Then $Ann_R(tM) = Ann_R(rtM)$. So by assumption, $(0 :_{tM} r) = 0$. Now let $mr = 0$ for some $m \in M$. As $t \notin W_R(M)$, $tM = M$. Thus $m = ty$ for some $y \in M$. Hence, $tyr = 0$ implies that $ty \in (0 :_{tM} r) = 0$. Thus $m = ty = 0$. Therefore, $(0 :_M r) = 0$.

(c) Let $r \in R$ such that $(0 :_M r) \neq M$. Clearly, $Ann_R(M) \subseteq Ann_R(rM)$. Now let $t \in Ann_R(rM)$. Then $tr \in Ann_R(M) = 0$. As R is an integral domain and $(0 :_M r) \neq M$, we have $t \in Ann_R(M)$. Hence $Ann_R(rM) \subseteq Ann_R(M)$. Therefore, $Ann_R(rM) = Ann_R(M)$. Thus $(0 :_M r) = 0$ since M is a quasi-multiplication R-module.

(d) Let $r \in R$. Then by assumption, $rM = 0$ or $Ann_R(M) = Ann_R(M/(0 :_M r)) = Ann_R(rM)$. Thus $rM = 0$ or $(0 :_M r) = 0$, as required. □

Remark 2.7. The converse of part (a) of Proposition 2.6, is not true in general because if it is true, then every multiplication module is prime by Lemma 2.2.

Theorem 2.8. *Let M be a finitely generated R-module and S be a multiplicatively closed subset of R such that $S \cap W_R(M) = \emptyset$. Then we have the following.*

(a) $Ann_{S^{-1}R}((r/1)S^{-1}M) = Ann_{S^{-1}R}(S^{-1}M)$ *implies that* $Ann_R(rM) = Ann_R(M)$.

(b) *M is a quasi-multiplication R-module if and only if $S^{-1}M$ is a quasi-multiplication $S^{-1}R$-module.*

Proof. (a) Let $t \in Ann_R(rM)$. As M is finitely generated,

$$S^{-1}Ann_R(rM) = Ann_{S^{-1}R}((r/1)S^{-1}M) =$$

$$Ann_{S^{-1}R}(S^{-1}M) = S^{-1}Ann_R(M).$$

Thus $t/1 \in S^{-1}Ann_R(M)$. Hence $ths = as$ for some $h, s \in S$ and $a \in Ann_R(M)$. Since $S \cap W_R(M) = \emptyset$, $hsM = M$. Therefore, $t \in Ann_R(M)$. Thus $Ann_R(rM) = Ann_R(M)$ because the reverse inclusion is clear.

(b) First assume that M is a quasi-multiplication R-module and

$$Ann_{S^{-1}R}((r/1)S^{-1}M) = Ann_{S^{-1}R}(S^{-1}M)$$

for some $r \in R$. Then $Ann_R(rM) = Ann_R(M)$ by part (a). Thus $(0 :_M r) = 0$. Therefore, $(0 :_{S^{-1}M} r/1) = 0$. Conversely, suppose that $Ann_R(rM) = Ann_R(M)$. Then as M is finitely generated,

$$S^{-1}Ann_R(rM) = Ann_{S^{-1}R}((r/1)S^{-1}M) =$$

$$Ann_{S^{-1}R}(S^{-1}M) = S^{-1}Ann_R(M).$$

Thus $(0 :_{S^{-1}M} r/1) = 0$. Now let $m \in (0 :_M r)$. Then $rm = 0$. Thus $(m/1)(r/1) = 0$. It follows that $m/1 \in (0 :_{S^{-1}M} r/1) = 0$. Hence, $sm = 0$ for some $s \in S$. As $S \cap W_R(M) = \emptyset$, and $Z_R(M) \subseteq W_R(M)$, we have $m = 0$, as desired. $\square$

3. Quasi-comultiplication modules

Definition 3.1. We say that an R-module M is a *quasi-comultiplication module* if whenever $Ann_R(rM) = Ann_R(M)$ for each $r \in R$, then $rM = M$. This can be regarded as a dual notion of quasi-multiplication module.

Remark 3.2. Every comultiplication R-module is a quasi-comultiplication module by [3, 3.2]. But we see in the Example 3.3 that the converse is not true in general.

Example 3.3. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since

$$(0 :_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} Ann_{\mathbb{Z}}(\mathbb{Z}_2 \oplus 0)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \neq \mathbb{Z}_2 \oplus 0,$$

M is not a comultiplication $\mathbb{Z}$-module. If r is an even integer, then we have

$$2\mathbb{Z} = Ann_{\mathbb{Z}}(M) \neq Ann_{\mathbb{Z}}(rM) = Ann_{\mathbb{Z}}(0) = \mathbb{Z}.$$

If r is an odd integer, then $Ann_{\mathbb{Z}}(rM) = Ann_{\mathbb{Z}}(M) = 2\mathbb{Z}$ and $rM = M$. Therefore, M is a quasi-comultiplication $\mathbb{Z}$-module.

The following example shows that not every quasi-multiplication R-module is a quasi-comultiplication R-module.

Example 3.4. Let M be the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. Then for each integer r, $Ann_{\mathbb{Z}}(rM) = Ann_{\mathbb{Z}}(M) = 0$. Since for each integer r, $(0 :_M r) = 0$, we have M is a quasi-multiplication $\mathbb{Z}$-module. But $2M \neq M$, implies that M is not a quasi-comultiplication $\mathbb{Z}$-module.

The following example shows that not every quasi-comultiplication R-module is a quasi-multiplication R-module.

Example 3.5. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. Then for each integer r, $Ann_{\mathbb{Z}}(rM) = Ann_{\mathbb{Z}}(M) = 0$. As $(0 :_M p) = (1/p + \mathbb{Z})\mathbb{Z} \oplus (1/p + \mathbb{Z})\mathbb{Z} \neq 0$, we have M is not a quasi-multiplication $\mathbb{Z}$-module. But since for each integer r, $rM = M$, M is a quasi-comultiplication $\mathbb{Z}$-module.

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [8].

Remark 3.6. [4] Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Proposition 3.7. Let M be an R-module. In either of the following cases, M is a quasi-comultiplication R-module.

(a) $rM = (0 :_M Ann_R(rM))$ for each $r \in R$.

(b) If $Ann_R(L) = Ann_R(M)$ for each completely irreducible submodule L of M, then $L = M$.

Proof. First assume that part (a) holds. Let $r \in R$ and $Ann_R(rM) = Ann_R(M)$. Then by part (a) and the fact that $(0 :_M Ann_R(M)) = M$ we have

$$rM = (0 :_M Ann_R(rM)) = (0 :_M Ann_R(M) = M.$$

Now assume that part (b) holds. Let $r \in R$ and $Ann_R(rM) = Ann_R(M)$. Let L be a completely irreducible submodule of M such that $rM \subseteq L$. Then $Ann_R(L) \subseteq Ann_R(rM) = Ann_R(M)$. It follows that $Ann_R(L) = Ann_R(M)$. Thus by assumption, $L = M$. Hence, $rM = M$ by Remark 3.6. □

Theorem 3.8. *Let M be an R-module. Then we have the following.*

(a) *If $Z_R(M) = Z_R(R/Ann_R(M))$ and M is a Hopfian R-module, then M is a quasi-comultiplication module.*

(b) *If M is a co-Hopfian quasi-comultiplication module, then*

$$Z_R(M) = Z_R(R/Ann_R(M)).$$

Proof. (a) Let $r \in R$ such that $Ann_R(rM) = Ann_R(M)$. If $(0 :_M r) = 0$, then $rM = M$ because M is Hopfian. So suppose that there exists $0 \neq m \in M$ such that $rm = 0$. Hence $r \in Z_R(M)$. Thus by assumption, there exists $t \in R \setminus Ann_R(M)$

such that $rt \in Ann_R(M)$. Therefore, $t \in Ann_R(rM) = Ann_R(M)$. This is a desired contradiction.

(b) Clearly, $Z_R(R/Ann_R(M)) \subseteq Z_R(M)$. Let $r \in Z_R(M)$. Then there exists $0 \neq m \in M$ such that $rm = 0$. This implies that $(0 :_M r) \neq 0$. Now as M is co-Hopfian, $rM \neq M$. So by assumption, we can choose $t \in Ann_R(rM) \setminus Ann_R(M)$. It follows that $r \in Z_R(R/Ann_R(M))$, as required. □

A non-zero submodule N of an R-module M is said to be *secondal* if $W_R(N) = \{a \in R : aN \neq N\}$ is an ideal of R [4].

Theorem 3.9. *Let M be an R-module and S be a multiplicatively closed subset of R such that $S \cap W_R(M) = \emptyset$. Then we have the following.*

(a) *If M is an Artinian quasi-comultiplication R-module, then*

$$W_{S^{-1}R}(Hom_R(S^{-1}R, Ann_R(M))) = S^{-1}(W_R(M)).$$

(b) *If M is an Artinian quasi-comultiplication secondal R-module, then $Hom_R(S^{-1}R, Ann_R(M))$ is a secondal $S^{-1}R$-module.*

(c) *If $Z_R(M) \subseteq W_R(M)$, then $S^{-1}(W_R(M)) = W_{S^{-1}R}(S^{-1}M)$.*

Proof. (a) Let $r/s \in W_{S^{-1}R}(Hom_R(S^{-1}R, Ann_R(M)))$. If $r/s \notin S^{-1}(W_R(M))$, then $r \notin W_R(M)$. Thus $rM = M$ and so $Ann_R(rM) = Ann_R(M)$. Hence

$$Hom_R(S^{-1}R, Ann_R(rM)) = Hom_R(S^{-1}R, Ann_R(M)).$$

This implies that

$$(r/s)Hom_R(S^{-1}R, Ann_R(M)) = (r/1)Hom_R(S^{-1}R, Ann_R(M)) =$$

$$Hom_R(S^{-1}R, Ann_R(M)).$$

This is a contradiction. Conversely, suppose that $r/s \in S^{-1}(W_R(M))$. Then $rM \neq M$. Now since M is a quasi-comultiplication secondal module, $Ann_R(rM) \neq Ann_R(M)$.

If $r/s \notin W_{S^{-1}R}(Hom_R(S^{-1}R, Ann_R(M)))$, then

$$(r/s)Hom_R(S^{-1}R, Ann_R(M)) = Hom_R(S^{-1}R, Ann_R(M)).$$

Thus

$$Hom_R(S^{-1}R, Ann_R(rM)) = Hom_R(S^{-1}R, Ann_R(M)).$$

It follows that

$$\phi(Hom_R(S^{-1}R, Ann_R(rM))) = \phi(Hom_R(S^{-1}R, Ann_R(M))),$$

where $\phi : Hom_R(S^{-1}R, Ann_R(rM))) \to Ann_R(rM))$ is the natural homomorphism defined by $f \mapsto f(1_{\overline{R}})$. Thus by [10], $tAnn_R(rM) = hAnn_R(M)$ for some $t, h \in S$. Hence, $tMAnn_R(rM) = 0$. Since $S \cap W_R(M) = \emptyset$, $tM = M$. Therefore, $MAnn_R(rM) = 0$. Thus $Ann_R(M) = Ann_R(rM)$, a contradiction.

(b) This follows from part (a).

(c) Let $r/s \in S^{-1}(W_R(M))$. Then $r/s = a/t$ for some $a \in W_R(M)$ and $s \in S$. Thus $aM \neq M$. Hence there exist $m \in M \setminus aM$. If $a/tS^{-1}M = S^{-1}M$, then $m/1 = (a/t)(m_1/h)$ for some $m_1 \in M$ and $h \in S$. Thus $kthm = kam_1$ for some $k \in S$. Since $S \cap W_R(M) = \emptyset$, $kthM = M$. Hence $m_1 = kthm_2$ for some $m_2 \in M$. It follows that $kth(m - kam_2) = 0$. If $(m - kam_2) \neq 0$, then $kth \in Z_R(M) \subseteq W_R(M)$, a contradiction. Thus $m \in aM$, which is a required contradiction. Therefore, $S^{-1}(W_R(M) \subseteq W_{S^{-1}R}(S^{-1}M)$. To see the reverse inclusion, let $r/s \in W_{S^{-1}R}(S^{-1}M)$. If $r \notin W_R(M)$, then $rM = M$. This implies that $(r/s)S^{-1}M = S^{-1}M$, a contradiction. □

Theorem 3.10. *Let M be a finitely generated R-module and S be a multiplicatively closed subset of R such that $S \cap W_R(M) = \emptyset$. Then M is a quasi-comultiplication R-module if and only if $S^{-1}M$ is a quasi-comultiplication $S^{-1}R$-module.*

Proof. First assume that M is a quasi-comultiplication R-module and

$$Ann_{S^{-1}R}((r/1)S^{-1}M) = Ann_{S^{-1}R}(S^{-1}M)$$

for some $r \in R$. Then $Ann_R(rM) = Ann_R(M)$ by part (a) of Theorem 2.8. Thus $rM = M$. Therefore, $(r/1)S^{-1}M = S^{-1}M$. Conversely, suppose that $Ann_R(rM) = Ann_R(M)$. Then as M is finitely generated,

$$S^{-1}Ann_R(rM) = Ann_{S^{-1}R}((r/1)S^{-1}M) =$$

$$Ann_{S^{-1}R}(S^{-1}M) = S^{-1}Ann_R(M).$$

hus $(r/1)S^{-1}M = S^{-1}M$. Now let $0 \neq m \in M$. Then $m/1 \in S^{-1}M = (r/1)S^{-1}M$. Thus $shm = srm_1$ for some $s, h \in S$ and $m_1 \in M$. As $S \cap W_R(M) = \emptyset$, we have $hM = M$. Therefore, $m_1 = hm_2$. Thus $m = rm_2$ because $sh \notin Z_R(M) \subseteq W_R(M)$, as required. □

Proposition 3.11. Let M be an R-module. Then we have the following.

(a) If M is a second module, then M is a quasi-comultiplication module.

(b) If $t \in R \setminus Z_R(M)$ and tM is a quasi-comultiplication R-module, then M is a quasi-comultiplication R-module.

(c) If M is a quasi-comultiplication Hopfian R-module, then M is a quasi multiplication module.

(d) If M is a quasi-multiplication co-Hopfian R-module, then M is a quasi-comultiplication module.

(e) If R is an integral domain and M is a faithful quasi-comultiplication R-module, then M is a second module.

(f) If M is a prime quasi-comultiplication R-module, then M is a second R-module.

Proof. (a), (c), and (d) These are clear.

(b) Let $r \in R$ such that $Ann_R(M) = Ann_R(rM)$. Then $Ann_R(tM) = Ann_R(rtM)$. So by assumption, $tM = trM$. Now let $m \in M$. Then $tm = rt\acute{m}$ for some $\acute{m} \in M$. Thus $t(m - r\acute{m}) = 0$. As $t \notin Z_R(M)$, $m = r\acute{m}$ and so $M \subseteq rM$, as needed.

(e) Let $r \in R$ such that $rM \neq 0$. As R is an integral domain and M is a faithful R-module, $Ann(rM) = Ann(Rr) = Ann_R(M) = 0$. Thus $rM = M$ since M is a quasi-comultiplication R-module.

(f) Let $r \in R$. Then by assumption, $rM = 0$ or $Ann_R(M) = Ann_R(rM)$. Thus $rM = 0$ or $M = rM$, as required. □

Remark 3.12. The converse of part (a) of Proposition 3.11, is not true in general because if it is true, then every comultiplication module is second by Remark 3.2.

Proposition 3.13. Let M be a finitely generated non-zero quasi-comultiplication R-module. Then $Ann_R(rM) \neq Ann_R(M)$ for each $r \in Jac(R)$, where $Jac(R)$ denotes the Jacobson radical of R.

Proof. This follows from Nakayama Lemma. □

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