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ON $G-$BANACH FRAMES

GHANSHYAM SINGH RATHORE [(1)] AND TRIPTI MITTAL [(2)]

ABSTRACT. Abdollahpour et.al [1] generalized the concepts of frames for Banach spaces and defined $g-$Banach frames in Banach spaces. In the present paper, we define various types of $g-$Banach frames in Banach spaces. Examples and counter examples to distinguish various types of $g-$Banach frames in Banach spaces have been given. It has been proved that if a Banach space $\mathcal{X}$ has a Banach frame, then $\mathcal{X}$ has a normalized tight $g-$Banach frame for $\mathcal{X}$. A characterization of an exact $g-$Banach frame has been given. Also, we consider the finite sum of $g-$Banach frames and give a sufficient condition for the finite sum of $g-$Banach frames to be a $g-$Banach frame. Finally, a sufficient condition for the stability of $g-$Banach frames in Banach spaces which provides optimal frame bounds has been given.

1. Introduction

Frames are main tools for use in signal processing, image processing, data compression and sampling theory etc. Today even more uses are being found for the theory such as optics, filter banks, signal detection as well as study of Besov spaces, Banach space theory etc. Frames for Hilbert spaces were introduced by Duffin and Schaeffer [18]. Later, in 1986, Daubechies, Grossmann and Meyer [17] reintroduced frames and found a new application to wavelet and Gabor transforms. For a nice introduction to frames, one may refer [7].

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Coifman and Weiss [14] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchenig [20] extended the notion of atomic decomposition to Banach spaces. Gröchenig [23] introduced a more general concept for Banach spaces called Banach frame. Banach frames were further studied in [5, 9].

Stability theorems for frames in Hilbert spaces were studied in [2, 6, 32] and for Banach frames were studied by Christensen and Heil [9]. Casazza and Christensen [4] investigated stability of Banach frames via perturbation of operators. Also, Jain, Kaushik and Vashisht [24, 25, 26] proved various stability results for Banach frames which in some sense are more general than similar results given by Christensen and Heil [9].

Over the last decade, various other generalizations of frames for Hilbert spaces have been introduced and studied. Some of them are bounded quasi-projectors by Fornasier [21]; Pseudo frames by Li and Ogawa [28]; Oblique frames by Eldar [19]; Christensen and Eldar [8]; $(p;Y)$-Operators frames by Cao et.al [3]. W. Sun [30] introduced and defined $g-$frames in Hilbert spaces and observed that bounded quasi-projectors, frames of subspaces, Pseudo frames and Oblique frames are particular cases of $g-$frames in Hilbert spaces. $G-$frames are further studied in [10, 11, 12, 13, 29, 31].

Abdollahpour et.al [1] generalized the concept of frames for Banach spaces and introduced $g-$Banach frames in Banach spaces. $G-$Banach frames in Banach spaces are generalized frames in Banach spaces which includes Banach frames, frames of subspaces (fusion frames) for Banach spaces as well as many recent generalization of frames in Banach spaces. Infact, $g-$Banach frames are natural generalizations of frames in Banach spaces which provides more choices to reconstruct vectors of the space from reconstruction operator. We further study $g-$Banach frames in Banach spaces and give a characterization of $g-$Banach frames and proved that $g-$Banach frames share many useful properties of Banach frames in Banach spaces.

In the present paper, we define various types of $g-$Banach frames in Banach spaces. Examples and counter examples to distinguish various types of $g-$Banach frames in Banach spaces have been given. It has been proved that if a Banach space $\mathcal{X}$ has a Banach frame, then $\mathcal{X}$ has a normalized tight $g-$Banach frame for $\mathcal{X}$. A characterization of an exact $g-$Banach frames has been given. Also, we consider the finite sum of $g-$Banach frames and give a sufficient condition for the finite sum of $g-$Banach frames to be a $g-$Banach frame. Finally, a sufficient condition for the stability of $g-$Banach frames in Banach spaces which provides optimal frame bounds has been given.

2. Preliminaries

Throughout this paper, $\mathcal{X}$ will denote a Banach space over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, $\mathcal{X}_d$ and $\mathcal{X}_{d_1}$, respectively, are associated Banach space of vector valued and scalar valued sequences indexed by $\mathbb{N}$. $\mathcal{H}$ is a separable Hilbert space, $B(\mathcal{X},\mathcal{H})$ is the collection of all bounded linear operators from $\mathcal{X}$ into $\mathcal{H}$, $\overline{span}\{\Lambda_n\}$ is closed linear span of $\{\Lambda_n\}$ in the strong operator topology in $B(\mathcal{X},\mathcal{H})$.

A Banach space of vector valued sequences (or BV-space) is a linear space of sequences with a norm which makes it a Banach space. Let $\mathcal{X}$ be a Banach space and $1 < p < \infty$, then

$$Y = \Big\{\{x_n\} : x_n \in \mathcal{X},\ \Big(\sum_{n\in\mathbb{N}} \|x_n\|^p\Big)^{1/p} < \infty\Big\}$$

and

$$\ell_\infty = \{\{x_n\} : \sup_{n\in\mathbb{N}} \|x_n\| < \infty,\ x_n \in \mathcal{X}\}$$

are BV spaces of $\mathcal{X}$ (a sequence space generated by the elements of $\mathcal{X}$).

It may verify that Y is a BV space (see [27], [15], [16]).

Definition 2.1. ([23]) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_{d_1}$ an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_{d_1} \to \mathcal{X}$ be given. Then the pair $(\{f_n\}, S)$ is called a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_{d_1}$, if

(1) $\{f_n(x)\} \in \mathcal{X}_{d_1}$, for each $x \in \mathcal{X}$.

(2) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{f_n(x)\}\|_{\mathcal{X}_{d_1}} \leq B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{2.1}$$

(3) S is a bounded linear operator such that $S(\{f_n(x)\}) = x, \quad x \in \mathcal{X}$.

The positive constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : \mathcal{X}_{d_1} \to \mathcal{X}$ is called the reconstruction operator (or the pre-frame operator). The inequality (2.1) is called the frame inequality.

Definition 2.2. ([1]) Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{X}_d$ be an associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{H})$ and $S : \mathcal{X}_d \to \mathcal{X}$ be given. Then the pair $(\{\Lambda_n\}, S)$ is called a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$, if

(1) $\{\Lambda_n(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.

(2) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{\Lambda_n(x)\}\|_{\mathcal{X}_d} \leq B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{2.2}$$

(3) S is a bounded linear operator such that $S(\{\Lambda_n(x)\}) = x, \quad x \in \mathcal{X}$.

The positive constants A and B, respectively, are called the lower and upper frame bounds of the $g-$Banach frame $(\{\Lambda_n\}, S)$. The operator $S : \mathcal{X}_d \to \mathcal{X}$ is called the reconstruction operator and the inequality (2.2) is called the frame inequality for $g-$Banach frame $(\{\Lambda_n\}, S)$.

3. Main Result

We begin this section with the following definition of total sequence of operators over $\mathcal{X}$.

Definition 3.1. Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. A sequence of operators $\{\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H})$ is called total over $\mathcal{X}$, if $\{x \in \mathcal{X} : \Lambda_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Next, we prove a basic result in the form of lemma which will be used in the subsequent results.

Lemma 3.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let $\{\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H})$ be a sequence of non-zero operators. If $\{\Lambda_n\}$ is total over $\mathcal{X}$, then $\mathcal{A} = \{\{\Lambda_n(x)\} : x \in \mathcal{X}\}$ is a Banach space with the norm given by $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}$.*

Proof. Clearly, $\mathcal{A}$ is a linear space under pointwise addition and scalar multiplication. Also

$$\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}} \geq 0 \ \ \forall\, x \in \mathcal{X}.$$

Let $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = 0$. Then $\|x\|_{\mathcal{X}} = 0$, i.e., $x = 0$. This gives $\{\Lambda_n(x)\} = 0$.
Also, if $\{\Lambda_n(x)\} = 0$. Then $\Lambda_n(x) = 0 \ \ \forall\, n \in \mathbb{N}$. Since $\{\Lambda_n\}$ is total over $\mathcal{X}$, we have $x = 0$. This gives $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = 0$.
Therefore, $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = 0 \iff \{\Lambda_n(x)\} = 0$.
Also, we have

$$\|\alpha\{\Lambda_n(x)\}\|_{\mathcal{A}} = |\alpha| \|\{\Lambda_n(x)\}\|_{\mathcal{A}}$$

and

$$\|\{\Lambda_n(x)\} + \{\Lambda_n(y)\}\|_{\mathcal{A}} \leq \|\{\Lambda_n(x)\}\|_{\mathcal{A}} + \|\{\Lambda_n(y)\}\|_{\mathcal{A}}.$$

Thus $\mathcal{A}$ is normed linear sapce.

Let $\{\{\Lambda_n(x_i)\}_n\}_i$ be a Cauchy sequence in $\mathcal{A}$. Then $\{x_i\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, then there exists an element $x \in \mathcal{X}$ such that $x_i \to x$ as $i \to \infty$. Thus, the Cauchy sequence $\{\{\Lambda_n(x_i)\}_n\}_i$ converges in $\mathcal{A}$. Hence $\mathcal{A}$ is a Banach space with the norm given by $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. □

Note. The Banach space $\mathcal{A}$ associated with a given Banach space $\mathcal{X}$ is a space of vector-valued sequences, indexed by $\mathbb{N}$. This associated Banach space may not be unique. Further, if $(\{\Lambda_n\}, S)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$, then $\{\Lambda_n(x)\} \in \mathcal{A},\ x \in \mathcal{X}$, i.e., $\{\{\Lambda_n(x)\} : x \in \mathcal{X}\} \subset \mathcal{A}$. Moreover, if $\{\Lambda_n\}$ is total over $\mathcal{X}$, then $\{\{\Lambda_n(x)\} : x \in \mathcal{X}\}$ behaves as the associated Banach space.

Next, we give definition of various types of $g-$Banach frames.

Definition 3.2. Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. A $g-$Banach frame $(\{\Lambda_n\}, S)$ for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$ with frame bounds A and B is called

- *Tight*, if it is possible to choose $A = B$ satisfying (2.2).
- *Normalized tight*, if it is possible to choose $A = B = 1$ satisfying (2.2).
- *Exact*, if there exist no reconstruction operator S_0 such that $(\{\Lambda_n\}_{n \neq m}, S_0)$ $(m \in \mathbb{N})$ is a $g-$Banach frame for $\mathcal{X}$.

Regarding the existence of these various types of $g-$Banach frames, we have the following examples:

Example 3.1. *Let $\mathcal{H}$ be a separable Hilbert space and let $E = \ell^\infty(\mathcal{H}) = \{\{x_n\} : x_n \in \mathcal{H};\ \sup_{1 \leq n < \infty} \|x_n\|_{\mathcal{H}} < \infty\}$ be a Banach space with norm given by*

$$\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_{\mathcal{H}}, \quad \{x_n\} \in E.$$

Now, for each $n \in I\!N$*, define* $\Lambda_n : E \to \mathcal{H}$ *by*

$$\Lambda_n(x) = x_n,\ x = \{x_n\} \in E.$$

Then $\{\Lambda_n\}$ *is a non-zero sequence of bounded linear operators on* E *into* $\mathcal{H}$ *such that*

$$\{x \in E : \Lambda_n(x) = 0, \quad \text{for all } n \in I\!N\} = \{0\}.$$

Therefore, by Lemma 3.1, there exists an associated Banach space $\mathcal{A} = \{\{\Lambda_n(x)\} : x \in E\}$ *with the norm given by* $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_E,\ x \in E$*. Define* $S : \mathcal{A} \to E$ *by* $S(\{\Lambda_n(x)\}) = x,\ x \in E$*.*

(1) ***Tight and Exact***. $(\{\Lambda_n\}, S)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}$ *with bounds* $A = B = 1$*. Also, there exists no reconstruction operator* S_0 *such* $(\{\Lambda_n\}_{n\neq m}, S_0)(m \in I\!N)$ *is a* $g-$ *Banach frame for* E*. Indeed, we have*

$$0 \neq x = (0, 0, \ldots, 1, \ldots, 0, ..) \in E, \text{ where } 1 \text{ is in the } m^{th} \text{ place,}$$
$$\text{such that } \Lambda_n(x) = 0,\ n \in I\!N,\ n \neq m.$$

(2) ***Non-Tight and Exact***. *Define the sequence of operators* $\{\Theta_n\}$ *on* E *into* $\mathcal{H}$ *by*

$$\begin{cases} \Theta_1(x) = \dfrac{1}{2}\Lambda_1(x), \\ \Theta_n(x) = \Lambda_n(x),\ n \geq 2,\ n \in I\!N, \quad x \in E. \end{cases}$$

Then, for each $x \in E$*,* $\{\Theta_n(x)\} \in \mathcal{A}$ *such that*

$$\frac{1}{2}\|x\|_E \leq \|\{\Theta_n(x)\}\|_{\mathcal{A}} \leq \|x\|_E,\ x \in E.$$

Now, define two operators U *and* V *on* E *into* $\mathcal{A}$ *by*

$$U(x) = \{\Lambda_n(x)\} \text{ and } V(x) = \{\Theta_n(x)\},\ x \in E.$$

Then $\|U - V\| < 1$ *and* $SU = I$*. Also* $\|I - SV\| < 1$*. So, the operator* SV *is invertible. Put* $T = (SV)^{-1}S$*. Then* $T : \mathcal{A} \to E$ *is a bounded linear operator* *such that* $T(\{\Theta_n(x)\}) = x,\ x \in E$. *Therefore,* $(\{\Theta_n\}, T)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}$ *which is non-tight and exact.*

(3) ***Tight and Non-Exact***. *Define the sequence of operators* $\{\Theta_n\}$ *on* E *into* $\mathcal{H}$ *by*

$$\begin{cases} \Theta_1(x) = \frac{1}{2}\Lambda_1(x), \\ \Theta_n(x) = \Lambda_{n-1}(x),\ n \geq 2,\ n \in I\!N, \quad x \in E. \end{cases}$$

Then, by Lemma 3.1, there exists an associated Banach space $\mathcal{A}_0 = \{\{\Theta_n(x)\} : x \in E\}$ *of vector-valued sequences indexed by* $I\!N$ *with the norm given by* $\|\{\Theta_n(x)\}\|_{\mathcal{A}_0} = \|x\|_E,\ x \in E$. *Define* $S_0 : \mathcal{A}_0 \to E$ *given by* $S_0(\{\Theta_n(x)\}) = x$, $x \in E$. *Then* $(\{\Theta_n\}, S_0)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}_0$ *with frame bounds* $A = B = 1$. *Also, by Lemma 3.1, there exists an associated Banach space* $\mathcal{A}_1 = \{\{\Theta_n(x)\}_{n\neq 1} : x \in E\}$ *and an operator* $S_1 : \mathcal{A}_1 \to E$ *defined by* $S_1(\{\Theta_n(x)\}_{n\neq 1}) = x,\ x \in E$ *such that* $(\{\Theta_n\}_{n\neq 1}, S_1)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}_1$. *Hence* $(\{\Theta_n\}, S_0)$ *is a tight and non-exact* $g-$*Banach frame for* E.

(4) ***Non-Tight and Non-Exact***. *Let* $\{\alpha_n\} \subset \mathbb{R}$ *be the sequence defined by*

$$\alpha_n = \begin{cases} \frac{1}{2},\ n = 3 \\ 1,\ n \neq 3,\ n \in I\!N. \end{cases}$$

Define a sequence of operators $\{\Theta_n\} \subset B(E, \mathcal{H})$ *as*

$$\begin{cases} \Theta_n(x) = \alpha_n\Lambda_n(x),\ n = 1 \\ \Theta_n(x) = \alpha_n\Lambda_{n-1}(x),\ n \geq 2,\ n \in I\!N,\ x \in E. \end{cases}$$

Then for each $x \in E$, $\{\Theta_n(x)\} \in \mathcal{A}$ *such that*

$$\frac{1}{2}\|x\|_E \leq \|\{\Theta_n(x)\}\|_{\mathcal{A}} \leq \|x\|_E,\ x \in E.$$

Then, proceeding as in (2), we can find a bounded linear operator $T : \mathcal{A} \to E$ *such that* $(\{\Theta_n\}, T)$ *is a non-tight* $g-$*Banach frame for* E *with respect to* $\mathcal{A}$. *Also, by*

Lemma 3.1, there exists an associated Banach space $\mathcal{A}_1 = \{\{\Theta_n(x)\}_{n\neq 1} : x \in E\}$ *and an operator* $S_1 : \mathcal{A}_1 \to E$ *defined by* $S_1(\{\Theta_n(x)\}_{n\neq 1}) = x,\ x \in E$ *such that* $(\{\Theta_n\}_{n\neq 1}, S_1)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}_1$. *Hence* $(\{\Theta_n\}, T)$ *is a non-tight and non-exact* $g-$*Banach frame for* E *with respect to* $\mathcal{A}$.

Example 3.2. *(**Not a** $g-$**Banach Frame**). Let* $\mathcal{H}$ *be a separable Hilbert space and let*

$$E = \ell^2(\mathcal{H}) = \{\{x_n\} : x_n \in \mathcal{H};\ \sum_{n=1}^{\infty} \|x_n\|^2_{\mathcal{H}} < \infty\}.$$

Define a norm $\|.\|_E$ *on* E *by*

$$\|\{x_n\}\|_E = \left(\sum_{n=1}^{\infty} \|x_n\|^2_{\mathcal{H}}\right)^{1/2},\ \{x_n\} \in E.$$

Thus E *is a Banach space.*

Now, for each $n \in I\!N$, *define* $\Lambda_n : E \to \ell^2(\mathcal{H})$ *by*

$$\begin{cases} \Lambda_1(x) = \delta_2^{x_2} \\ \Lambda_n(x) = \delta_n^{x_n},\ n \geq 2,\ x = \{x_n\} \in E, \end{cases}$$

where $\delta_n^x = (0, \ldots, 0 \underbrace{x}_{n^{th} place}, 0, \ldots)$, *for all* $n \in I\!N$ *and* $x \in \mathcal{H}$.

Then, there exists no associated Banach space $\mathcal{A}$ *such that* $(\{\Lambda_n\}, S)$ $(S : \mathcal{A} \to E)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}$. *Indeed, we have* $x = (1, 0, 0, \ldots)$ *in* E *such that* $\Lambda_n(x) = 0$, *for all* $n \in I\!N$.

Regarding the existence of $g-$Banach frame, Abdollahpour et.al [1] prove that every separable Banach space has $g-$Banach frame with frame bounds $A = B = 1$.

In the next theorem, we give more general result regarding the existence of $g-$Banach frame in Banach spaces.

Theorem 3.1. *Let* $\mathcal{X}$ *be a Banach space and* $\mathcal{H}$ *be a separable Hilbert space. If* $\mathcal{X}$ *has a Banach frame, then* $\mathcal{X}$ *has a* $g-$*Banach frame with frame bounds* $A = B = 1$.

Proof. . Let $\mathcal{X}$ has a Banach frame. Then, by the Theorem 4.2 [24], $\mathcal{X}$ has an exact Banach frame. Let $(\{f_n\}, S)$ $(\{f_n\} \subset \mathcal{X}^*,\ S : \mathcal{X}_{d_1} \to \mathcal{X})$ be an exact Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_{d_1}$. Then, there exist constants $A,\ B$ with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{f_n(x)\}\|_{\mathcal{X}_{d_1}} \leq B\|x\|_{\mathcal{X}}, \text{ for all } x \in \mathcal{X}. \tag{3.1}$$

Also, by Lemma 4.1 [24], there exists a sequence $\{x_n\} \subset \mathcal{X}$ such that

$$f_i(x_j) = \delta_{ij}, \quad \text{for all } i,\ j \in \mathbb{N}. \tag{3.2}$$

Let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$.
Now, for each $n \in \mathbb{N}$, define $\Lambda_n : \mathcal{X} \to \mathcal{H}$ as

$$\Lambda_n(x) = f_n(x)e_n,\ x = \{x_n\} \in \mathcal{X}.$$

Then $\{\Lambda_n\}$ is a sequence of non-zero bounded and linear operators from $\mathcal{X}$ into $\mathcal{H}$. Also, $\{\Lambda_n\}$ is total over $\mathcal{X}$. Therefore, by Lemma 3.1, there exist an associated Banach space $\mathcal{A} = \Big\{\{\Lambda_n(x)\} : x \in \mathcal{X}\Big\}$ with the norm given by $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Define $T : \mathcal{A} \to \mathcal{X}$ by $T(\{\Lambda_n(x)\}) = x,\ x \in \mathcal{X}$. Thus $(\{\Lambda_n\}, T)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$. □

Corollary 3.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{X}$ has a Banach frame, then $\mathcal{X}$ has a normalized tight and exact $g-$Banach frame.*

Proof. . Let $\mathcal{X}$ has a Banach frame. Then, by the Theorem 3.1, $(\{\Lambda_n\}, T)$ is a normalized tight $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$.
Now, we shall show that $\mathcal{X}$ has an exact $g-$Banach frame for $\mathcal{X}$.
Let, if possible $(\{\Lambda_n\}, T)$ be not an exact. Then, there exists a positive integer m and a reconstruction operator $T_0 : \mathcal{A}_0 \to \mathcal{X}$ such that $(\{\Lambda_n\}_{n \neq m}, T_0)$ is a $g-$Banach frame for $\mathcal{X}$, where $\mathcal{A}_0$ is some associated Banach space of vector-valued sequences

indexed by $\mathbb{N}$.

Let A_0 and B_0 be the choice of bounds for $(\{\Lambda_n\}_{n\neq m}, T_0)$. Then,

$$A_0\|x\|_{\mathcal{X}} \leq \|\{\Lambda_n(x)\}_{n\neq m}\|_{\mathcal{A}_0} \leq B_0\|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{3.3}$$

Let $x = (0, 0, \ldots, 0, \underbrace{1}_{m^{th}\text{place}}, 0, \ldots)$ be an element in E.

Then $\Lambda_n(x) = 0,\ n \in \mathbb{N},\ n \neq m$. Which implies $x = 0$, which is a contradiction. Hence $(\{\Lambda_n\}, T)$ is normalized tight and exact $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$. □

Remark 1. (I) The converse of the Theorem 3.1 need not be true, in general. Let $E = \ell^\infty(\mathcal{H})/c_0(\mathcal{H})$. Let, if possible $(\{f_n\}, S)$ be a Banach frame for E with respect to E_d. Then, there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E,\ x \in E.$$

Then $\{x \in E : \ f_n(x) = 0, \ \text{for all} \ n \in \mathbb{N}\} = \{0\}$. But, this contradicts the Theorem 2 in [16]. However, E does have a $g-$Banach frame. Indeed, let $\{x_n\}$ be a basic sequence in E.

Now, for each $n \in \mathbb{N}$, define $\Lambda_n : E \to \mathcal{H}$ as

$$\Lambda_n(x) = x_n,\ x = \{x_n\} \in E.$$

Then, by Lemma 3.1, there exists an associated Banach space $\mathcal{A} = \{\{\Lambda_n(x)\} : x \in E\}$ together with a bounded linear operator $S_1 : \mathcal{A} \to E$ defined by $S_1(\{\Lambda_n(x)\}) = x,\ x \in E$ such that $(\{\Lambda_n\}, S_1)$ is a $g-$Banach frame for E with respect to $\mathcal{A}$.

(II) We observe that for a sequence of operators $\{\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H})$, if the coefficient mapping $T : \mathcal{X} \to \mathcal{X}_d$ defined by $T(x) = \{\Lambda_n(x)\},\ x \in \mathcal{X}$ is a topological isomorphism onto $\mathcal{X}_d$. Then there exists a reconstruction operator $S : \mathcal{X}_d \to \mathcal{X}$ such that $(\{\Lambda_n\}, S)$

is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$ with bounds $\|T^{-1}\|^{-1}$ and $\|T\|$. Indeed, for each $x \in E$, we have

$$\|\{\Lambda_n(x)\}\|_{\mathcal{X}_d} = \|Tx\|_{\mathcal{X}_d} \leq \|T\|\|x\|_{\mathcal{X}}$$

and

$$\|x\|_{\mathcal{X}} \leq \|T^{-1}\|\|\{\Lambda_n(x)\}\|_{\mathcal{X}_d}.$$

Therefore

$$\|T^{-1}\|^{-1}\|x\|_{\mathcal{X}} \leq \|\{\Lambda_n(x)\}\|_{\mathcal{X}_d} \leq \|T\|\|x\|_{\mathcal{X}},\ x \in \mathcal{X}.$$

Put $S = T^{-1}$. Then $S : \mathcal{X}_d \to \mathcal{X}$ is a bounded linear operator such that $S(\{\Lambda_n(x)\}) = x,\ x \in \mathcal{X}$. Hence $(\{\Lambda_n\}, S)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ with frame bounds $\|T^{-1}\|^{-1}$,$\|T\|$.

In the next theorem, we provide a characterization of an exact $g-$Banach frame.

Theorem 3.2. *Let $(\{\Lambda_n\}, S)(\{\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H}),\ S : \mathcal{X}_d \to \mathcal{X})$ be a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then $(\{\Lambda_n\}, S)$ is exact if and only if $\Lambda_i \notin \overline{span}\{\Lambda_n\}_{n\neq i}$, for all $i \in \mathbb{N}$.*

Proof. Let $(\{\Lambda_n\}, S)$ be an exact $g-$Banach frame for $\mathcal{X}$ and let $\Lambda_i \in \overline{span}\{\Lambda_n\}_{n\neq i}$, for some $i \in \mathbb{N}$.

Then, $\Lambda_i(x) = lim_{k\to\infty} \sum_{\substack{n=1 \\ n\neq i}}^{k} \alpha_n^{(k)} \Lambda_n(x),\ x \in \mathcal{X}$.

Thus, by frame inequality of $g-$Banach frame $(\{\Lambda_n\}, S)$, $\{\Lambda_n\}_{n\neq i}$ is total over $\mathcal{X}$.

So, by Lemma 3.1, there exists an associated Banach space $\mathcal{A}_0 = \{\{\Lambda_n(x)\}_{n\neq i} : x \in \mathcal{X}\}$ with the norm given by $\|\{\Lambda_n(x)\}_{n\neq i}\|_{\mathcal{A}_0} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Let $S_0 : \mathcal{A}_0 \to \mathcal{X}$ be given by $S_0(\{\Lambda_n(x)\})_{n\neq i} = x,\ x \in \mathcal{X}$. Then $(\{\Lambda_n\}_{n\neq i}, S_0)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}_0$, which is a contradiction to given hypothesis.

Therefore, $\Lambda_i \notin \overline{span}\{\Lambda_n\}_{n\neq i}$, for all $i \in \mathbb{N}$.

Conversely, let $\Lambda_i \notin \overline{span}\{\Lambda_n\}_{n\neq i}$, for all $i \in \mathbb{N}$. Let $(\{\Lambda_n\}, S)$ be not an exact. Then there exists a positive integer m and a reconstruction operator S_1 such that $(\{\Lambda_n\}_{n\neq m}, S_1)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}_1$, where $\mathcal{A}_1$ is some associated Banach space of vector valued sequences indexed by $\mathbb{N}$. Let A_1 and B_1 be choice of bounds for $(\{\Lambda_n\}_{n\neq m}, S_1)$. Then

$$A_1\|x\|_{\mathcal{X}} \leq \|\{\Lambda_n(x)\}_{n\neq m}\|_{\mathcal{A}_1} \leq B\|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{3.4}$$

This gives $\{\Lambda_n\}_{n\neq m}$ is total over $\mathcal{X}$, since otherwise there exists a non zero element $x \in \mathcal{X}$ such that $\Lambda_n(x) = 0, n \in \mathbb{N},\ n \neq m$. Then, by frame inequality (3.4), $x = 0$, which is a contradiction.

Therefore $\overline{span}\{\Lambda_n\}_{n\neq m} = B(\mathcal{X}, \mathcal{H})$. This gives $\Lambda_m \in \overline{span}\{\Lambda_n\}_{n\neq m}$. Which is again a contradiction. Hence $(\{\Lambda_n\}, S)$ is an exact $g-$Banach frame for $\mathcal{X}$. □

Now, in the next result, we give a condition under which the Banach space $\mathcal{Y}$ possesses a $g-$Banach frame, when $\mathcal{Y}$ is related in some sense to a Banach space $\mathcal{X}$ which is having a $g-$Banach frame.

Theorem 3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. Let $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H})$, $S : \mathcal{X}_d \to \mathcal{X})$ be a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\{\Theta_n\} \subset B(\mathcal{Y}, \mathcal{H})$ be any sequence of operators. If there exists a continuous linear mapping U from $\mathcal{X}$ onto $\mathcal{Y}$ such that $\Theta_n o U = \Lambda_n, n \in \mathbb{N}$. Then $\mathcal{Y}$ has a $g-$Banach frame with respect to some associated Banach space $\mathcal{A}$.*

Proof. Since $U : \mathcal{X} \to \mathcal{Y}$ is an onto map. Then, for each $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ such that $U(x) = y$. Suppose that $\Theta_n(y) = 0$ for all $n \in \mathbb{N}$. Then, $\Lambda_n(x) = 0$ for all $n \in \mathbb{N}$. This gives $y = 0$. Thus $\{\Theta_n\}$ is total over $\mathcal{Y}$. Therefore, by Lemma 3.1, there exists an associated Banach space $\mathcal{A} = \{\{\Theta_n(y)\} : y \in \mathcal{Y}\}$ with the norm given by $\|\{\Theta_n(y)\}\|_{\mathcal{A}} = \|y\|_{\mathcal{Y}}$, $y \in \mathcal{Y}$. Define $T : \mathcal{A} \to \mathcal{Y}$ by $T(\{\Theta_n(y)\}) = y$, $y \in \mathcal{Y}$. Then

T is a bounded linear operator such that $(\{\Theta_n\}, T)$ is a $g-$Banach frame for $\mathcal{Y}$ with respect to $\mathcal{A}$. □

4. Finite Sum of $G-$Banach Frames

Let $(\{\Lambda_{i,n}\}, S_i)$, $i = 1, 2, ..., k$ be $g-$Banach frames for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and $\mathcal{H}$. Then, there exists, in general no reconstruction operator S such that $(\{\sum_{i=1}^{k} \Lambda_{i,n}\}, S)$ is a $g-$Banach frame for $\mathcal{X}$. Regarding this, we give the following example:

Example 4.1. *Let $\mathcal{H}$ be a separable Hilbert space and let $E = \ell^\infty(\mathcal{H}) = \{\{x_n\} : x_n \in \mathcal{H};\ \sup_{1\leq n<\infty} \|x_n\|_{\mathcal{H}} < \infty\}$ be a Banach space with norm given by*

$$\|\{x_n\}\|_E = \sup_{1\leq n<\infty} \|x_n\|_{\mathcal{H}}, \quad \{x_n\} \in E.$$

For each $n \in I\!N$, define $\Lambda_{1,n} : E \to \ell^2(\mathcal{H})$ by

$$\Lambda_{1,n}(x) = \sum_{i=n}^{n+1} \delta_i^{x_i},\ x = \{x_n\} \in E,$$

where $\delta_n^x = (0, \ldots, 0 \underbrace{x}_{n^{th} place}, 0, \ldots)$, for all $n \in I\!N$ and $x \in \mathcal{H}$.
Then, by Lemma 3.1, there exists an associated Banach space $\mathcal{A} = \{\{\Lambda_{1,n}(x)\} : x \in E\}$ with the norm given by $\|\{\Lambda_{1,n}(x)\}\|_{\mathcal{A}} = \|x\|_E,\ x \in E$. Define $S_1 : \mathcal{A} \to E$ by $S_1(\{\Lambda_{1,n}(x)\}) = x,\ x \in E$. Then S_1 is a bounded linear operator such that $(\{\Lambda_{1,n}\}, S_1)$ is a $g-$Banach frame for E with respect to $\mathcal{A}$.
Now, we define a sequence of operators $\{\Lambda_{2,n}\} \subset B(E, \ell^2(\mathcal{H}))$ by

$$\begin{cases} \Lambda_{2,n}(x) = -\Lambda_{1,n}(x),\ n = 1 \\ \Lambda_{2,n}(x) = \Lambda_{1,n}(x),\ n \geq 2,\ n \in I\!N,\ x \in E. \end{cases}$$

Then, there exists a reconstruction operator $S_2 : \mathcal{A} \to E$ such that $(\{\Lambda_{2,n}\}, S_2)$ is a $g-$Banach frame for E with respect to $\mathcal{A}$. But there exists no reconstruction operator S_0 such that $\left(\left\{ \sum_{i=1}^{2} \Lambda_{i,n}\right\}, S_0\right)$ $(S_0 : \mathcal{A}_0 \to E)$ is a $g-$Banach frame for E with respect to $\mathcal{A}_0$, where $\mathcal{A}_0$ is some associated Banach space of vector valued sequences

indexed by $I\!N$.
Indeed, let $\Big(\Big\{\sum_{i=1}^{2}\Lambda_{i,n}\Big\},S_0\Big)$ *be a $g-$Banach frame for E with respect to $\mathcal{A}_0$. Let A and B be choice of bounds for* $\Big(\Big\{\sum_{i=1}^{2}\Lambda_{i,n}\Big\},S_0\Big)$*. Then*

$$A\|x\|_E\leq\Big\|\Big\{\Big(\sum_{i=1}^{2}\Lambda_{i,n}\Big)(x)\Big\}\Big\|_{\mathcal{A}_0}\leq B\|x\|_E,\ x\in E. \tag{4.1}$$

Let $x=(1,0,0,\dots)$ be a non-zero element in E such that

$$\Big(\sum_{i=1}^{2}\Lambda_{i,n}\Big)(x)=0,\ \textit{for all } n\in I\!N.$$

Then, by frame inequality (4.1), we get $x=0$. This is a contradiction. Hence $\Big(\Big\{\sum_{i=1}^{2}\Lambda_{i,n}\Big\},S_0\Big)$ *is not a $g-$Banach frame for E with respect to $\mathcal{A}_0$.*

In view of Example 4.1, we give a condition under which the finite sum of $g-$Banach frames for a Banach space $\mathcal{X}$ is a $g-$Banach frame for $\mathcal{X}$.

Theorem 4.1. *Let $(\{\Lambda_{i,n}\},S_i)(\{\Lambda_{i,n}\}\subset B(\mathcal{X},\mathcal{H}),\ S_i:\mathcal{X}_d\to\mathcal{X}),\ i=1,2,3,\dots,k$ be a $g-$Banach frames for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then, there exists a reconstruction operator S_0 such that* $\Big(\Big\{\sum_{i=1}^{k}\Lambda_{i,n}\Big\},S_0\Big)$ *is a normalized tight $g-$Banach frame for $\mathcal{X}$, provided*

$$\|\{\Lambda_{p,n}(x)\}\|_{\mathcal{X}_d}\leq\Big\|\Big\{\Big(\sum_{i=1}^{k}\Lambda_{i,n}\Big)(x)\Big\}\Big\|_{\mathcal{X}_d},\ x\in\mathcal{X},\ \textit{for some } p\in\{1,2,\dots,k\}.$$

Proof. Since for each $i=1,2,\dots,k$, $(\{\Lambda_{i,n}\},S_i)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Thus

$$\begin{aligned}\|x\|_{\mathcal{X}}&=\|S_p(\{\Lambda_{p,n}(x)\})\|_{\mathcal{X}}\\&\leq\|S_p\|\Big\|\Big\{\Big(\sum_{i=1}^{k}\Lambda_{i,n}\Big)(x)\Big\}\Big\|_{\mathcal{X}_d},\ x\in\mathcal{X}.\end{aligned}$$

Then

$$\|S_p\|^{-1}\|x\|_{\mathcal{X}} \leq \left\|\left\{\left(\sum_{i=1}^{k}\Lambda_{i,n}\right)(x)\right\}\right\|_{\mathcal{X}_d},\ x\in\mathcal{X}.$$

Therefore, by Lemma 3.1, there exists an associated Banach space $\mathcal{A}_0 = \left\{\left\{\left(\sum_{i=1}^{k}\Lambda_{i,n}\right)(x)\right\} : x\in\mathcal{X}\right\}$ with the norm given by $\left\|\left\{\left(\sum_{i=1}^{k}\Lambda_{i,n}\right)(x)\right\}\right\|_{\mathcal{A}_0} = \|x\|_{\mathcal{X}},\ x\in\mathcal{X}$. Define $S_0 : \mathcal{A}_0 \to \mathcal{X}$ by $S_0\left(\left\{\left(\sum_{i=1}^{k}\Lambda_{i,n}\right)(x)\right\}\right) = x,\ x\in\mathcal{X}$. Thus S_0 is a bounded linear operator such that $\left(\left\{\sum_{i=1}^{k}\Lambda_{i,n}\right\}, S_0\right)$ is a normalized tight $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}_0$. □

Remark 2. Towards the converse of Theorem 4.1, we observe that $\left(\left\{\sum_{i=1}^{k}\Lambda_{i,n}\right\}, S\right)$ $(S:\mathcal{X}_d\to\mathcal{X})$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$, where $\{\Lambda_{i,n}\}\subset B(\mathcal{X},\mathcal{H})$ $i=1,2,\ldots,k$. Then there exists, in general, no reconstruction operator S_i, for $i=1,2,\ldots,k$ such that $(\{\Lambda_{i,n}\}, S_i)$, $i=1,2,\ldots,k$ is a $g-$Banach frame for $\mathcal{X}$ with respect to some associated Banach space $\mathcal{A}_0$. Regarding this, we give the following example:

Example 4.2. *Let $\mathcal{H}$ be a separable Hilbert space and let $E=\ell^\infty(\mathcal{H})=\{\{x_n\}: x_n\in\mathcal{H};\ \sup_{1\leq n<\infty}\|x_n\|_{\mathcal{H}}<\infty\}$ be a Banach space with norm given by*

$$\|\{x_n\}\|_E = \sup_{1\leq n<\infty}\|x_n\|_{\mathcal{H}},\quad \{x_n\}\in E.$$

For each $n\in I\!N$, define $\{\Lambda_{1,n}\}$, $\{\Lambda_{2,n}\}\subset B(E,\ell^2(\mathcal{H}))$ by

$$\begin{cases}\Lambda_{1,1}(x)=0,\ \Lambda_{1,n}(x)=\delta_n^{x_n},\ n>1\\ \Lambda_{2,1}(x)=\delta_1^{x_1},\ \Lambda_{2,n}(x)=0,\ n>1,\ x=\{x_n\}\in E,\end{cases}$$

where $\delta_n^x=(0,\ldots,0\underbrace{x}_{n^{th}place},0,\ldots)$, for all $n\in I\!N$ and $x\in\mathcal{H}$.

Then $\{\sum_{i=1}^{2}\Lambda_{i,n}\}$ is total over E. Therefore, by Lemma 3.1, there exists an associated Banach space $\mathcal{A}=\left\{\left\{\left(\sum_{i=1}^{2}\Lambda_{i,n}\right)(x)\right\}: x\in E\right\}$ with the norm given by

$$\left\|\left\{\left(\sum_{i=1}^{2}\Lambda_{i,n}\right)(x)\right\}\right\|_{\mathcal{A}} = \|x\|_E,\ x \in E.$$

Now, define $S : \mathcal{A} \to E$ *by* $S\left(\left\{\left(\sum_{i=1}^{2}\Lambda_{i,n}\right)(x)\right\}\right) = x,\ x \in E$. *Thus* S *is a bounded linear operator such that* $\left(\left\{\sum_{i=1}^{2}\Lambda_{i,n}\right\}, S\right)$ *is a* $g-$*Banach frame for* E *with respect to* $\mathcal{A}$. *But there exist no reconstruction operator* S_1 *and* S_2 *such that* $(\{\Lambda_{1,n}\}, S_1)$ *and* $(\{\Lambda_{2,n}\}, S_2)$ *are* $g-$*Banach frame for* E.

5. Stability of $G-$Banach Frames

In 2011, W. Gang [22] studied stability of $g-$Banach frames in Banach spaces and obtained various stability conditions for $g-$Banach frames in Banach spaces. But, the stability of $g-$Banach frame in Theorem 2 and 3 in [22] depends on the value of positive constant M. Since for large value of M, the $g-$Banach frame inequality becomes redundant. Therefore, we still need stability conditions which gives optimal frame bounds. The following theorem gives such stability conditions.

Theorem 5.1. *Let* $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H}), S : \mathcal{X}_d \to \mathcal{X})$ *be a* $g-$*Banach frame for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$. *Let* $\{\Theta_n\} \subset B(\mathcal{X}, \mathcal{H})$ *be such that* $\{\Theta_n(x)\} \in \mathcal{X}_d,\ x \in \mathcal{X}$ *and let* $V : \mathcal{X} \to \mathcal{X}_d$ *be coefficient mapping given by* $V(x) = \{\Theta_n(x)\},\ x \in \mathcal{X}$. *If there exist non-negative constants* λ, μ, ν *and* ξ *such that*

(i) $\left(\dfrac{\|T\| + \|V\| + 1}{(\|S\|)^{-1}}\right)\sqrt{\max\{\lambda, \mu, \nu, \xi\}} < 1$

(ii)
$$\|\{(\Lambda_n - \Theta_n)(x)\}\|^2_{\mathcal{X}_d} \le \lambda\|\{\Lambda_n(x)\}\|^2_{\mathcal{X}_d} + 2\mu\|\{\Lambda_n(x)\}\|_{\mathcal{X}_d}\|\{\Theta_n(x)\}\|_{\mathcal{X}_d} + \nu\|\{\Theta_n(x)\}\|^2_{\mathcal{X}_d} + \xi\|x\|^2_{\mathcal{X}},\quad x \in \mathcal{X},$$

then there exists a reconstruction operator U *such that* $(\{\Theta_n\}, U)$ *is a* $g-$*Banach frame for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$ *and with frame bounds*

$$\left(\frac{(\|S\|)^{-1} - ((\|S\|)^{-1} + 1)\sqrt{\max\{\lambda, \mu, \nu, \xi\}}}{1 + \sqrt{\max\{\lambda, \mu, \nu, \xi\}}}\right)$$

and

$$\left(\frac{(\|T\|) + ((\|T\|) + 1)\sqrt{\max\{\lambda, \mu, \nu, \xi\}}}{1 - \sqrt{\max\{\lambda, \mu, \nu, \xi\}}}\right),$$

where T is the coefficient mapping given by $T(x) = \{\Lambda_n(x)\}, \quad x \in \mathcal{X}$.

Proof. Let $\eta = \max\{\lambda, \mu, \nu, \xi\}$. Then (ii) may be restated as

$$\|Tx - Vx\|_{\mathcal{X}_d} \leq \sqrt{\eta}\bigg(\|Tx\|_{\mathcal{X}_d} + \|Vx\|_{\mathcal{X}_d} + \|x\|_{\mathcal{X}}\bigg), \quad x \in \mathcal{X}.$$

Now,

$$\begin{aligned}\|Vx\|_{\mathcal{X}_d} &\leq \|Tx\|_{\mathcal{X}_d} + \|Vx - Tx\|_{\mathcal{X}_d}\\ &\leq \|Tx\|_{\mathcal{X}_d} + \sqrt{\eta}\bigg(\|Tx\|_{\mathcal{X}_d} + \|Vx\|_{\mathcal{X}_d} + \|x\|_{\mathcal{X}}\bigg).\end{aligned}$$

This gives

$$(1 - \sqrt{\eta})\|Vx\|_{\mathcal{X}_d} \leq \bigg((1 + \sqrt{\eta})\|T\| + \sqrt{\eta}\bigg)\|x\|_{\mathcal{X}}.$$

Also, since $ST : \mathcal{X} \to \mathcal{X}$ is an identity operator, then

$$\|x\|_{\mathcal{X}} = \|ST(x)\|_{\mathcal{X}} \leq \|S\|\|Tx\|_{\mathcal{X}_d}. \tag{5.1}$$

Thus,

$$\begin{aligned}\|Vx\|_{\mathcal{X}_d} &\geq \|Tx\|_{\mathcal{X}_d} - \|Tx - Vx\|_{\mathcal{X}_d}\\ &\geq \|Tx\|_{\mathcal{X}_d} - \sqrt{\eta}\bigg(\|Tx\|_{\mathcal{X}_d} + \|Vx\|_{\mathcal{X}_d} + \|x\|_{\mathcal{X}}\bigg)\end{aligned}$$

which implies

$$\begin{aligned}(1 + \sqrt{\eta})\|Vx\|_{\mathcal{X}_d} &\geq (1 - \sqrt{\eta})\|Tx\|_{\mathcal{X}_d} - \sqrt{\eta}\|x\|_{\mathcal{X}}\\ &\geq \bigg((1 - \sqrt{\eta})\|S\|^{-1} - \sqrt{\eta}\bigg)\|x\|_{\mathcal{X}} \ \ \text{(by using(5.1))}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\frac{(1-\sqrt{\eta})(\|S\|)^{-1}-\sqrt{\eta}}{1+\sqrt{\eta}}\right)\|x\|_{\mathcal{X}} &\leq \|\{\Theta_n(x)\}\|_{\mathcal{X}_d}\\ &\leq \left(\frac{(1+\sqrt{\eta})\|T\|+\sqrt{\eta}}{1-\sqrt{\eta}}\right)\|x\|_{\mathcal{X}}, \quad x\in\mathcal{X}.\end{aligned}$$

Also, $ST=I$, where I is an identity operator on $\mathcal{X}$ and

$$\begin{aligned}\|I-SV\| &\leq \|S\|\|T-V\|\\ &\leq \|S\|\sqrt{\eta}(\|T\|+\|V\|+1)\\ &< 1.\end{aligned}$$

Thus, SV is an invertible operator. Let $U=(SV)^{-1}S$. Then $U:\mathcal{X}_d\to\mathcal{X}$ is a bounded linear operator such that $U(\{\Theta_n(x)\})=x,\ x\in\mathcal{X}$. Hence $(\{\Theta_n\},U)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ with desire frame bounds. □

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(1) Department of Mathematics and Statistics, University College of Science, M.L.S. University, Udaipur, INDIA.

E-mail address: ghanshyamsrathore@yahoo.co.in

(2) Department of Mathematics, Dr. Akhilesh Das gupta Institute of Technology and Management, G.G.S. Inderprastha University, Delhi, INDIA.

E-mail address: triptimittal.05@gmail.com