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ON ALTERNATE DUALS OF GENERALIZED FRAMES

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Abstract. In this paper we give a sufficient condition as to when the difference of two g-frames is a g-frame and characterize an alternate dual g-frame of a given g-frame in a Hilbert space.

1. Introduction

Frames in Hilbert spaces have been introduced in 1952 by J. Duffin and A.C. Schaeffer [5] while studying non harmonic Fourier series. The work of Daubechies, Grossmann and Meyer [4] in 1986 reintroduced the Frames. Since then Frames have been widely developed and applied in signal and image processing [1], characterizing functional spaces [4], wireless communications [10, 11], probabilistic [6, 13] and coding theory [12]. The concept of generalized frames (or g-frames) in Hilbert spaces was introduced by W. Sun in [15]. G-frames are a natural generalizations of frames which cover many other recent generalizations of frames such as bounded quasi-projections [7], fusion frames [2] and pseudo frames [14]. In this paper, we obtain a sufficient condition [Theorem 3.1] for difference of g-frames to be a g-frame and characterization [Theorem3.3] of alternate dual g-frames.

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2. Preliminaries

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces and $\{\mathscr{V}_i\}_{i\in I}$ is a sequence of closed subspaces of $\mathscr{K}$, where I is a subset of Z and $L(\mathscr{H},\mathscr{V}_i)$ is the collection of all bounded linear operators from $\mathscr{H}$ into $\mathscr{V}_i$. And we denote by $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$.

Definition 2.1. A sequence $\{f_i : i \in I\}$ of elements in $\mathscr{H}$ is called a frame for $\mathscr{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|\mathbf{f}\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|\mathbf{f}\|^2 \ \ \forall\, f \in \mathscr{H} \tag{1}$$

The constants A and B are called lower and upper frame bounds.

Definition 2.2. A sequence $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{V}_i) : i \in I\}$ of bounded operators is said to be a generalized frame or simply a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|\mathbf{f}\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|\mathbf{f}\|^2 \ \ \forall\, f \in \mathscr{H} \tag{2}$$

we call A and B the lower and upper g-frame bounds, respectively. We call $\{\Lambda_i\}_{i\in I}$ a tight g-frame if $A = B$ and a Parseval g-frame if $A = B = 1$. If only the second inequality is required, we call it a g-Bessel sequence.

Definition 2.3. $\left(\sum_{i\in I} \oplus \mathscr{V}_i\right)_{l^2}$ is a Hilbert space and is defined by

$$\left(\sum_{i\in I} \oplus \mathscr{V}_i\right)_{l^2} = \left\{\{f_i\}_{i\in I} : f_i \in \mathscr{V}_i, i \in I,\ \|\{f_i\}_{i\in I}\|^2 = \sum_{i\in I} \|f_i\|^2 < \infty\right\}.$$

with the inner product defined by: $\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle$.

The synthesis(g-pre frame) operator of $\{\Lambda_i\}_{i\in I}$ is defined by

$$T_\Lambda : \left(\sum_{i\in I} \oplus \mathscr{V}_i\right)_{l^2} \to \mathscr{H},\ \ T_\Lambda\Big(\{f_i\}_{i\in I}\Big) = \sum_{i\in I} \Lambda_i^* f_i,$$

We call the adjoint T_Λ^*, where $T_\Lambda^* : \mathscr{H} \to \left(\sum_{i\in I} \oplus \mathscr{V}_i\right)_{l^2}$, of the synthesis operator, the analysis operator which is given by

$$T_\Lambda^* f = \{\Lambda_i f\}_{i\in I} \ \ \forall\, f \in \mathscr{H}.$$

By composing T_Λ and T_Λ^*, we obtain the g-frame operator $S_\Lambda : \mathscr{H} \to \mathscr{H}$ given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f \tag{3}$$

which is bounded, positive, self adjoint, invertible operator and satisfies $AI_{\mathscr{H}} \leq S_\Lambda \leq BI_{\mathscr{H}}$. Then the following reconstruction formula takes place for all $f \in \mathscr{H}$

$$f = S_\Lambda^{-1} S_\Lambda f = S_\Lambda S_\Lambda^{-1} f.$$

$\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is also a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_i\}_{i\in I}$ with bounds B^{-1} and A^{-1} and is said to be the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Definition 2.4. A g-frame $\{\Theta_i\}_{i\in I}$ of $\mathscr{H}$ is called an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$ if it satisfies

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \qquad \forall\, f \in \mathscr{H}. \tag{4}$$

It is easy to show that if $\{\Theta_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i\}_{i\in I}$ will be an alternate dual g-frame of $\{\Theta_i\}_{i\in I}$.

Definition 2.5. We call two g-Bessel sequences $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ to be orthogonal if

$$\sum_{i\in I} \Lambda_i^* \Theta_i f = 0 \ \text{ or } \ \sum_{i\in I} \Theta_i^* \Lambda_i f = 0 \qquad \forall\, f \in \mathscr{H}. \tag{5}$$

In terms of synthesis operators

$$T_\Lambda T_\Theta^* = 0 \quad \text{or} \quad T_\Theta T_\Lambda^* = 0. \tag{6}$$

where T_Λ and T_Θ are the synthesis operators for $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ respectively.

Definition 2.6. Let $\{\Xi_i \in \mathscr{L}(\mathscr{H}, \mathscr{W}_i) : i \in I\}$ be a sequence of operators. Then

(1) $\{\Xi_i\}_{i\in I}$ is a g-complete set for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$ if $\mathscr{H} = \overline{Span}\ \{\Xi_i^*(\mathscr{W}_i)\}_{i\in I}$;

(2) $\{\Xi_i\}_{i\in I}$ is a g-orthonormal system for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$ if $\Xi_i\Xi_j^* = \delta_{ij} I_{\mathscr{W}j}$ for all $i, j \in I$;

(3) a g-complete and g-orthonormal system $\{\Xi_i\}_{i\in I}$ is called a g-orthonormal basis for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$.

For more details about g-frames theory and it's applications Ref. [9] and [15].

Definition 2.7. (8):Let $\{\Xi_i\}_{i\in I}$ and $\{\Psi_i\}_{i\in I}$ be g-orthonormal basis for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$ and $\{\mathscr{V}_i\}_{i\in I}$, respectively. Let $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{V}_i) : i \in I\}$ be such that the series $\sum_{i\in I} \Lambda_i^* g_i^{'}$ is convergent for all $\{g_i^{'}\}_{i\in I} \in \left(\sum_{i\in I} \oplus \mathscr{V}_i\right)_{l^2}$

The g-R-dual sequence for the sequence $\{\Lambda_i\}_{i\in I}$ is $\Gamma_j^{\Lambda} : \mathscr{H} \to W_j$ and is defined as

$$\Gamma_j^{\Lambda} = \sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i \qquad \forall\, j \in I.$$

The following results which are referred to in this paper are listed in the form of lemmas.

Lemma 2.8.(3): *Let $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{V}_i) : i \in I\}$ be a g-frame for $\mathscr{H}$ with g-frame operator S and bounds A and B. Let L be a bounded linear operator on $\mathscr{H}$. Then $\{\Lambda_i L \in \mathscr{L}(\mathscr{H}, \mathscr{V}_i) : i \in I\}$ is a g-frame for $\mathscr{H}$ if and only if L is invertible on $\mathscr{H}$. Moreover, in this case, the g-frame operator for $\{\Lambda_i L\}_{i\in I}$ is L^*SL and new bounds are $A\|L^{-1}\|^2$ and $B\|L\|^2$.*

Lemma 2.9.(9): *Suppose that $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are g-frames for Hilbert space $\mathscr{H}$, T_1 and T_2 are g-pre frame operators associated with $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ respectively. If $T_2T_1^* = 0$, then $\{\Lambda_i + \Theta_i : i \in I\}$ is a g-frame for $\mathscr{H}$. Moreover, if $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are normalized tight g-frames and $T_2T_1^* = 0$, then $\{\Lambda_i + \Theta_i : i \in I\}$ is a tight g-frame for $\mathscr{H}$ with bounds 2.*

3. Characterization of Duality

Theorem 3.1. *Let* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{V}_i) : i \in I\}$ *and* $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{V}_i) : i \in I\}$ *be two g-frames for* $\mathscr{H}$ *and* S *be the g-frame operator of* $\{\Lambda_i\}_{i\in I}$*. Then* $\{(\Lambda_i - \Theta_i) \in \mathscr{L}(\mathscr{H}, \mathscr{V}_i) : i \in I\}$ *is a g-frame for* $\mathscr{H}$*, if* $\Theta_i = \Lambda_i S^a, i \in I, a \in \mathscr{R} - \{0\}$

Proof. Let A and B be the frame bounds of the g-frame $\{\Lambda_i\}_{i\in I}$. Note that

$$\sum_{i\in I} \|(\Lambda_i - \Theta_i)f\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2 + \sum_{i\in I} \|\Theta_i f\|^2 - 2\langle S^{a+1}f, f\rangle \quad \forall\, f \in \mathscr{H}.$$

For $a > 0$, we have for all $f \in \mathscr{H}$

$$\big(A + A^{2a+1} - 2B^{a+1}\big)\|f\|^2 \le \sum_{i\in I} \|(\Lambda_i - \Theta_i)f\|^2 \le \big(B + B^{2a+1} - 2A^{a+1}\big)\|f\|^2.$$

and for $a < 0$, we have for all $f \in \mathscr{H}$

$$\big(A + B^{2a+1} - 2A^{a+1}\big)\|f\|^2 \le \sum_{i\in I} \|(\Lambda_i - \Theta_i)f\|^2 \le \big(B + A^{2a+1} - 2B^{a+1}\big)\|f\|^2.$$

Hence $\{\Lambda_i - \Theta_i\}_{i\in I}$ is a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_i\}_{i\in I}$.

Corollary 3.2. *Let* $\{\Lambda_i\}_{i\in I}$ *be a g-frame for* $\mathscr{H}$ *and* S *be the g- frame operator of* $\{\Lambda_i\}_{i\in I}$ *.Then* $\{\Lambda_i - \Lambda_i S^{-1}\}_{i\in I}$ *is a g-frame.*

Proof. This follows immediately from Theorem 3.1 if we take $a = -1$.

We state our main result □

Theorem 3.3. *Let* $\{\Lambda_i\}_{i\in I}$ *be a g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_i\}_{i\in I}$ *and* S_Λ *be the g-frame operator of* $\{\Lambda_i\}_{i\in I}$*. Then* $\{\Theta_i\}_{i\in I}$ *is an alternate dual g-frame of* $\{\Lambda_i\}_{i\in I}$ *if and only if* $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ *and* $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$ *are orthogonal.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ are g-frames, there exist $0 < A_1 \le B_1 < \infty$ and $0 < A_2 \le B_2 < \infty$ such that

$$A_1\|\mathbf{f}\|^2 \le \sum_{i\in I} \|\Lambda_i \mathbf{f}\|^2 \le B_1\|\mathbf{f}\|^2 \qquad \forall\, f \in \mathscr{H} \tag{7}$$

and

$$A_2\|\mathbf{f}\|^2 \leq \sum_{i\in I} \|\Theta_i \mathbf{f}\|^2 \leq B_2\|\mathbf{f}\|^2 \qquad \forall\ f \in \mathscr{H}.$$

Let $\{\Theta_i\}_{i\in I}$ be an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$. Then we have

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f = \sum_{i\in I} \Theta_i^* \Lambda_i f \qquad \forall\ f \in \mathscr{H}. \tag{8}$$

By corollary 3.2, the sequence $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is a g-frame.

And

$$\begin{aligned}\|(\Lambda_i S_\Lambda^{-1} - \Theta_i)f\|^2 &= \|\Lambda_i S_\Lambda^{-1} f\|^2 + \|\Theta_i f\|^2 - 2Re\langle \Lambda_i S_\Lambda^{-1} f, \Theta_i f\rangle \\ &= \|\Lambda_i S_\Lambda^{-1} f\|^2 + \|\Theta_i f\|^2 - 2Re\langle \Theta_i^* \Lambda_i S_\Lambda^{-1} f, f\rangle\end{aligned}$$

Therefore

$$\sum_{i\in I} \|(\Lambda_i S_\Lambda^{-1} - \Theta_i)f\|^2 = \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1} f\|^2 + \sum_{i\in I} \|\Theta_i f\|^2 - 2Re\left\langle \sum_{i\in I} \Theta_i^* \Lambda_i S_\Lambda^{-1} f, f\right\rangle$$

By eq.(8), we get

$$\sum_{i\in I} \|(\Lambda_i S_\Lambda^{-1} - \Theta_i)f\|^2 = \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1} f\|^2 + \sum_{i\in I} \|\Theta_i f\|^2 - 2\langle S_\Lambda^{-1} f, f\rangle \tag{9}$$

Since $\{\Theta_i\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ are g-frames with bounds A_2, B_2 and B_1^{-1}, A_1^{-1} respectively, we have for each $f \in \mathscr{H}$

$$B_1^{-1}\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1} f\|^2 \leq A_1^{-1}\|f\|^2$$

$$\text{and} \qquad A_2\|f\|^2 \leq \sum_{i\in I} \|\Theta_i f\|^2 \leq B_2\|f\|^2$$

$$\Rightarrow B_1^{-1}\|f\|^2 + A_2\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1} f\|^2 + \sum_{i\in I} \|\Theta_i f\|^2$$

(10)

$$\Rightarrow B_1^{-1}\|f\|^2 + A_2\|f\|^2 - 2\langle S_\Lambda^{-1} f, f\rangle \leq \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1} f\|^2 + \sum_{i\in I} \|\Theta_i f\|^2 - 2\langle S_\Lambda^{-1} f, f\rangle$$

Now

$$\langle S_\Lambda^{-1}f,f\rangle = \left\langle S_\Lambda^{-1/2}S_\Lambda^{-1/2}f,f\right\rangle = \left\langle S_\Lambda^{-1/2}f,S_\Lambda^{-1/2}f\right\rangle = \left\|S_\Lambda^{-1/2}f\right\|^2 \leq \left\|S_\Lambda^{-1/2}\right\|^2 \|f\|^2$$

Therefore

$$\langle S_\Lambda^{-1}f,f\rangle \leq \left\|S_\Lambda^{-1/2}\right\|^2 \|f\|^2$$

$$\Rightarrow -2\langle S_\Lambda^{-1}f,f\rangle \geq -2\left\|S_\Lambda^{-1/2}\right\|^2 \|f\|^2. \tag{11}$$

Therefore from equations (9),(10) and (11) we get

$$\sum_{i\in I}\|(\Lambda_i S_\Lambda^{-1}-\Theta_i)f\|^2 \geq \left(B_1^{-1}+A_2-2\left\|S_\Lambda^{-1/2}\right\|^2\right)\|f\|^2.$$

Similarly, we can show that

$$\sum_{i\in I}\|(\Lambda_i S_\Lambda^{-1}-\Theta_i)f\|^2 \leq \left(A_1^{-1}+B_2\right)\|f\|^2.$$

Thus $\{\Lambda_i S_\Lambda^{-1}-\Theta_i\}_{i\in I}$ is a g-frame.

To prove the orthogonality of $\{\Lambda_i-\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1}-\Theta_i\}_{i\in I}$, we compute $\sum_{i\in I}\left(\Lambda_i-\Lambda_i S_\Lambda^{-1}\right)^*\left(\Lambda_i S_\Lambda^{-1}-\Theta_i\right)f$.

$$\begin{aligned}\sum_{i\in I}\left(\Lambda_i-\Lambda_i S_\Lambda^{-1}\right)^*\left(\Lambda_i S_\Lambda^{-1}-\Theta_i\right)f &= \sum_{i\in I}\left(\Lambda_i^*-S_\Lambda^{-1}\Lambda_i^*\right)\left(\Lambda_i S_\Lambda^{-1}-\Theta_i\right)f\\ &= \sum_{i\in I}\left(\Lambda_i^*\Lambda_i S_\Lambda^{-1}-\Lambda_i^*\Theta_i-S_\Lambda^{-1}\Lambda_i^*\Lambda_i S_\Lambda^{-1}+S_\Lambda^{-1}\Lambda_i^*\Theta_i\right)f\\ &= \left(S_\Lambda S_\Lambda^{-1}-I-S_\Lambda^{-1}S_\Lambda S_\Lambda^{-1}+S_\Lambda^{-1}I\right)f\\ &= \left(I-I-S_\Lambda^{-1}+S_\Lambda^{-1}\right)f\\ &= 0\end{aligned}$$

Thus $\{\Lambda_i-\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is orthogonal to $\{\Lambda_i S_\Lambda^{-1}-\Theta_i\}_{i\in I}$.

Conversely, let $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ be orthogonal to $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$. We note that sequences $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$ are g-Bessel sequences .

$$\begin{aligned}\sum_{i\in I} \|(\Lambda_i S_\Lambda^{-1} - \Theta_i)f\|^2 &\leq 2(\sum_{i\in I} \|\Lambda_i S_\Lambda^{-1} f\|^2 + \sum_{i\in I} \|\Theta_i f\|^2)\\ &\leq 2(A_1^{-1} + B_2)\,\|f\|^2.\end{aligned}$$

We shall prove that $\{\Theta_i\}_{i\in I}$ is an alternate dual of $\{\Lambda_i\}_{i\in I}$. Since $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is orthogonal to $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$, we have

$$\begin{aligned}\sum_{i\in I} \big(\Lambda_i - \Lambda_i S_\Lambda^{-1}\big)^* \big(\Lambda_i S_\Lambda^{-1} - \Theta_i\big) f &= 0\\ \Big(I - \sum_{i\in I} \Lambda_i^* \Theta_i - S_\Lambda^{-1} + S_\Lambda^{-1} \sum_{i\in I} \Lambda_i^* \Theta_i\Big) f &= 0\\ \Rightarrow \big(I - S_\Lambda^{-1}\big)\Big(I - \sum_{i\in I} \Lambda_i^* \Theta_i\Big) f &= 0\end{aligned}$$

Since $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I} = \{\Lambda_i(I - S_\Lambda^{-1})\}_{i\in I}$ is a g-frame, we have by lemma [2.8] that $(I - S_\Lambda^{-1})$ is invertible. Therefore for all $f \in \mathscr{H}$

$$\Big(I - \sum_{i\in I} \Lambda_i^* \Theta_i\Big) f = 0$$

which implies that $\sum_{i\in I} \Lambda_i^* \Theta_i f = f$. Thus $\{\Theta_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Corollary 3.4. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_i\}_{i\in I}$ and S be the g-frame operator of $\{\Lambda_i\}_{i\in I}$. If $\{\Theta_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i - \Theta_i\}_{i\in I}$ is also a g-frame for $\mathscr{H}$.*

Proof. Let T_1 and T_2 be synthesis operators of $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$. Since $\{\Theta_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$, we have by theorem 3.3 $\{\Lambda_i -$

$\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$ are orthogonal that is $T_2T_1^* = 0$. Therefore by lemma 2.9[9], $\{\Lambda_i - \Theta_i\}_{i\in I}$ is a g-frame for $\mathscr{H}$.

Remark 3.5. In [8] authors discussed the duality of frames in terms of their R-dual sequences. We note that the g-R-dual sequences of orthogonal g-Bessel sequences are orthogonal.

By theorem 3.3 if $\{\Theta_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$ then $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$ are orthogonal.

Let the g-R-dual sequences of $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1} - \Theta_i\}_{i\in I}$ be denoted by $\Gamma_j^{(1)}$ and $\Gamma_j^{(2)}$ respectively and are given by

$$\Gamma_j^{(1)} = \sum_{i\in I} \Xi_j(\Lambda_i - \Lambda_i S_\Lambda^{-1})^*\Psi_i \qquad \forall\, j\in I.$$

and

$$\Gamma_j^{(2)} = \sum_{i\in I} \Xi_j(\Lambda_i S_\Lambda^{-1} - \Theta_i)^*\Psi_i \qquad \forall\, j\in I.$$

where $\{\Xi_i\}_{i\in I}$ and $\{\Psi_i\}_{i\in I}$ are g-orthonormal bases for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$ and $\{\mathscr{V}_i\}_{i\in I}$.

Now for every $i,j\in I$ and $\{g_j\}_{j\in I} \in \Big(\sum_{j\in I}\oplus \mathscr{W}_j\Big)_{l^2}$

$$\begin{aligned}
\Gamma_i^{(1)}\Big(\Gamma_j^{(2)}\Big)^* g_j &= \sum_{k\in I}\Xi_i\big(\Lambda_k - \Lambda_k S_\Lambda^{-1}\big)^*\Psi_k\Big\{\sum_{m\in I}\Xi_j\big(\Theta_m - \Lambda_m S_\Lambda^{-1}\big)^*\Psi_m\Big\}^* g_j\\
&= \sum_{k\in I}\sum_{m\in I}\Xi_i\big(\Lambda_k^* - S_\Lambda^{-1}\Lambda_k^*\big)\Psi_k\Psi_m^*\big(\Theta_m - \Lambda_m S_\Lambda^{-1}\big)\Xi_j^* g_j\\
&= \sum_{k\in I}\Xi_i\big(\Lambda_k^* - S_\Lambda^{-1}\Lambda_k^*\big)\big(\Theta_k - \Lambda_k S_\Lambda^{-1}\big)\Xi_j^* g_j\\
&= \Xi_i\sum_{k\in I}\big(\Lambda_k^*\Theta_k - \Lambda_k^*\Lambda_k S_\Lambda^{-1} - S_\Lambda^{-1}\Lambda_k^*\Theta_k + S_\Lambda^{-1}\Lambda_k^*\Lambda_k S_\Lambda^{-1}\big)\Xi_j^* g_j\\
&= \Xi_i\big(I - S_\Lambda S_\Lambda^{-1} - S_\Lambda^{-1} I + S_\Lambda^{-1} S_\Lambda S_\Lambda^{-1}\big)\Xi_j^* g_j\\
&= 0
\end{aligned}$$

Therefore g-R-dual sequences of orthogonal g-Bessel sequences are orthogonal.

Remark 3.6. If $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ are dual g-frames then the difference of their g-R-dual sequences is a g-R-dual sequence of the g-frame $\{\Lambda_i - \Theta_i\}_{i\in I}$.
g-R-dual of $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ are

$$\Gamma_j^{\Lambda} = \sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i \qquad \forall\, j \in I.$$

and

$$\Gamma_j^{\Theta} = \sum_{i\in I} \Xi_j \Theta_i^* \Psi_i, \qquad \forall\, j \in I.$$

Now

$$\begin{aligned}\Gamma_j^{\Lambda} - \Gamma_j^{\Theta} &= \sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i - \sum_{i\in I} \Xi_j \Theta_i^* \Psi_i, \qquad \forall\, j \in I \\ &= \sum_{i\in I} \Xi_j (\Lambda_i - \Theta_i)^* \Psi_i, \qquad \forall\, j \in I.\end{aligned}$$

which is the g-R-dual sequence of the g-frame $\{\Lambda_i - \Theta_i\}_{i\in I}$.

References

[1] Benedetto JJ, Powell AM, Yilmaz O., *Sigma-delta (sigma delta) quantization and finite frames*, IEEE Trans Inform Theory, 2006; **52(5)**: 1990-2005.

[2] Casazza PG, Kutyniok G, Li S., *Fusion frames and distributed processing*, Applied Comput Harmonic Anal. 2008; **25(1)**: 114-132.

[3] Chugh Renu and Goel Shashank, *On sums of G-frames in Hilbert spaces*, Jordan Journal of Mathematics and Statistics (JJMS). 2012; **5(2)**: 115-124.

[4] Daubechies I, Grossmann A, Meyer Y, *Painless nonorthogonal expansions*, J Math Phys. **27** (1986) 1271-1283.

[5] Duffin RJ, Schaeffer AC, *A class of nonharmonic Fourier series*, Trans Amer Math Soc. **72** (1952) 341-366.

[6] Ehler M., *Random tight frames*, J Fourier Anal Appl. 2011; **18(1)**: 1-20.

[7] Fornasier M, *Quasi-orthogonal decompositions of structured frames*, J Math Anal Appl. 2004; **289(1)**: 180-199.

[8] F. Enayati and M.S. Asgari, *Duality properties for generalized frames*, Banach J. Math. Anal., Volume 11, Number 4 (2017), 880-898.

[9] Guo X., *Operator characterizations and some properties of g-frames on Hilbert spaces*, Journal of Function Spaces and Applications, vol. 2013, Article ID 931367, 9 pages, 2013.

[10] Heath RW, Paulraj AJ., *Linear dispersion codes for mimo systems based on frame theory*, IEEE Trans Signal Process.2002; **50(10)**: 2429-2441.

[11] Leng J, Han D, Huang T., *Optimal dual frames for communication coding with probabilistic erasures*, IEEE Trans Signal Process. 2011; **59(11)**: 5380-5389.

[12] Leng J, Han D., *Optimal dual frames for erasures in Linear Algebra Appl.* 2011; **435(6)**: 1464-1472.

[13] Li D, Leng J, Huang T, et al. *Some equalities and inequalities for probabilistic frames*, J Inequal Appl. 2016; **2016(1)**: 245.

[14] Li S, Ogawa H. *Pseudoframes of subspaces with applications*, J Fourier Anal Appl. 2004; **10(4)**: 409-431.

[15] Sun W. *G-frames and g-riesz bases*, J Math Anal Appl. 2006; **322(1)**: 437-452.

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