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GPF-PROPERTIES OF GROUP RINGS

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ABSTRACT. All rings R in this article are assumed to be commutative with unity $1 \neq 0$. A ring R is called a $GPF-$ring if for every $a \in R$ there exists a positive integer n such that the annihilator ideal $Ann_R(a^n)$ is pure. We prove that for a ring R and an Abelian group G, if the group ring RG is a $GPF-$ring then so is R. Moreover, if G is a finite Abelian group then $|G|$ is a unit or a zero-divisor in R. We prove that if G is a group such that for every nontrivial subgroup H of G, $[G : H] < \infty$, then the group ring RG is a $GPF-$ring if and only if RH is a $GPF-$ring for each finitely generated subgroup H of G. It is proved that if R is a local ring and RG is a $U-$group ring, then RG is a $GPF-$ring if and only if R is a $GPF-$ring and $p \in Nil(R)$. Finally, we prove that if R is a semisimple ring and G is a finite group such that $|G|^{-1} \in R$, then RG is a $GPF-$ring if and only if RG is a $PF-$ring.

1. Introduction

All rings considered in this paper are assumed to be commutative with unity $1 \neq 0$, and all groups are Abelian. Recall that a ring R is called a $PF-$ring if every principal ideal is a flat $R-$module. An ideal I of a ring R is called pure if for every $a \in I$, there exists $b \in I$ such that $ab = a$. It is well known that a ring R is a $PF-$ring if and only

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if $Ann_R(a) = \{x \in R : xa = 0\}$ is a pure ideal for every $a \in R$, see [2]. There are different characterizations of $PF-$rings, see [7] and [3]. A ring R is called a $PP-$ring if for each $a \in R$, $Ann_R(a)$ is generated by an idempotent element in R. These rings were studied extensively in literatures, see [6], [9], and [3]. As a generalization of $PP-$ring, Hirano in [10] introduced a new class of rings called $GPP-$rings. A ring R is called a $GPP-$ring if for each $a \in R$, there exists a positive integer n such that $a^n R$ is projective. It is known that a ring R is a $GPP-$ring if and only if for every $a \in R$ there exists a positive integer n such that the annihilator ideal $Ann_R(a^n)$ is generated by an idempotent. Some properties of $GPP-$rings were investigated in [10] and [11].

Al-Ezeh in [4] introduced a new class of rings called $GPF-$rings. A ring R is called a $GPF-$ring if for each $a \in R$, there exists a positive integer n such that $a^n R$ is a flat $R-$module. Also, a ring R is a $GPF-$ring if and only if for every $a \in R$, there exists a positive integer n such that the annihilator ideal $Ann_R(a^n)$ is a pure ideal. A study of this class of rings and the relationship between $GPP-$rings and $GPF-$rings were carried by Al-Ezeh [4].

Clearly, every $PF-$ring is a $GPF-$ring. Every $PF-$ring is a reduced ring (a ring with no nonzero nilpotent elements), see [2, Lemma 2], but not every $GPF-$ring is a reduced ring. Al-Ezeh in [4] proved that a ring R is a reduced $GPF-$ring if and only if R is a $PF-$ring. Let $C(X)$ be the ring of all real valued continuous functions defined on X. Then $C(\mathbb{N})$ is $PF-$ring, and so it is $GPF-$ring, while $C(\mathbb{R})$ is not $GPF-$ring, being a reduced non $PF-$ring. Later on we will give an example of a $GPF-$ring which is not a $PF-$ring.

It was shown in [14] that if the group ring RG is $PP-$ring ($PF-$ring), then so is R, and if G is torsion free, then the converse is also true. Other cases for the converse can be found also in [16].

The $PP-$rings has been studied on group rings by Zan and Chen in [17]. Our aim in this paper is to characterize when group rings RG are $GPF-$rings. Our work would be parallel to the work done in [17], with modifications when needed.

2. $GPF-$Rings

In this section, we establish general new results on $GPF-$rings.

Definition 2.1 ([15]). *An ideal I of the ring R is said to be generalized pure if for every $a \in I$ there exists $b \in I$ and a positive integer n such that $a^n = a^n b$.*

Proposition 2.1 ([15, Theorem 2.5]). *Let R be a $GPF-$ring. Then for every $a \in R$, $Ann_R(a)$ is a generalized pure ideal.*

Recall that a ring S is called an over ring of R if $R \subset S \subset Q(R)$, where $Q(R)$ is the total quotient ring of R.

Theorem 2.1. *If a ring R is a $GPF-$ring, then every over ring of R is a $GPF-$ring.*

Proof. Let S be an over ring of R. Let $\frac{a}{t} \in S$. Since R is a $GPF-$ring, there exists a positive integer n such that $Ann_R(a^n)$ is a pure ideal. Let $\frac{x}{s} \in Ann_S\left(\frac{a^n}{t^n}\right)$. Then $xa^n = 0$, and so there exists $y \in Ann_R(a^n)$ such that $x = xy$. Hence, $y \in Ann_S\left(\frac{a^n}{t^n}\right)$ and $\frac{x}{s} = \frac{x}{s}y$. So, S is a $GPF-$ring. □

Theorem 2.2. *Let R be a subring of a ring S both with the same identity. Suppose that S is a free $R-$module with a basis G such that $1 \in G$. If S is $GPF-$ring, then so is R.*

Proof. Let $a \in R \subseteq S$. Then there exists a positive integer $n \geq 1$ such that $Ann_S(a^n)$ is pure.

Let $b \in Ann_R(a^n) \subseteq Ann_S(a^n)$. Then there exists $c = c_0 + c_1 g_1 + ... + c_m g_m \in Ann_S(a^n)$ such that $b = bc$, where $c_0, c_1, ..., c_m \in R$.

So, $b = bc = bc_0 + bc_1 g_1 + ... + bc_m g_m$. Thus, $b = bc_0$.

Moreover, since $0 = a^n c = a^n c_0 + a^n c_1 g_1 + ... + a^n c_m g_m$, $a^n c_i = 0$ for all $i = 0, 1, ..., m$, we have $c_0 \in Ann_R(a^n)$ and $b = bc_0$.

Thus, R is $GPF-$ring. □

Corollary 2.1. *If $R[x]$ or $R[x, x^{-1}]$ is a $GPF-$ring, then so is R.*

Proof. $R[x]$ and $R[x, x^{-1}]$ are free $R-$module with bases $\{x^i : i = 0, 1, ...\}$ and $\{x^i : i = 0, \pm 1, ...\}$, respectively, satisfying the assumptions of Theorem 2.2. □

Corollary 2.2. *Let $f(x) = x^m + a_1 x^{m-1} + ... + a_m \in R[x]$ be a monic polynomial. If $R[x]/(f(x))$ is a $GPF-$ring, then R is so.*

Proof. $S = R[x]/(f(x))$ is a free $R-$module with a basis $\{1, x, ..., x^{m-1}\}$ satisfying the assumptions of Theorem 2.2. □

Lemma 2.1. *Let $R_1, R_2, ..., R_m$ be rings. Then $R = \prod_{i=1}^{m} R_i$ is a $GPF-$ring if and only if R_i is a $GPF-$ring for all $i = 1, 2, ..., m$.*

Proof. Assume that R_i is a $GPF-$ring for all $i = 1, 2, ..., m$.

Let $x = (x_1, x_2, ..., x_m) \in R$. Then there exist positive integers $n_i \geq 1$ such that $Ann_{R_i}(x_i^{n_i})$ is pure ideal for all $i = 1, 2, ..., m$.

Hence, $Ann_{R_i}\left(x_i^k\right)$ is pure ideal for all $k \geq n_i$; $i = 1, 2, ..., m$, see [15, Lemma 3.2].

Let $k = \max\{n_1, n_2, ..., n_m\}$ and let $y = (y_1, y_2, ..., y_m) \in Ann_R\left(x^k\right)$.

Then $y_i \in Ann_{R_i}\left(x_i^k\right)$ and so, there exists $c_i \in Ann_{R_i}\left(x_i^k\right)$ such that $y_i = y_i c_i$ for all $i = 1, 2, ..., m$.

So, $c = (c_1, c_2, ..., c_m) \in Ann_R\left(x^k\right)$ and $y = yc$.

Thus, R is a $GPF-$ring.

Conversely, assume that $R = \prod_{i=1}^{m} R_i$ is a $GPF-$ring and let $x_i \in R_i$; $i = 1, 2, ..., m$.

Consider $x = (\alpha_1, \alpha_2, ..., \alpha_m)$, with $\alpha_j = \begin{cases} x_i & j = i \\ 0 & j \neq i \end{cases}$. Then since R is a $GPF-$ring, there exists a positive integer n such that $Ann_R(x^n)$ is pure.

Let $y_i \in Ann_{R_i}(x_i^n)$. Consider $y = (\beta_1, \beta_2, ..., \beta_m)$, with $\beta_j = \begin{cases} y_i & j = i \\ 0 & j \neq i \end{cases}$. Then $y \in Ann_R(x^n)$ and hence there exists $c = (c_1, c_2, ..., c_m) \in Ann_R(x^n)$ such that $y = yc$.

Thus, $c_i \in Ann_{R_i}(x_i^n)$ and $y_i = y_i c_i$.

Therefore, R_i is a $GPF-$ring for all $i = 1, 2, ..., m$. □

3. Group Rings

Given a ring R and a group G, we will denote the group ring of G over R by RG. Elements of the ring RG are just formal finite sums of the form $\sum_{g \in G} a_g g$ with all but a finite number of a_g are 0_R. We write C_n for the cyclic group of order n, $\mathbb{Z}$ for the ring of integers, $\mathbb{Z}_n$ for the ring of integers modulo n, and $\mathbb{C}$ is the field of complex numbers. The imaginary unit is denoted by $\mathbf{i}$.

Theorem 3.1. *Let R be a ring and G be a group. If RG is a $GPF-$ring, then so is R.*

Proof. $S = RG$ is a free $R-$module with a basis G satisfying the assumptions of Theorem 2.2. □

However, the converse of this theorem is not true, as well be seen in Example 3.4, and the question is what extra conditions on R or G can be added to ensure that if R is a $GPF-$ring, then so is RG.

Theorem 3.2. *If RG is a $GPF-$ring and H is a subgroup of G, then RH is a $GPF-$ring too.*

Proof. RH is a subring of RG and RG is a free $RH-$module on the set $\{g_1, g_2, ...\}$, the coset representatives of H in G. So, by Theorem 2.2, RH is a $GPF-$ring. □

Corollary 3.1. *Let G be a group such that for every non trivial finitely generated subgroup H of G, $[G : H] < \infty$. Then the group ring RG is a $GPF-$ring if and only if RH is a $GPF-$ring for each finitely generated subgroup H of G.*

Proof. If RG is $GPF-$ring, then by Theorem 3.2 RH is a $GPF-$ring for each finitely generated subgroup H of G.

Assume that RH is a $GPF-$ring for each finitely generated subgroup H of G. Let $u = \sum_{i=1}^{m} a_i h_i \in RG$, and let $H = \langle h_1, ..., h_m \rangle$ be the finitely generated subgroup of G generated by $h_1, ..., h_n$. Then RH is a $GPF-$ring. Since $u \in RH$, there exists a positive integer n such that $Ann_{RH}(u^n)$ is pure ideal.

Let $v \in Ann_{RG}(u^n)$. Since $[G : H] < \infty$, let $\{g_0, g_1, ..., g_r\}$ be a left coset representative of H in G, $g_0 = 1$. That is $G = H \cup g_1 H \cup ... \cup g_r H$. Now, v can be written as $v = \sum_{j=0}^{r} g_j b_j$, where $b_j \in RH$. Since $0 = vu^n = \sum_{j=0}^{r} g_j (b_j u^n)$, we obtain that $b_j u^n = 0$ for all $j = 0, 1, ..., r$. So, $b_j \in Ann_{RH}(u^n)$ and thus $b_j = b_j c_j$ for some $c_j \in Ann_{RH}(u^n)$, $j = 0, 1, ..., r$.

Let $1 - c = \prod_{j=0}^{r} (1 - c_j)$. Then $c \in Ann_{RH}(u^n)$ and $b_j = b_j c$ for all $j = 0, 1, ..., r$. It follows that $v = \sum_{j=0}^{r} g_j b_j = \sum_{j=0}^{r} g_j (b_j c) = \left(\sum_{j=0}^{r} g_j b_j \right) c = vc$ and $c \in Ann_{RH}(u^n) \subseteq Ann_{RG}(u^n)$.

Hence, $Ann_{RG}(u^n)$ is pure ideal for some positive integer n.

Therefore, RG is $GPF-$ring. □

Theorem 3.3. *If RG is a $GPF-$ring, then the order of each finite order element $g \in G$ is either a unit or a zero-divisor in R.*

Proof. Let $g \in G$ with $|g| = m$. Let H be the cyclic subgroup generated by g. Then, by Theorem 3.2, RH is a $GPF-$ring too. Now $1+g+g^2+...+g^{m-1} \in Ann_{RH}(1-g)$.

Since RH is a $GPF-$ring, $Ann_{RH}(1-g)$ is generalized pure ideal in RH. So, there exists a positive integer $n \geq 1$ and $a = a_0 + a_1 g + ... + a_{m-1}g^{m-1} \in Ann_{RH}(1-g)$ such that $(1+g+g^2+...+g^{m-1})^n a = (1+g+g^2+...+g^{m-1})^n$ and $a(1-g) = 0$.

Thus, $a_0 = a_1 = ... = a_{m-1}$ since $a = ag$.

So, $(1+g+...+g^{m-1})^n (a_0 + a_0 g + ... + a_0 g^{m-1}) = (1+g+...+g^{m-1})^n$.

Hence, $a_0 (1+g+...+g^{m-1})^{n+1} = (1+g+...+g^{m-1})^n$.

So, $a_0 m^n = m^{n-1}$ and hence $(a_0 m - 1) m^{n-1} = 0$.

Therefore, m is either a unit or a zero-divisor in R. □

Corollary 3.2. *If G is a finite group and RG is a $GPF-$ring, then $|G|$ is either a unit or a zero-divisor in R.*

Proof. Let G be a finite group and $|G| = n = \prod_{i=1}^{k} p_i^{\alpha_i}$ where p_i are distinct primes and $\alpha_i \geq 1$ are positive integers for all $i = 1, ..., k$. Then by Cauchy Theorem, there exists $g_i \in G$ such that $|g_i| = p_i$, for all $i = 1, ..., k$.

Thus, since RG is a $GPF-$ring and by Theorem 3.3, p_i is either a unit or a zero-divisor in R for all $i = 1, ..., k$.

So, $|G| = \prod_{i=1}^{k} p_i^{\alpha_i}$ is either a unit or a zero-divisor in R. □

Example 3.1. *Let $R = \mathbb{Z}_4$ and $G = C_2$. Then RG is not a $PF-$ring because RG is not a reduced ring. Since every non-zero element in RG is either a unit or a nilpotent, then it follows by* [4, Lemma 1.5] *that RG is a $GPF-$ring. Moreover, 2 is a zero-divisor in $R = \mathbb{Z}_4$.*

Example 3.2. *$\mathbb{Z}G$ is not a $GPF-$ring for any nontrivial finite group G.*

A group ring is called a $U-$group ring if $a = \sum a_i g_i \in RG$ is a unit if and only if $\epsilon(a) = \sum a_i$ is a unit in R. It was proved in [1] that RG is a $U-$group ring if and only if G is a $p-$group and $p \in J(R)$, the Jacobson Radical of R.

Theorem 3.4. *Let R be a local ring and RG be a $U-$group ring. Then RG is a $GPF-$ring if and only if R is a $GPF-$ring and $p \in Nil\,(R)$.*

Proof. By [13, page 138], RG is a local ring.

Since RG is a local ring, RG is a $GPF-$ring if and only if RG is a domainlike ring, $(Z(R) = Nil(R))$, see [4, Lemma 1.5].

Now, G is a torsion group because G is an Abelian $p-$group and so by [5, Theorem 3.8] RG is a domainlike ring if and only if R is a domainlike ring and $p \in Nil\,(R)$.

But since R is a local ring, R is a domainlike ring if and only if R is a $GPF-$ring.

Thus, RG is a $GPF-$ring if and only if R is a $GPF-$ring and $p \in Nil\,(R)$. □

Example 3.3. *$\mathbb{Z}_{p^r}G$ is a $GPF-$ring for any Abelian $p-$group because $p \in Nil\,(\mathbb{Z}_{p^r})$ and $\mathbb{Z}_{p^r}$ is local domainlike ring.*

The following lemma exists in [12, page 134].

Lemma 3.1. $(R_1 \times R_2 \times \cdots \times R_m)\,G \cong \prod_{i=1}^{m} R_i G$

Theorem 3.5. *If $R = R_1 \times R_2 \times \cdots \times R_m$, then RG is a $GPF-$ring if and only if R_iG is a $GPF-$ring for all $i = 1, 2, ..., m$.*

Proof. The proof follows from Lemma 2.1 and Lemma 3.1. □

Theorem 3.6. *Let $R = \prod_{i=1}^{n} R_i$ where each R_i is local ring. Assume that RG is a $U-$group ring. Then RG is a $GPF-$ring if and only if R is $GPF-$ring and $p \in Nil\,(R_i)$ for all $i = 1, ..., n$.*

Proof. RG is a $U-$group ring if and only if R_iG is a $U-$group ring for all $i = 1, ..., n$, see [1].

But since R_i is local and R_iG is a $U-$group ring, R_iG is a $GPF-$ring if and only if R_i is a $GPF-$ring and $p \in Nil\,(R_i)$ for all $i = 1, ..., n$.

By Theorem 3.5, RG is a $GPF-$ring if and only if R_iG is a $GPF-$ring for all $i = 1, ..., n$.

So, RG is a $GPF-$ring if and only if R_iG is a $GPF-$ring for all $i = 1, ..., n$ if and only if R_i is a $GPF-$ring and $p \in Nil\,(R_i)$ for all $i = 1, ..., n$ if and only if R is a $GPF-$ring and $p \in Nil\,(R_i)$ for all $i = 1, ..., n$. □

Recall that, every Artinian ring R is isomorphic to a finite direct product of Artinian local rings R_i.

Corollary 3.3. *Let R be an Artinian ring and RG be a $U-$group ring. Then RG is a $GPF-$ring if and only if R is a $GPF-$ring and $p \in Nil\,(R_i)$, where $R = \prod_{i=1}^{n} R_i$, for all $i = 1, ..., n$.*

Theorem 3.7. *If R is a semisimple ring and G is a finite group such that $|G|^{-1} \in R$, then RG is a $GPF-$ring if and only if RG is a $PF-$ring.*

Proof. By [12, Theorem 3.4.7], RG is a simisimple ring and hence it is a reduced ring.

Thus, RG is a $GPF-$ring if and only if RG is a $PF-$ring. □

Theorem 3.8. *Let R be a ring and G be a torsion free group such that $|Spec\,(RG)| < \infty$. Then RG is a $GPF-$ring if and only if R is a $GPF-$ring.*

Proof. By Theorem 3.1, if RG is a $GPF-$ring then R is a $GPF-$ring.

Conversely, assume R is a $GPF-$ring. Then R_p is a domainlike for every $p \in Spec\,(R)$.

Since G is a torsion free, it follows that R_pG is a domainlike, see [5, Theorem 3.11].

Let $P \in Spec\,(RG)$. Then $q = P \cap R \in Spec\,(R)$, and $(RG)_P = (R_q G)_{PR_qG}$. Consequently, $(RG)_P$ is a domainlike for all $P \in Spec\,(RG)$. Since $|Spec\,(RG)| < \infty$ and by [4, Theorem 1.8], RG is a $GPF-$ring. □

The following proposition exists in [16].

Proposition 3.1. *Let R be a ring. Then*

(1) *If $2^{-1} \in R$, then $RC_2 \cong R \times R$ and $RC_4 \cong R \times R \times (R\,[x]\,/(x^2+1))$*

(2) *If $R \subseteq \mathbb{C}$ and $3^{-1} \in R$, then $RC_3 \cong R \times (R\,[x]\,/(x^2+x+1))$.*

Corollary 3.4. *If $2^{-1} \in R$, then RC_2 is a $GPF-$ring if and only if R is a $GPF-$ring.*

Proof. The proof follows from Theorem 3.1, Proposition 3.1 and Lemma 2.1. □

Corollary 3.5. *If $2^{-1} \in R$, then RC_4 is a $GPF-$ring if and only if $R\,[x]\,/(x^2+1)$ is a $GPF-$ring.*

Proof. The proof follows from Theorem 3.1, Proposition 3.1, Corollary 2.2 and Lemma 2.1. □

Corollary 3.6. *If $R \subseteq \mathbb{C}$ and $3^{-1} \in R$, then RC_3 is a $GPF-$ring if and only if $R\,[x]\,/(x^2+x+1)$ is a $GPF-$ring.*

Proof. The proof follows from Theorem 3.1, Proposition 3.1, Corollary 2.2 and Lemma 2.1. □

Note that, if $G = H \times K$, then $RG = R\,(H \times K) \cong (RH)\,K$.

Corollary 3.7. *If $R \subseteq \mathbb{C}$ and $6^{-1} \in R$, then RC_6 is a $GPF-$ring if and only if $R\,[x]\,/(x^2+x+1)$ is a $GPF-$ring.*

Proof. Since $C_6 \cong C_3 \times C_2$, then $RC_6 \cong (RC_3)\, C_2$.

So, because $2^{-1} \in R \subseteq RC_3$, RC_6 is a $GPF-$ring if and only if RC_3 is a $GPF-$ring.

Since $R \subseteq \mathbb{C}$ and $3^{-1} \in R$, RC_3 is a $GPF-$ring if and only if $R\,[x]\,/(x^2+x+1)$ is a $GPF-$ring.

Hence, RC_6 is a $GPF-$ring if and only if $R\,[x]\,/(x^2+x+1)$ is a $GPF-$ring. $\square$

We now investigate a case at which $R\,[x]\,/(x^2+x+1)$ is a $GPF-$ring.

Lemma 3.2. *Let R_1 and R_2 be two integral domains, and let T be a non-integral domain subring of $R = R_1 \times R_2$ containing the identity element $(1,1)$. Then T is a $GPF-$ring if and only if $(0,1) \in T$.*

Proof. Assume that T is a $GPF-$ring. Since T is not an integral domain, there are non-zero elements $(a,b)\,,(c,d) \in T$ such that $(a,b)\,.\,(c,d) = (0,0)$. Since $(a,b) \neq (0,0)$, either $a \neq 0$ or $b \neq 0$, say $a \neq 0$. Thus $c = 0$ and $d \neq 0$. Since $(c,d) \in Ann_T\,((a,b))$, by [15, Theorem 2.5] there exists a positive integer n and $(x,y) \in Ann_T\,((a,b))$ such that $(c,d)^n\,(x,y) = (c,d)^n$. So, $d^n y = d^n$ and $d \neq 0$ in R_2. Thus $y = 1$. Since $x^n a = 0$ and $a \neq 0$ in R_1, $x = 0$. So, $(x,y) = (0,1) \in T$.

Now, assume that $(0,1) \in T$. Then, $(1,1) - (0,1) = (1,0) \in T$. Consider any $(0,0) \neq (a,b) \in T$. If $a \neq 0, b \neq 0$, $Ann_T\,((a,b)) = \{(0,0)\}$. If $a = 0$, $b \neq 0$, $Ann_T\,((a,b)) = (1,0)\,T$ and if $a \neq 0$, $b = 0$, $Ann_T\,((a,b)) = (0,1)\,T$. Also if $a = b = 0$, then $Ann_T\,((a,b)) = T$ So, T is a $PP-$ring and hence T is a $GPF-$ring. $\square$

Theorem 3.9. *Let R be an integral domain and let $Q(R)$ denotes the quotient field of R. Consider the polynomial $x^2 + a_1 x + a_2 \in R[x]$ with α, β are its roots in some field extension, and $\alpha - \beta$ is a unit in R. Then $R\,[x]\,/(x^2 + a_1 x + a_2)$ is a $GPF-$ring if and only if either $\alpha \in R$ or $\alpha \notin Q(R)$.*

Proof. Let $T = R\,[x]\,/(x^2 + a_1 x + a_2)$ and $x^2 + a_1 x + a_2 = (x-\alpha)\,(x-\beta)\,.$ By hypothesis, $\alpha \neq \beta$. First suppose $\alpha \notin Q\,(R)\,.$ Then $x^2 + a_1 x + a_2$ is irreducible over

$Q(R)$ and hence it is irreducible over R since the polynomial is monic. Thus, T is an integral domain. In particular T is a $GPF-$ring.

If $\alpha \in Q(R)$, then define $\Phi : R[x] \longrightarrow Q(R) \times Q(R)$ by $\Phi(f(x)) = (f(\alpha), f(\beta)) \in Q(R) \times Q(R)$. Then Φ is a ring homomorphism with $Ker(\Phi) = (x^2 + a_1 x + a_2)$. Hence, T is a subring of $Q(R) \times Q(R)$.

Assume now that T is a $GPF-$ring, and so it follows by Lemma 3.2 that $(0,1) \in T$. Thus there exists $ax + b \in R[x]$ such that $a\alpha + b = 0$ and $a\beta + b = 1$. But since $x^2 + a_1 x + a_2 = (x - \alpha)(x - \beta)$, $\alpha + \beta = -a_1$ and $\alpha\beta = a_2$. So,

$$2(b-1)b = 2(-a\beta)(-a\alpha) = 2a^2\alpha\beta = 2a^2 a_2.$$

Also,

$$2(b-1)b = (2b-2)b = -(1 + a(\alpha+\beta)b = -(1 - aa_1)b = (aa_1 - 1)b.$$

So,

$$2a^2 a_2 = (aa_1 - 1)b.$$

Thus

$$-b = 2a^2 a_2 - aa_1 b.$$

Hence

$$\alpha = -\frac{b}{a} = 2aa_2 - a_1 b \in R.$$

So, $\alpha \in R$.

Now, assume that $\alpha \in R$, and define $p(x) = \dfrac{x-\alpha}{\beta-\alpha}$. Then $p(x) \in R[x]$, since $\beta - \alpha$ is a unit. But $\Phi(p(x)) = (p(\alpha), p(\beta) = (0,1)$. Thus it follows by Lemma 3.2 that T is a $GPF-$ring. □

Example 3.4. *Let* $S = \left\{\frac{n}{3^k} : n, k \in \mathbb{Z},\ k \geq 0\right\}$. *Then* S *is a subring of* $\mathbb{Q}$. *Set* $R = \left\{a + \sqrt{3}b\mathbf{i} : a, b \in S\right\}$. *Then* R *is a subring of* $\mathbb{C}$ *with* $\frac{1}{3} \in R$. *Because* R *is a domain, it is certainly a* $GPF-$*ring.*

Let $f(x) = x^2 + x + 1 \in R[x]$. *Then* $\alpha = \dfrac{-1+\sqrt{3}\mathbf{i}}{2} \notin R$.
Let $r = 2\sqrt{3}\mathbf{i},\ s = -\left(3+\sqrt{3}\mathbf{i}\right)$. *Then* $r, s \in R$ *and* $\alpha = \dfrac{s}{r} \in Q(R)$.
Since $(\alpha-\beta)^{-1} = \left(\sqrt{3}\mathbf{i}\right)^{-1} = -\dfrac{\sqrt{3}}{3}\mathbf{i} \in R,\ RC_3 \cong R \times \left(R[x]/(x^2+x+1)\right)$ *is not a* $GPF-$*ring.*

The above example shows that RC_3 is not a $GPF-$ring although 3 is a unit in R and R is a $GPF-$ring.

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References

[1] E. Abu Osba, H. Al-Ezeh and M. Ghanem, On U-group rings, Comm. Korean Math. Soc.**33**(4) (2018), 1075-1082.

[2] H. Al-Ezeh, On Some Properties of Polynomial Rings, Int. J. Math. and Math. Sci. **10** (1987), 311-314.

[3] H. Al-Ezeh, The Pure Spectrum of a *PF*-Ring, Comment. Math. Univ. St. Paul **37** (1988), 179-183.

[4] H. Al-Ezeh, On Generalized $PF-$Rings, Math. J. Okayama Univ. **31** (1989), 25-29.

[5] D. Anderson and O. Al-Mallah, Commutative Group Rings That are Presimplifiable or Domainlike, J. Algebra and its App. **16** (2017), 1750019 (9 pages).

[6] G. Bergman, Hereditary Commutative Rings and Centres of Hereditary Rings, Proc. London Math. Soc. **23** (1971), 214-236.

[7] F. Cheniour and N. Mahdou, When Every Principal Ideal Is Flat, Portugaliae Math. **70** (2011), 51-58.

[8] J. Chen, Y. Li and Y. Zhou, Morphic Group Rings, J. of Pure and Applied Algebra **205** (2006), 621-639.

[9] S. Endo, Note on $PP-$Rings, Nagoya Math. J. **17** (1960), 167-170.

[10] Y. Hirano, On Generlized $PP-$Rings, Math. J. Okayama Univ. **25** (1983), 7-11.

[11] C. Huh, H. K. Kim and Y. Lee, $PP-$Rings and Generalized $PP-$Rings, J. Pure Appl. Algebra **167** (2002), 37-52.

[12] C. Milies and S. Sehgal, An Introduction to Group Rings, Kluwer Academic Publishers, Dordrecht, 2002.

[13] W. Nicholson, Local Group Rings, Cand. Math. Bull. **15** (1972), 137-138.

[14] R. Schwarz and S. Glaz, Commutative group rings with von Neumann regular total rings of quotients, J. Algebra **388** (2013), 287-293.

[15] N. Shuker and H. Mahammad, On Generalized $PF-$Rings, Raf. J. of Comp. and Math. **1** (2004), 39-45.

[16] Z. Yi and Y. Zhou, Baer and Quasi-Baer Properties of Group Rings, J. Austral. Math. Soc. **83** (2007), 285-296.

[17] L. Zan and J. Chen, PP- Properties of Group Rings, Int. Electronic J. of Algebra **3** (2008), 117-124.

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