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MEROMORPHIC FUNCTIONS CONCERNING DIFFERENCE OPERATOR

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Abstract. We deal with a uniqueness question of meromorphic functions sharing a polynomial with their difference operators and obtain some results, which generalize and improve the recent result of Sujoy Majumder [11].

1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We will use the standard notations of Nevanlinna's value distribution theory such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, and $m(r,f)$, as explained in Hayman [4], Yang [14], and Yang and Yi [13]. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$ possibly outside a set of finite linear measures. We denote $\rho(f)$ for order of f. $\rho_2(f)$ is hyper order of $f(z)$, defined by

$$\rho_2(f) = \lim_{r\to\infty} \sup \frac{\log\log T(r,f)}{\log r}.$$

Let f and g be two nonconstant meromorphic functions and a be a finite complex number. We say that f, g share the value a CM (counting multiplicities) if f, g have

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the same a-points with the same multiplicities, and we say that f, g share the value a IM(ignoring multiplicities) if we do not consider the multiplicities.

Recently, people have raised great interest in difference analogues of Nevanlinna's theory and obtained many profound results. A number of papers have focused on value distribution and uniqueness of difference polyomials, which are analogues of Nevanlinna theory. For a meromorphic function $f(z)$ and a constant c, $f(z+c)$ is called the shift of f, where $f(z)$ is not periodic function with period c. We define the difference operator $\Delta_c f = f(z+c) - f(z)$ and $\Delta_c^k f = \Delta_c^{k-1}(\Delta_c f)$ for any positive integer k.
In 2011, K. Liu, X. L. Liu and T. B. Cao studied the uniqueness of the difference monomials and obtained the following results.

Theorem 1.1. [9] *Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f \equiv tg$ or $fg \equiv t$, where $t^{n+1} = 1$.*

Theorem 1.2. [9] *Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f \equiv tg$ or $fg \equiv t$, where $t^{n+1} = 1$.*

We now explain the notation of weighted sharing as introduced in [6].

Definition 1.1. [6] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times is $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. For each meromorphic function f on complex plane, by a difference product, we mean a difference monomial and its shifts, that is, an expression of type

$$\prod_{\nu=1}^{m} f(z + c_\nu)^{l_\nu},$$

where $c_1, ..., c_m$ are distinct complex numbers and $l_1, ..., l_m$ are natural numbers.

In 2015, Y.Liu, J. P. Wang and F. H. Liu improved Theorems 1.1, 1.2 and obtained the following results.

Theorem 1.3. [10] *Let $c \in \mathbb{C} \setminus \{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n\,(\geq 14)$, $k\,(\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 1.4. [10] *Let $c \in \mathbb{C}\setminus\{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n\,(\geq 16)$ be a positive integer. If $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 1.5. [10] *Let $c \in \mathbb{C}\setminus\{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n\,(\geq 22)$ be a positive integer. If $E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Recently, Sujoy Majumder has replaced the sharing value 1 in Theorems 1.3, 1.4 and 1.5 by a nonzero polynomial and obtained the following results

Theorem 1.6. [11] *Let f and g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p\,(\not\equiv 0)$ be a polynomial such that deg(p) $< (n-1)/2$. If $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share* $(0, 2)$, *then one of the following two cases holds:*
(1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,
(2)$fg \equiv t$, where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+1} = c^2$.

Theorem 1.7. [11] *Let f and g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p\,(\not\equiv 0)$ be a polynomial such that deg(p) $< (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share* $(0, 1)$. *Then conclusion of Theorem 1.6 holds.*

Theorem 1.8. [11] *Let f and g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p\,(\not\equiv 0)$ be a polynomial such that deg(p) $< (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share* $(0, 0)$. *Then conclusion of Theorem 1.6 holds.*

Now it is quite natural to ask the following question.
Question 1.1. What can be said about the uniqueness of finite order meromorphic functions f and g of the difference polynomials $f^n(z)\Delta_c f(z)$ and $g^n(z)\Delta_c g(z)$ when they share a non-zero polynomial?
We give a positive answer to the above question by using notion of weighted sharing values which generalize and improves Theorems 1.6, 1.7 and 1.8.

Theorem 1.9. *Let f and g be two transcendental meromorphic functions of finite order and n be a positive integer such that $n \geq 12$. Suppose that c is a non-zero complex constant such that $\Delta_c f(z) \not\equiv 0$ and $\Delta_c g(z) \not\equiv 0$. Let $f^n(z)\Delta_c f(z) - p(z)$ and $g^n(z)\Delta_c g(z) - p(z)$ share (0, 2), where $p(z)$ be a nonzero polynomial such that*

deg(p) $< (n-2)/2$ *and* $g(z)$, $g(z+c)$ *share 0 CM, then one of the following two cases holds:*
(1) $f \equiv tg$ *for some constant* t *such that* $t^{n+1} = 1$,
(2) $f(z) = c_1 e^{az}$ *and* $g(z) = c_2 e^{-az}$, *where* a, c_1 *and* c_2 *are non-zero constants such that* $(c_1c_2)^{n+1}(e^{ac} + e^{-ac} - 2) = -d^2$.

Theorem 1.10. *Let* f *and* g *be two transcendental meromorphic functions of finite order and* n *be a positive integer such that* $n \geq 27/2$. *Suppose that* c *is a non-zero complex constant such that* $\Delta_c f(z) \not\equiv 0$ *and* $\Delta_c g(z) \not\equiv 0$. *Let* $f^n(z)\Delta_c f(z) - p(z)$ *and* $g^n(z)\Delta_c g(z) - p(z)$ *share (0, 1), where* $p(z)$ *be a nonzero polynomial such that deg(p)* $< (n-2)/2$ *and* $g(z)$, $g(z+c)$ *share 0 CM. Then conclusion of Theorem 1.9 holds.*

Theorem 1.11. *Let* f *and* g *be two transcendental meromorphic functions of finite order and* n *be a positive integer such that* $n \geq 17$. *Suppose that* c *is a non-zero complex constant such that* $\Delta_c f(z) \not\equiv 0$ *and* $\Delta_c g(z) \not\equiv 0$. *Let* $f^n(z)\Delta_c f(z) - p(z)$ *and* $g^n(z)\Delta_c g(z) - p(z)$ *share (0, 0), where* $p(z)$ *be a nonzero polynomial such that deg(p)* $< (n-2)/2$ *and* $g(z)$, $g(z+c)$ *share 0 CM. Then conclusion of Theorem 1.9 holds.*

2. Lemmas

Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [12] *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0), a_{n-1}(z), ..., a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r, f)$ *for* $i = 0, 1, 2, ..., n$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. [14] *Let f and g be two non-constant meromorphic functions. Then*

$$N\left(r,\infty;\frac{f}{g}\right)-N\left(r,\infty;\frac{g}{f}\right)=N(r,\infty;f)+N(r,0;g)-N(r,\infty;g)-N(r,0;f).$$

Lemma 2.3. [3] *Let f be a meromorphic function of finite order σ, and let $c\in\mathbb{C}\backslash\{0\}$ be fixed. Then for each $\epsilon>0$, we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=O(r^{\sigma-1+\epsilon})=S(r,f).$$

The following lemma has little modifications of the original version (Theorem 2.1 of [3]*)*

Lemma 2.4. [3] *Let f be a transcendental meromorphic function of finite order, $c\in\mathbb{C}\setminus\{0\}$ be fixed. Then*

$$T(r,f(z+c))=T(r,f)+S(r,f).$$

Lemma 2.5. [5] *Let f be a non-constant meromorphic function of finite order and $c\in\mathbb{C}$. Then*

$$N(r,0;f(z+c))\le N(r,0;f(z))+S(r,f),\quad N(r,\infty;f(z+c))\le N(r,\infty;f)+S(r,f),$$

$$\overline{N}(r,0;f(z+c))\le\overline{N}(r,0;f(z))+S(r,f),\quad \overline{N}(r,\infty;f(z+c))\le\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.6. *Let f be a transcendental meromorphic function of finite order and let $F=f^n(z)\Delta_c f(z)$, where n is positive integer. Then*

$$(n-2)T(r,f)\le T(r,F)+S(r,f).$$

Proof. *From Lemma 2.1, Lemma 2.3 and first fundamental theorem, we obtain*

$$\begin{aligned}(n+1)T(r,f)&=T(r,f^{n+1})+S(r,f)\\&\le T\left(r,\frac{f(z)F}{\Delta_c f}\right)+S(r,f)\\&\le T(r,F)+T\left(r,\frac{f(z)}{\Delta_c f}\right)+S(r,f)\end{aligned}$$

$$\leq T(r,F) + m\left(r, \frac{\Delta_c f}{f(z)}\right) + N\left(r, \frac{\Delta_c f}{f(z)}\right) + S(r,f)$$

$$\leq T(r,F) + 3T(r,f) + S(r,f)$$

$$(n-2)T(r,f) \leq T(r,F) + S(r,f).$$

Hence, we get Lemma 2.5.

Lemma 2.7. *Let f, g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 3$. Let $p(z)$ be a nonzero polynomial such that $deg(p) < (n-2)/2$. Then*

(1) if $deg(p) \geq 1$, then $f^n(z)\Delta_c f(z) g^n(z)\Delta_c g(z) \not\equiv p^2(z)$;

(2) if $p(z)$ is a nonconstant d and $f^n(z)\Delta_c f(z) g^n(z)\Delta_c g(z) \equiv p^2(z)$, then

$$f(z) = c_1 e^{az}, \;\; g(z) = c_2 e^{-az},$$

where a, c_1 and c_2 are non-zero constants such that $(c_1c_2)^{n+1}(e^{ac} + e^{-ac} - 2) = d^2$.

Proof. Suppose

$$f^n(z)\Delta_c f(z) g^n(z)\Delta_c g(z) \equiv p^2(z). \tag{2.2}$$

Let $h_1 = fg$. Then by (2.2), we have

$$h_1^n(z) \equiv \frac{p^2(z)}{\Delta_c f(z)\Delta_c g(z)}. \tag{2.3}$$

We now consider following three cases.

Case 1. *Suppose h_1 is a transcendental meromorphic function. Now by Lemmas 2.1, 2.3 and 2.5, we get*

$$nT(r,h_1) = T(r,h_1^n) + S(r,h_1) = T\left(r, \frac{p^2}{\Delta_c f(z)\Delta_c g(z)}\right) + S(r,h_1)$$

$$\leq N(r,0;\Delta_c f(z)\Delta_c g(z)) + m(r,0;\Delta_c f(z)\Delta_c g(z))$$

$$+ S(r,h_1)$$

$$\leq 2[T(r,f)+T(r,g)]+S(r,h_1)$$

$$n[T(r,f)+T(r,g)] \leq 2[T(r,f)+T(r,g)]+S(r,h_1),$$

which is a contradiction.

Case 2. *Suppose h_1 is a rational function. Let*

$$h_1 = \frac{h_2}{h_3}, \tag{2.4}$$

where h_2 and h_3 are two nonzero relatively prime polynomials. By (2.4), we have

$$T(r,h_1) = \max\{\deg(h_2), \deg(h_3)\}\log r + O(1). \tag{2.5}$$

Now by (2.3)-(2.5), we have

$$n\max\{\deg(h_2),\deg(h_3)\}\log r = T(r,h_1^n)+O(1)$$
$$\leq 2[T(r,f)+T(r,g)]+2T(r,p)+O(1)$$

$$n\max\{\deg(h_2),\deg(h_3)\}\log r \leq 2\max\{\deg(h_2),\deg(h_3)\}\log r + 2\deg(p)\log r + O(1). \tag{2.6}$$

We see that $\max\{\deg(h_2),\deg(h_3)\} \geq 1$. *Now by (2.6), we deduce that* $(n-2)/2 \leq \deg(p)$, *which contradicts our assumption that* $\deg(p) < (n-2)/2$. *Hence h_1 must be a nonzero constant. Let*

$$h_1 = t \in \mathbb{C}\setminus\{0\}. \tag{2.7}$$

Now when deg(p) ≥ 1, by (2.3) and (2.7), we arrive at a contradiction. Therefore in this case we have $f^n(z)\Delta_c f(z) g^n(z)\Delta_c g(z) \not\equiv p^2(z)$.

Case 3. *Let $p(z)$ be a non-zero constant d. In this case we see that $f(z)$ and $g(z)$ have no zeros and so we can take $f(z)$ and $g(z)$ as follows:*

$$f(z) = e^{\alpha(z)},\ g(z) = e^{\beta(z)}, \tag{2.8}$$

where α and β are non-constant polynoimals. Now from (2.8) *we get*

$$(e^{\alpha(z+c)-\alpha(z)}-1)(e^{\beta(z+c)-\beta(z)}-1)\equiv d^2e^{-(n+1)[\alpha(z)+\beta(z)]}. \tag{2.9}$$

We conclude from (2.9) *that $e^{\alpha(z+c)-\alpha(z)}-1$ has no zeros. Let $\phi(z)=e^{\alpha(z+c)-\alpha(z)}$. Then $\phi(z)\neq 0,1,\infty$ for any $z\in\mathbb{C}$. By Picard's theorem, ϕ is a constant and so deg$(\alpha)=1$. Similarly we can prove that deg$(\beta)=1$. Assume now that*

$$f(z)=c_1e^{az},\ g(z)=c_2e^{bz},$$

where a, b, c_1 and c_2 are non-zero constants. Applying (2.2) *again we get $a=-b$ and*

$$(c_1c_2)^{n+1}(e^{ac}+e^{-ac}-2)=-d^2.$$

Finally $f(z)$ and $g(z)$ take the form

$$f(z)=c_1e^{az},\,g(z)=c_2e^{-az},$$

where a, c_1 and c_2 are non-zero constants such that $(c_1c_2)^{n+1}(e^{ac}+e^{-ac}-2)=-d^2$. This completes the proof.

Lemma 2.8. [7] *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}(z)$ which are not the zeros of $f(z)$, where a zero of $f^{(k)}(z)$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geq k)+S(r,f).$$

Lemma 2.9. *Let f and g be two transcendental meromorphic functions of finite order, $c\in\mathbb{C}\setminus\{0\}$ be finite complex constant such that $\Delta_cf(z)\not\equiv 0$ and $\Delta_cg(z)\not\equiv 0$ and let n be an integer such that $n>8$. Let $F(z)=\frac{f^n(z)\Delta_cf(z)}{p(z)}$ and $G(z)=\frac{g^n(z)\Delta_cg(z)}{p(z)}$, where $p(z)$ is non-zero polynomial. If $g(z)$, $g(z+c)$ share 0 CM and $H\equiv 0$, then one of the following conclusions occur*

(i) $f^n(z)\Delta_cf(z)g^n(z)\Delta_cg(z)\equiv p^2(z)$, where $f^n(z)\Delta_cf(z)-p(z)$ and $g^n(z)\Delta_cg(z)-p(z)$ share 0 CM;

(ii) $f(z) \equiv tg(z)$ *for a constant* t *with* $t^{n+1} = 1$.

Proof. Since $H \equiv 0$, *by integration we get*

$$\frac{1}{F-1} = \frac{BG + A - B}{G-1}, \tag{2.10}$$

where A, B *are constants and* $A \neq 0$. *From* (2.10) *it is clear that* F *and* G *share* $(1, \infty)$. *We now consider following cases.*

Case 1. *Let* $B \neq 0$ *and* $A \neq B$.

If $B = -1$, *then from* (2.10) *we have*

$$F = \frac{-A}{G - A - 1}.$$

Therefore

$$\overline{N}(r, A+1; G) = N(r, 0; p) = S(r, g).$$

So in view of Lemma 2.6 and the second fundamental theorem we get

$$\begin{aligned}(n-2)T(r,g) &\leq T(r, g^n \Delta_c g) + S(r,g)\\ &\leq T(r,G) + s(r,g)\\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, A+1; G) + S(r,g)\\ &\leq \overline{N}(r,\infty; g^n\Delta_c g) + \overline{N}(r,0;g^n\Delta_c g) + S(r,g)\\ &\leq 3T(r,g) + S(r,g),\end{aligned}$$

which is a contradiction since $n > 5$.

If $B \neq -1$, *from* (2.10) *we obtain that*

$$F - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2[G + \frac{A-B}{B}]}.$$

So

$$\overline{N}\left(r, \frac{(B-A)}{B}; G\right) = S(r,g).$$

Using Lemma 2.6 and the same argument as used in the case when $B = -1$ *we can get a contradiction.*

Case 2. *Let* $B \neq 0$ *and* $A = B$.

If $B = -1$, *then from* (2.10) *we have*

$$F(z)G(z) \equiv 1,$$

i.e.,

$$f^n(z)\Delta_c f(z) g^n(z)\Delta_c g(z) \equiv p^2(z),$$

where $f^n(z)\Delta_c f(z) - p(z)$ *and* $g^n(z)\Delta_c g(z) - p(z)$ *share 0 CM.*

If $B \neq -1$, *from* (2.10) *we have*

$$\frac{1}{F} = \frac{BG}{(1+B)G - 1}.$$

Therefore

$$\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r, 0; F) + S(r, f).$$

So in view of Lemmas 2.3, 2.6 and the second fundamental theorem we get

$$\begin{aligned}
(n-2)T(r,g) &\leq T(r, g^n\Delta_c g) + S(r,g) \\
&\leq T(r,G) + S(r,g) \\
&\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{1}{1+B}; G) + S(r,g) \\
&\leq \overline{N}(r,\infty; g^n\Delta_c g) + \overline{N}(r,0;g^n\Delta_c g) + \overline{N}(r,0;f^n\Delta_c f) \\
&+ S(r,f) + S(r,g) \\
&\leq 3T(r,g) + 3T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

So for $r \in I$ *we have*

$$(n-8)T(r,g) \leq S(r,g),$$

which is a contradiction since $n > 8$.

Case 3. *Let* $B = 0$. *From* (2.10) *we obtain*

$$F = \frac{G + A - 1}{A}. \tag{2.11}$$

If $A \neq 1$, *then from* (2.11) *we obtain*

$$\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $A = 1$ *and from* (2.11) *we obtain*

$$F(z) \equiv G(z),$$

i.e.,

$$f^n(z)\Delta_c f(z) \equiv g^n(z)\Delta_c g(z) \tag{2.12}$$

Let $h = \frac{f}{g}$, *and then substituting* $f = gh$ *in* (2.12) *we deduce*

$$h^{n+1} = \frac{f}{\Delta_c f}.\frac{\Delta_c g}{g}$$

If h *is not a constant, then we have*

$$\begin{aligned}(n+1)T(r, h) &\leq T\left(r, \frac{f}{\Delta_c f}\right) + T\left(r, \frac{\Delta_c g}{g}\right) + S(r, f) + S(r, g)\\ &\leq T\left(r, \frac{\Delta_c f}{f}\right) + T\left(r, \frac{\Delta_c g}{g}\right) + S(r, f) + S(r, g)\\ &\leq N\left(r, \frac{\Delta_c f}{f}\right) + N\left(r, \frac{\Delta_c g}{g}\right) + S(r, f) + S(r, g)\\ &\leq 3[T(r, f) + T(r, g)] + S(r, f) + S(r, g).\end{aligned}$$

Combining above inequality with $T(r, h) = T(r, \frac{f}{g}) = T(r, f) + T(r, g) + S(r, f) + S(r, g,)$, *we obtain* $(n-2)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g)$ *which is impossible. Therefore,* h *is a constant, then substitute* $f = gh$ *in to* (2.12), *we have* $h^{n+1} \equiv 1$. *Therefore* $f = tg$, *where* t *is a constant with* $t^{n+1} = 1$.

Lemma 2.10. [1] *If* f, g *be two non-constant meromorphic functions such that they share* $(1, 1)$. *Then*

$$2\overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) - \overline{N}_{f>2}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Lemma 2.11. [2] *Let f, g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f^{'}) + S(r,f),$$

where $N_0(r,0;f^{'})$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)$.

Lemma 2.12. [2] *Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f) \leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.13. [2] *Let f, g share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$$

Lemma 2.14. [2] *Let f, g share $(1,0)$. Then*
(i)$\overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f^{'}) + S(r,f)$
(ii)$\overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g^{'}) + S(r,f)$.

3. Proofs of the Theorems.

Proof of Theorem 1.9. Let $F(z) = \frac{f^n(z)\Delta_c f(z)}{p(z)}$ and $G(z) = \frac{g^n(z)\Delta_c g(z)}{p(z)}$. It follows that F and G share (1,2) except for the zeros of $p(z)$.

Case 1. Let $H \not\equiv 0$.

From (2.1) we obtain

$$N(r,\infty;H) \leq \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_0(r,0;F^{'})$$
$$+ \overline{N}_0(r,0;G^{'}). \tag{3.1}$$

Let z_0 be a simple zero of $F-1$ such that $p(z_0) \neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F| = 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,f) + S(r,g). \tag{3.2}$$

Using (3.1) and (3.2) we get

$$\overline{N}(r,1;F) \leq \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F|\geq 2)$$

$$+ \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g). \tag{3.3}$$

Now in view of Lemma 2.8 we get

$$\begin{aligned}\overline{N}_0(r,0;G^{'}) + \overline{N}(r,1;F|\geq 2) + \overline{N}_*(r,1;F,G) &\leq \overline{N}_0(r,0;G^{'}) + \overline{N}(r,1;F|\geq 2)\\ &+ \overline{N}(r,1;F|\geq 3)\\ &\leq N(r,0;G^{'}|G\neq 0)\\ &\leq \overline{N}(r,0;G) + S(r,g). \qquad (3.4)\end{aligned}$$

Note that since $g(z)$ and $g(z+c)$ share 0 CM, it follows that $N\left(r,\infty;\frac{\Delta_c g}{g}\right) = 0$.

Hence using (3.3), (3.4), Lemmas 2.3 and 2.6 we get from second fundamental theorem that

$$\begin{aligned}(n-2)T(r,f) &\leq T(r,F) + S(r,f)\\ &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) - N_0(r,0;F^{'}) + S(r,f)\\ &\leq \overline{N}(r,\infty;F) + N_2(r,0;F) + N_2(r,0;G) + S(r,f) + S(r,g)\\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\Delta_c f) + 2\overline{N}(r,0;f) + N(r,0;\Delta_c f)\\ &+ N_2\left(r,0;g^{n+1}\frac{\Delta_c g}{g}\right) + S(r,f) + S(r,g)\\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\Delta_c f) + 2\overline{N}(r,0;f) + N(r,0;\Delta_c f)\\ &+ N_2\left(r,0;g^{n+1}\right) + N_2\left(r,0;\frac{\Delta_c g}{g}\right) + S(r,f) + S(r,g)\\ &\leq 7T(r,f) + 2T(r,g) + T\left(r,\frac{\Delta_c g}{g}\right) + S(r,f) + S(r,g)\\ &\leq 7T(r,f) + 2T(r,g) + m\left(r,\frac{\Delta_c g}{g}\right) + S(r,f) + S(r,g)\end{aligned}$$

$$\leq 7T(r,f)+2T(r,g)+S(r,f)+S(r,g). \tag{3.5}$$

In a similar way we can obtain

$$(n-2)T(r,g)\leq 7T(r,g)+2T(r,f)+S(r,f)+S(r,g). \tag{3.6}$$

Combining (3.5) and (3.6) we see that

$$(n-2)[T(r,f)+T(r,g)]\leq 9[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \tag{3.7}$$

Since $n\geq 12$, (3.7) leads to a contradiction.

Case 2. Let $H\equiv 0$. Then the theorem follows from Lemmas 2.7 and 2.9. This completes the proof.

Proof of Theorem 1.10. Let $F(z)=\frac{f^n(z)\Delta_c f(z)}{p(z)}$ and $G(z)=\frac{g^n(z)\Delta_c g(z)}{p(z)}$. Then F and G share (1,1) except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $H\not\equiv 0$.

Using Lemmas 2.8, 2.10, 2.11, (3.1) and (3.2) we get

$$\begin{aligned}
\overline{N}(r,1;F) &\leq N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)\\
&+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\
&+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>2}(r,1;G)+N(r,1;G)\\
&-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq \overline{N}(r,0;F|\geq 2)+\frac{1}{2}\overline{N}(r,0;F)+\overline{N}(r,0;G|\geq 2)+N(r,1;G)$$
$$-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,0;F|\geq 2)+\frac{1}{2}\overline{N}(r,0;F)+\overline{N}(r,0;G|\geq 2)+N(r,0;G^{'}|G\neq 0)$$
$$+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,0;F|\geq 2)+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}_0(r,0;F^{'})$$
$$+S(r,f)+S(r,g). \tag{3.8}$$

Hence using (3.8), Lemmas 2.3 and 2.6 we get from second fundamental theorem that

$$(n-2)T(r,f)\leq T(r,F)+S(r,f)$$
$$\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)-N_0(r,0;F^{'})+S(r,f)$$
$$\leq \overline{N}(r,\infty;F)+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;F)+N_2(r,0;G)+S(r,f)$$
$$+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;\Delta_c f)+\frac{1}{2}\overline{N}(r,0;f^n\Delta_c f)+2\overline{N}(r,0;f)$$
$$+N(r,0;\Delta_c f)+N_2\left(r,0;g^{n+1}\frac{\Delta_c g}{g}\right)+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;\Delta_c f)+\frac{1}{2}\overline{N}(r,0;f^n\Delta_c f)+2\overline{N}(r,0;f)$$
$$+N(r,0;\Delta_c f)+N_2\left(r,0;g^{n+1}\right)+N_2\left(r,0;\frac{\Delta_c g}{g}\right)+S(r,f)+S(r,g)$$
$$\leq \frac{17}{2}T(r,f)+2T(r,g)+S(r,f)+S(r,g) \tag{3.9}$$

In a similar way we can obtain

$$(n-2)T(r,g)\leq \frac{17}{2}T(r,g)+2T(r,f)+S(r,f)+S(r,g). \tag{3.10}$$

Combining (3.9) and (3.10) we see that

$$(n-2)[T(r,f)+T(r,g)] \leq \frac{21}{2}[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \tag{3.11}$$

Since $n \geq \frac{27}{2}$, (3.11) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 2.7 and 2.9. This completes the proof.

Proof of Theorem 1.11. Let $F(z) = \frac{f^n(z)\Delta_c f(z)}{p(z)}$ and $G(z) = \frac{g^n(z)\Delta_c g(z)}{p(z)}$. Then F and G share (1, 0) except for the zeros of $p(z)$.

Here (3.2) changes to

$$N_E^{1)}(r,1;F) \leq N(r,0;H) \leq N(r,\infty;H)+S(r,F)+S(r,G). \tag{3.12}$$

Using Lemmas 2.8, 2.12, 2.13, 2.14 , (3.2) and (3.12) we get

$$\begin{aligned}
\overline{N}(r,1;F) &\leq N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)\\
&+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\
&+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\
&+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)\\
&+\overline{N}_L(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\
&+S(r,f)+S(r,g)\\
&\leq N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)+N(r,1;G)-\overline{N}(r,1;G)\\
&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)+N(r,0;G^{'}|G\neq 0)$$
$$+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)$$
$$\leq N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})$$
$$+S(r,f)+S(r,g). \tag{3.13}$$

Hence using (3.13), Lemmas 2.3 and 2.6 we get from second fundamental theorem that

$$(n-2)T(r,f)\leq T(r,F)+S(r,f)$$
$$\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)-N_0(r,0;F^{'})+S(r,f)$$
$$\leq \overline{N}(r,\infty;F)+2N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;G)+S(r,f)$$
$$+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;\Delta_c f)+4\overline{N}(r,0;f)+2N(r,0;\Delta_c f)$$
$$+N_2\left(r,0;g^{n+1}\frac{\Delta_c g}{g}\right)+\overline{N}\left(r,0;g^{n+1}\frac{\Delta_c g}{g}\right)+S(r,f)+S(r,g)$$
$$\leq 11T(r,f)+3T(r,g)+T\left(r,\frac{\Delta_c g}{g}\right)+S(r,f)+S(r,g)$$
$$\leq 11T(r,f)+3T(r,g)+S(r,f)+S(r,g) \tag{3.14}$$

In a similar way we can obtain

$$(n-2)T(r,g)\leq 11T(r,g)+3T(r,f)+S(r,f)+S(r,g). \tag{3.15}$$

Combining (3.14) and (3.15) we see that

$$(n-2)[T(r,f)+T(r,g)]\leq 14[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \tag{3.16}$$

Since $n \geq 17$, (3.16) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 2.7 and 2.9. This completes the proof.

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