

TRIANGULAR FUNCTIONS WITH CONVERGENCE FOR SOLVING LINEAR SYSTEM OF TWO-DIMENSIONAL FUZZY FREDHOLM INTEGRAL EQUATION

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ABSTRACT. In this paper, we present a review on triangular functions (TFs) to solve linear two-dimensional fuzzy Fredholm integral equations system of the second kind (2D-FFIES-2). The properties of triangular functions are utilized to reduce the 2D-FFIES-2 to a linear system of algebraic equations. Moreover, we state the convergence analysis of the method. Finally, some examples show the simplicity and the validity of the present numerical method.

1. INTRODUCTION

It is well known that the fuzzy differential equations and the fuzzy integral equations are one of the important parts of numerical analysis and applied mathematics. Usually in many mathematical models, some of problems are represented by fuzzy Volterra and Fredholm integral equations. For example, Nanda, in his book, [1] introduced the integration of fuzzy mappings. Kaleva [2], Wu and Ma [3] introduced the differential equations and integral equations of fuzzy set-valued functions. There are several numerical methods to solve Fredholm integral equations, fuzzy Fredholm integral equations and fuzzy integro-differential equations. Otadi and Mosleh

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[4] considered fuzzy nonlinear integral equations of the second kind and obtained an approximate solution to the fuzzy nonlinear integral equations. The existence of solution of nonlinear fuzzy Fredholm integro-differential equations is discussed in [5]. Rivaz and Yousefi [6] and Ezzati and Ziari [7] used homotopy perturbation method and fuzzy Bivariate Bernstein polynomials method for solving two-dimensional fuzzy Fredholm integral equations of the second kind, respectively. Recently, Babolian et al. [8, 9, 10], Maleknejad et al. [11], Mirzaee et al. [12] and Hengamian Asl et al. [13, 14] have used triangular functions for solving of Fredholm integral equations and fuzzy Fredholm integral equations. Since the triangular functions method is a successful numerical method for solving Fredholm integral equations, we will develop this method for following general form of linear system of two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIES-2):

$$(1.1) \quad \begin{cases} u_1(x, y) = g_1(x, y) \oplus \sum_{j=1}^n \lambda_{1j} \otimes \int_0^1 \int_0^1 k_{1j}(x, y, s, t) \otimes u_j(s, t) ds dt, \\ u_2(x, y) = g_2(x, y) \oplus \sum_{j=1}^n \lambda_{2j} \otimes \int_0^1 \int_0^1 k_{2j}(x, y, s, t) \otimes u_j(s, t) ds dt, \\ \vdots \\ u_n(x, y) = g_n(x, y) \oplus \sum_{j=1}^n \lambda_{nj} \otimes \int_0^1 \int_0^1 k_{nj}(x, y, s, t) \otimes u_j(s, t) ds dt, \end{cases}$$

where $k_{ij}(x, y, s, t)$, $i, j = 1, \dots, n$, are an arbitrary kernel function over $(\Omega \times \Omega)$ and $\lambda_{ij} \neq 0$, $i, j = 1, \dots, n$ are real constants and $u_i(x, y)$ and $g_i(x, y)$ are fuzzy real valued functions for $i = 1, \dots, n$ and $u_1(x, y), u_2(x, y), \dots, u_n(x, y)$ are the solutions to be determined.

This paper is organized as follows. Review of triangular functions and their properties which will be used later, is briefly provided in Section 2. Also in this section, we give an overview of elementary concepts of the fuzzy calculus. Section 3 presents a numerical method for solving system of two-dimensional fuzzy Fredholm integral equations of the second kind. Convergence analysis for this method is established in

Section 4. Finally, we illustrate in Section 5 some numerical examples to show the efficiency and accuracy of the proposed method.

2. PRELIMINARIES

2.1. One-dimensional triangular functions.

Definition 2.1. ([9]) Two m -sets of one-dimensional triangular functions (1D-TFs) are defined over the interval $[0, T]$ as:

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \leq t < (i+1)h, \\ 0, & o.w, \end{cases}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \leq t < (i+1)h, \\ 0, & o.w, \end{cases}$$

where $i = 0, 1, \dots, m-1$, $h = \frac{T}{m}$, with a positive integer value for m .

Moreover, if

$$(2.1) \quad T1(t) = [T1_0(t), T1_1(t), \dots, T1_{m-1}(t)]^T,$$

$$(2.2) \quad T2(t) = [T2_0(t), T2_1(t), \dots, T2_{m-1}(t)]^T,$$

then $T(t)$, the TF vector, can be defined as:

$$(2.3) \quad T(t) = [T1(t) \quad T2(t)]^T.$$

2.2. Two-dimensional triangular functions.

Definition 2.2. ([11]) An $(m_1 \times m_2)$ -set of two-dimensional triangular functions (2D-TFs) are defined on $\Omega = [0, 1] \times [0, 1]$ as:

$$\begin{aligned}
T_{i,j}^{1,1}(s,t) &= \begin{cases} (1 - \frac{s-ih_1}{h_1})(1 - \frac{t-jh_2}{h_2}), & ih_1 \leq s < (i+1)h_1, \\ & jh_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \\
T_{i,j}^{1,2}(s,t) &= \begin{cases} (1 - \frac{s-ih_1}{h_1})(\frac{t-jh_2}{h_2}), & ih_1 \leq s < (i+1)h_1, \\ & jh_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \\
T_{i,j}^{2,1}(s,t) &= \begin{cases} (\frac{s-ih_1}{h_1})(1 - \frac{t-jh_2}{h_2}), & ih_1 \leq s < (i+1)h_1, \\ & jh_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \\
T_{i,j}^{2,2}(s,t) &= \begin{cases} (\frac{s-ih_1}{h_1})(\frac{t-jh_2}{h_2}), & ih_1 \leq s < (i+1)h_1, \\ & jh_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where $i = 0, 1, \dots, m_1 - 1, j = 0, 1, \dots, m_2 - 1, h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}$. m_1 and m_2 are arbitrary positive integers.

Moreover, if

$$\begin{aligned}
T11(s,t) &= [T_{0,0}^{1,1}(s,t), \dots, T_{0,m_2-1}^{1,1}, T_{1,0}^{1,1}(s,t), \dots, T_{m_1-1,m_2-1}^{1,1}(s,t)]^T, \\
T12(s,t) &= [T_{0,0}^{1,2}(s,t), \dots, T_{0,m_2-1}^{1,2}, T_{1,0}^{1,2}(s,t), \dots, T_{m_1-1,m_2-1}^{1,2}(s,t)]^T, \\
T21(s,t) &= [T_{0,0}^{2,1}(s,t), \dots, T_{0,m_2-1}^{2,1}, T_{1,0}^{2,1}(s,t), \dots, T_{m_1-1,m_2-1}^{2,1}(s,t)]^T, \\
T22(s,t) &= [T_{0,0}^{2,2}(s,t), \dots, T_{0,m_2-1}^{2,2}, T_{1,0}^{2,2}(s,t), \dots, T_{m_1-1,m_2-1}^{2,2}(s,t)]^T,
\end{aligned}$$

then $T(s, t)$, the 2D-TF vector, can be defined as:

$$(2.4) \quad T(s, t) = \begin{bmatrix} T11(s, t) \\ T12(s, t) \\ T21(s, t) \\ T22(s, t) \end{bmatrix}_{4m_1m_2 \times 1}.$$

Also, we have

$$(2.5) \quad \int_0^1 \int_0^1 T^T(s, t) T(s, t) ds dt = D,$$

where D is $(4m_1m_2 \times 4m_1m_2)$ -matrix as follows:

$$(2.6) \quad D = \begin{bmatrix} \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 \\ \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 \\ \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 \\ \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 \end{bmatrix},$$

where $I_1 = I_{m_1 \times m_1}$ and $I_2 = I_{m_2 \times m_2}$ (see [12]).

2.3. Function expansion with 1D-TFs and 2D-TFs. Let $f(t)$ be an $L^2[0, 1)$ function, the expansion of $f(t)$ with respect to 1D-TFs, can be defined as follows:

$$(2.7) \quad f(t) \simeq \sum_{i=0}^{m-1} [f_i T1_i(t) + f_{i+1} T2_i(t)] = F1^T T1(t) + F2^T T2(t) = \mathcal{F}^T \cdot T(t),$$

where the sequence of constant coefficients $\{f_i\}_{i=0}^m$ are the samples of $f(t)$ function such that $f_i = f(ih)$ for $i = 0, 1, \dots, m$.

Let $f(s, t)$ be a function of two variables on $\Omega = [0, 1] \times [0, 1]$. It can be approximated with respect to 2D-TFs as follows:

$$f(s, t) \simeq \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} c_{i,j} T_{i,j}^{1,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} d_{i,j} T_{i,j}^{1,2}(s, t)$$

$$\begin{aligned}
& + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} e_{i,j} T_{i,j}^{2,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} l_{i,j} T_{i,j}^{2,2}(s, t) \\
& = F1^T.T11(s, t) + F2^T.T12(s, t) + F3^T.T21(s, t) + F4^T.T22(s, t) \\
& = [F1^T \quad F2^T \quad F3^T \quad F4^T]. \begin{bmatrix} T11(s, t) \\ T12(s, t) \\ T21(s, t) \\ T22(s, t) \end{bmatrix} = F^T.T(s, t),
\end{aligned}$$

or

$$(2.8) \quad f(s, t) \simeq T^T(s, t).F,$$

where $F1, F2, F3$ and $F4$ can be computed by sampling the function $f(s, t)$ at grid points s_i and t_j such that $s_i = ih_1$ and $t_j = jh_2$, for various values of i and j . So we have

$$(F1)_k = c_{i,j} = f(s_i, t_j),$$

$$(F2)_k = d_{i,j} = f(s_i, t_{j+1}),$$

$$(F3)_k = e_{i,j} = f(s_{i+1}, t_j),$$

$$(F4)_k = l_{i,j} = f(s_{i+1}, t_{j+1}),$$

where $k = im_2 + j$ and $i = 0, 1, \dots, m_1 - 1, j = 0, 1, \dots, m_2 - 1$.

Let $u(s, t, r)$ be a function of three variables on $\Omega \times [0, 1]$. It can be approximated with respect to 2D-TFs and 1D-TFs as follows:

$$(2.9) \quad u(s, t, r) = T^T(s, t).U.T(r),$$

where $T(s, t)$ and $T(r)$ are 2D-TF vector and 1D-TF vector of dimension $4m_1m_2$ and $2m_3$, respectively and U is a $(4m_1m_2 \times 2m_3)$ 2D-TF coefficient matrix. This matrix

can be represented as

$$(2.10) \quad U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \\ U_{41} & U_{42} \end{bmatrix},$$

where each block of U is an $(m_1 m_2 \times m_3)$ -matrix that can be computed by sampling the function $u(s, t, r)$ at grid points (s_i, t_j, r_k) such that

$$\begin{aligned} s_i &= ih_1, & i &= 0, 1, \dots, m_1 - 1, & h_1 &= \frac{1}{m_1}, \\ t_j &= jh_2, & j &= 0, 1, \dots, m_2 - 1, & h_2 &= \frac{1}{m_2}, \\ r_k &= kh_3, & k &= 0, 1, \dots, m_3 - 1, & h_3 &= \frac{1}{m_3}. \end{aligned}$$

Choosing $l = im_2 + j$, we have

$$\begin{aligned} (U_{11})_{l,k} &= u(s_i, t_j, r_k), \\ (U_{12})_{l,k} &= u(s_i, t_j, r_{k+1}), \\ (U_{21})_{l,k} &= u(s_i, t_{j+1}, r_k), \\ (U_{22})_{l,k} &= u(s_i, t_{j+1}, r_{k+1}), \end{aligned}$$

$$\begin{aligned} (U_{31})_{l,k} &= u(s_{i+1}, t_j, r_k), \\ (U_{32})_{l,k} &= u(s_{i+1}, t_j, r_{k+1}), \\ (U_{41})_{l,k} &= u(s_{i+1}, t_{j+1}, r_k), \\ (U_{42})_{l,k} &= u(s_{i+1}, t_{j+1}, r_{k+1}). \end{aligned}$$

Let $k(s, t, x, y)$ be a function of four variables on $(\Omega \times \Omega)$. It can be approximated with respect to 2D-TFs as follows:

$$(2.11) \quad k(s, t, x, y) \simeq T^T(s, t).K.T(x, y),$$

where $T(s, t)$ and $T(x, y)$ are 2D-TF vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively and K is a $(4m_1m_2 \times 4m_3m_4)$ 2D-TF coefficient matrix. This matrix can be represented as

$$(2.12) \quad K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix},$$

where each block of K is an $(m_1m_2 \times m_3m_4)$ -matrix that can be computed by sampling the function $k(s, t, x, y)$ at grid points $(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2})$ such that

$$\begin{aligned} s_{i_1} &= i_1 h_1, & i_1 &= 0, 1, \dots, m_1 - 1, & h_1 &= \frac{1}{m_1}, \\ t_{j_1} &= j_1 h_2, & j_1 &= 0, 1, \dots, m_2 - 1, & h_2 &= \frac{1}{m_2}, \\ x_{i_2} &= i_2 h_3, & i_2 &= 0, 1, \dots, m_3 - 1, & h_3 &= \frac{1}{m_3}, \\ y_{j_2} &= j_2 h_4, & j_2 &= 0, 1, \dots, m_4 - 1, & h_4 &= \frac{1}{m_4}. \end{aligned}$$

Choosing $p = i_1 m_2 + j_1$ and $q = i_2 m_4 + j_2$, we get

$$(K11)_{p,q} = k(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2}),$$

$$(K12)_{p,q} = k(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2+1}),$$

$$(K13)_{p,q} = k(s_{i_1}, t_{j_1}, x_{i_2+1}, y_{j_2}),$$

$$(K14)_{p,q} = k(s_{i_1}, t_{j_1}, x_{i_2+1}, y_{j_2+1}),$$

$$(K21)_{p,q} = k(s_{i_1}, t_{j_1+1}, x_{i_2}, y_{j_2}),$$

$$(K22)_{p,q} = k(s_{i_1}, t_{j_1+1}, x_{i_2}, y_{j_2+1}),$$

$$(K23)_{p,q} = k(s_{i_1}, t_{j_1+1}, x_{i_2+1}, y_{j_2}),$$

$$(K24)_{p,q} = k(s_{i_1}, t_{j_1+1}, x_{i_2+1}, y_{j_2+1}),$$

$$(K31)_{p,q} = k(s_{i_1+1}, t_{j_1}, x_{i_2}, y_{j_2}),$$

$$(K32)_{p,q} = k(s_{i_1+1}, t_{j_1}, x_{i_2}, y_{j_2+1}),$$

$$(K33)_{p,q} = k(s_{i_1+1}, t_{j_1}, x_{i_2+1}, y_{j_2}),$$

$$(K34)_{p,q} = k(s_{i_1+1}, t_{j_1}, x_{i_2+1}, y_{j_2+1}),$$

$$(K41)_{p,q} = k(s_{i_1+1}, t_{j_1+1}, x_{i_2}, y_{j_2}),$$

$$(K42)_{p,q} = k(s_{i_1+1}, t_{j_1+1}, x_{i_2}, y_{j_2+1}),$$

$$(K43)_{p,q} = k(s_{i_1+1}, t_{j_1+1}, x_{i_2+1}, y_{j_2}),$$

$$(K44)_{p,q} = k(s_{i_1+1}, t_{j_1+1}, x_{i_2+1}, y_{j_2+1}).$$

In this paper for convergence, we supposed that $m_1 = m_2 = m_3 = m_4 = m$. More details about the properties of functions expansion with TFs are given in [12, 13, 14].

2.4. The basic concepts of fuzzy equations.

In this Section the most basic used notations in fuzzy calculus and integral equations are briefly introduced. We started by defining the fuzzy number.

Definition 2.3. ([15]) A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \rightarrow [0, 1]$ such that:

- (a): u is upper semi-continuous,
- (b): $u(x) = 0$ outside some interval $[a, d]$,
- (c): There are real numbers b, c such as $a \leq b \leq c \leq d$ and
 - (i) $u(x)$ is monotonic increasing on $[a, b]$,
 - (ii) $u(x)$ is monotonic decreasing on $[c, d]$,
 - (iii) $u(x) = 1, b \leq x \leq c$.

An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [2] as follows:

Definition 2.4. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, such that

- (a): $\underline{u}(r)$ is abounded monotonic increasing left continuous function,
- (b): $\overline{u}(r)$ is abounded monotonic decreasing left continuous function,
- (c): $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and real number k , we define

- (a): $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$,
- (b): addition, $u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,
- (c): scalar multiplication and, $k \otimes u = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0. \end{cases}$

Definition 2.5. For arbitrary numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$,

$$(2.13) \quad D(u, v) = \max \left\{ \sup_{0 \leq r \leq 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leq r \leq 1} |\overline{u}(r) - \overline{v}(r)| \right\}$$

is the distance between u and v . It is proved that (E^1, D) is a complete metric space with the properties [1], and

- (a): $\forall u, v, w \in E^1; D(u \oplus w, v \oplus w) = D(u, v)$,
- (b): $\forall u, v \in E^1, \forall k \in \mathbb{R}; D(k \otimes u, k \otimes v) = |k|D(u, v)$,
- (c): $\forall u, v, w, e \in E^1; D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$.

More details about the properties of the fuzzy integral are given in [1, 2, 3, 15].

3. SOLVING LINEAR 2D-FFIES-2

In this section, we present a 2D-TFs method to solve a linear 2D-FFIES-2. First consider $(\underline{g}_i(x, y, r), \overline{g}_i(x, y, r))$ and $(\underline{u}_i(x, y, r), \overline{u}_i(x, y, r))$, $0 \leq r \leq 1$ be parametric forms of $g_i(x, y)$ and $u_i(x, y)$ in system (1.1), respectively. In this paper, we assumed that $\lambda_{ij} = 1$ and $k_{ij}(x, y, s, t) \geq 0$. Therefore, by using definition (4), we can write

$$k_{ij}(x, y, s, t) \otimes u_j(x, y) = (k_{ij}(x, y, s, t)\underline{u}_j(x, y, r), k_{ij}(x, y, s, t)\overline{u}_j(x, y, r)),$$

Now, for solving (1.1) we write the parametric form of the given fuzzy integral equations system as follows:

$$(3.1) \quad \left\{ \begin{array}{l} \underline{u}_1(x, y, r) = \underline{g}_1(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{1j}(x, y, s, t) \underline{u}_j(s, t, r) ds dt, \\ \overline{u}_1(x, y, r) = \overline{g}_1(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{1j}(x, y, s, t) \overline{u}_j(s, t, r) ds dt, \\ \\ \underline{u}_2(x, y, r) = \underline{g}_2(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{2j}(x, y, s, t) \underline{u}_j(s, t, r) ds dt, \\ \overline{u}_2(x, y, r) = \overline{g}_2(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{2j}(x, y, s, t) \overline{u}_j(s, t, r) ds dt, \\ \vdots \\ \underline{u}_n(x, y, r) = \underline{g}_n(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{nj}(x, y, s, t) \underline{u}_j(s, t, r) ds dt, \\ \overline{u}_n(x, y, r) = \overline{g}_n(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{nj}(x, y, s, t) \overline{u}_j(s, t, r) ds dt. \end{array} \right.$$

For convenience, we consider the i th equation of system (3.1) as

$$(3.2) \quad \underline{u}_i(x, y, r) = \underline{g}_i(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{ij}(x, y, s, t) \underline{u}_j(s, t, r) ds dt,$$

$$(3.3) \quad \overline{u}_i(x, y, r) = \overline{g}_i(x, y, r) + \sum_{j=1}^n \int_0^1 \int_0^1 k_{ij}(x, y, s, t) \overline{u}_j(s, t, r) ds dt.$$

For solving (3.2) by using TFs, first let us expand $\underline{u}_i(x, y, r)$, $\underline{g}_i(x, y, r)$ and $k_{ij}(x, y, s, t)$ by using Eqs. (2.9) and (2.11) as follows:

$$(3.4) \quad \begin{aligned} \underline{u}_i(x, y, r) &\simeq T^T(x, y) \cdot \underline{U}_i \cdot T(r), \\ \underline{g}_i(x, y, r) &\simeq T^T(x, y) \cdot \underline{G}_i \cdot T(r), \\ k_{ij}(x, y, s, t) &\simeq T^T(x, y) \cdot K_{ij} \cdot T(s, t), \end{aligned}$$

where \underline{U}_i and \underline{G}_i for $i = 1, \dots, n$ are similar to Eq. (2.10) as follows:

$$\underline{U}_i = \begin{bmatrix} U11_i & U12_i \\ U21_i & U22_i \\ U31_i & U32_i \\ U41_i & U42_i \end{bmatrix}, \quad \underline{G}_i = \begin{bmatrix} G11_i & G12_i \\ G21_i & G22_i \\ G31_i & G32_i \\ G41_i & G42_i \end{bmatrix},$$

and K_{ij} for $i, j = 1, \dots, n$ are similar to Eq. (2.12) as follows:

$$K_{ij} = \begin{bmatrix} K11_{ij} & K12_{ij} & K13_{ij} & K14_{ij} \\ K21_{ij} & K22_{ij} & K23_{ij} & K24_{ij} \\ K31_{ij} & K32_{ij} & K33_{ij} & K34_{ij} \\ K41_{ij} & K42_{ij} & K43_{ij} & K44_{ij} \end{bmatrix}.$$

Substituting the Eqs. (3.4) in Eq. (3.2), we have

$$\begin{aligned} T^T(x, y) \underline{U}_i T(r) &\simeq T^T(x, y) \underline{G}_i T(r) \\ &+ \sum_{j=1}^n \int_0^1 \int_0^1 (T^T(x, y) K_{ij} T(s, t) T^T(s, t) \underline{U}_j T(r)) ds dt \end{aligned}$$

$$(3.5) \quad \begin{aligned} &= T^T(x, y) \underline{G}_i T(r) \\ &\quad + T^T(x, y) \sum_{j=1}^n K_{ij} \left(\int_0^1 \int_0^1 T(s, t) T^T(s, t) ds dt \right) \underline{U}_j T(r). \end{aligned}$$

Substituting the Eq. (2.5) in Eq. (3.5), we can write

$$T^T(x, y) \underline{U}_i \simeq T^T(x, y) \underline{G}_i + T^T(x, y) \sum_{j=1}^n \lambda_{ij} K_{ij} D \underline{U}_j.$$

Thus we have

$$\underline{U}_i = \underline{G}_i + \sum_{j=1}^n \lambda_{ij} K_{ij} D \underline{U}_j,$$

therefore we get the following system

$$(3.6) \quad \sum_{j=1}^n (\Delta_{ij} - \lambda_{ij} K_{ij} D) \underline{U}_j = \underline{G}_i,$$

where

$$\Delta_{ij} = \begin{cases} I, & i = j, \\ 0, & i \neq j, \end{cases}$$

for $i, j = 1, 2, \dots, n$ and I is a $4m^2 \times 4m^2$ identity matrix. By solving matrix system (3.6), we can find \underline{U}_i for $i = 1, 2, \dots, n$. So $\underline{u}_i(x, y, r) \simeq T^T(x, y) \underline{U}_i T(r)$. The same trend hold for $\overline{u}_i(x, y, r)$ in Eq. (3.3) as follows:

$$\overline{u}_i(x, y, r) \simeq T^T(x, y) \overline{U}_i T(r).$$

For solving system (3.1), we need to solve two systems of (3.6).

4. CONVERGENCE ANALYSIS

In this Section, we prove that the present numerical method converges to the exact solution.

Theorem 4.1. *If $k_{ij}(x, y, s, t)$, $i, j = 1, 2, \dots, n$ and $0 \leq x, y, s, t \leq 1$ are bounded and continuous, then approximate solution of system (1.1), converges to the exact solution.*

Proof. Suppose that $u_{i,m}(x, y), i = 1, \dots, n$ is approximate solution of exact solution $u_i(x, y)$. Therefore

$$(4.1) \quad \begin{aligned} u_{i,m}(x, y) = & \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} c_{p,q}^i T_{p,q}^{1,1}(s, t) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} d_{p,q}^i T_{p,q}^{1,2}(s, t) \\ & + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} e_{p,q}^i T_{p,q}^{2,1}(s, t) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} l_{p,q}^i T_{p,q}^{2,2}(s, t). \end{aligned}$$

By using Eqs. (1.1) and (4.1), we can write

$$\begin{aligned} D(u_{i,m}(x, y) - u_i(x, y)) = & D\left(\sum_{j=1}^n \lambda_{ij} \int_0^1 \int_0^1 k_{ij}(x, y, s, t) \left(\sum_{p=0}^{m-1} \sum_{q=0}^{m-1} c_{p,q}^j T_{p,q}^{1,1}(s, t) \right. \right. \\ & + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} d_{p,q}^j T_{p,q}^{1,2}(s, t) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} e_{p,q}^j T_{p,q}^{2,1}(s, t) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} l_{p,q}^j T_{p,q}^{2,2}(s, t) \Big) ds dt \\ & \left. - \sum_{j=1}^n \lambda_{ij} \int_0^1 \int_0^1 k_{ij}(x, y, s, t) u_j(s, t) ds dt\right) \\ \leq & M \sum_{j=1}^n \int_0^1 \int_0^1 D(u_{j,m}(s, t) - u_j(s, t)) ds dt, \end{aligned}$$

where

$$M = \max_{0 \leq x, y, s, t \leq 1} |\lambda_{ij} k_{ij}(x, y, s, t)| < \infty.$$

Also, we have $\lim_{m \rightarrow \infty} u_{j,m}(x, y) = u_j(x, y)$, so $D(u_{j,m}(x, y) - u_j(x, y)) \rightarrow 0$ as $m \rightarrow \infty$ for $j = 1, \dots, n$, and since M is bounded, thus

$$\lim_{m \rightarrow \infty} D(u_{i,m}(x, y) - u_i(x, y)) \rightarrow 0.$$

So the proof is completed.

5. NUMERICAL ILLUSTRATION

In this section, we present two examples of linear 2D-FFIES-2 and results will be compared with the exact solutions. All results are computed by using a program written in the Matlab. in this regard, the result presented in the following Tables and Figures.

Example 5.1. Consider the system of linear two-dimensional fuzzy Fredholm integral equations with

$$\begin{aligned}\underline{g}_1(x, y, r) &= xy\left(\frac{8}{9}r - \frac{1}{12}(r^5 + 2r)\right), \\ \overline{g}_1(x, y, r) &= xy\left(\frac{8}{9}(2 - r) - \frac{1}{12}(6 - 3r^3)\right), \\ \underline{g}_2(x, y, r) &= xy\left(-\frac{1}{12}r - \frac{15}{16}(r^5 + 2r)\right), \\ \overline{g}_2(x, y, r) &= xy\left(-\frac{1}{12}(2 - r) - \frac{15}{16}(6 - 3r^3)\right),\end{aligned}$$

and kernel functions:

$$k_{ij}(x, y, s, t) = xys^i t^j, \quad i, j = 1, 2.$$

One can easily verify that,

$$\begin{aligned}(\underline{u}_1(x, y, r), \overline{u}_1(x, y, r)) &= xy(r, 2 - r), \\ (\underline{u}_2(x, y, r), \overline{u}_2(x, y, r)) &= xy(r^5 + 2r, 6 - 3r^3),\end{aligned}$$

is an exact solution of the given problem.

The results will be compared with the exact solutions. The accuracy of present method is estimated by the absolute errors $\underline{E}_i^m(x, y, r)$ and $\overline{E}_i^m(x, y, r)$, which are given as follows:

$$\underline{E}_i^m(x, y, r) = |\underline{u}_i(x, y, r) - T^T(x, y)\underline{U}_i T(r)|,$$

$$\overline{E}_i^m(x, y, r) = |\overline{u}_i(x, y, r) - T^T(x, y)\overline{U}_i T(r)|.$$

We have applied the numerical method given in this paper to this equation. The absolute errors $\underline{E}_i^m(x, y, r)$ and $\overline{E}_i^m(x, y, r)$, are listed in Tables 1 and 2. We see that the proposed method is accurate for this example. As it can be observed in Table 1, the absolute error is greater at grid points far from the $(x, y) = (0, 0)$. The dependence of the absolute error on m is displayed in Table 2. We see that the absolute error decreases by increasing m . Also Fig. 1 shows a comparison between the exact solution and the approximate solution by the presented method. Moreover, Absolute error functions obtained by the present method for $\underline{u}_1(x, y, r)$, $\overline{u}_1(x, y, r)$, $\underline{u}_2(x, y, r)$ and $\overline{u}_2(x, y, r)$ are shown in Figs. 2 and 3. We can see that the absolute error converges to zero as $m \rightarrow \infty$ (see the absolute error functions obtained for $\underline{u}_1(x, y, r)$ for $m = 5, 10, 15, 20$. in the Fig. 2).

TABLE 1. Absolute errors $\underline{E}_i^m(x, y, r)$ and $\overline{E}_i^m(x, y, r)$, for Example 1, with $r = 0.5, m = 20$.

(x, y)	$\underline{E}_1^{20}(x, y, r)$	$\overline{E}_1^{20}(x, y, r)$	$\underline{E}_2^{20}(x, y, r)$	$\overline{E}_2^{20}(x, y, r)$
(0.0, 0.0)	0.0000e-00	0.0000e-00	0.0000e-00	0.0000e-00
(0.1, 0.1)	9.5597e-07	5.1283e-06	1.6019e-06	7.8208e-06
(0.2, 0.2)	3.8239e-06	2.0513e-05	6.4076e-06	3.1283e-05
(0.3, 0.3)	8.6037e-06	4.6155e-05	1.4417e-05	7.0387e-05
(0.4, 0.4)	1.5295e-05	8.2054e-05	2.5630e-05	1.2513e-04
(0.5, 0.5)	2.3899e-05	1.2821e-04	4.0048e-05	1.9552e-04
(0.6, 0.6)	3.4415e-05	1.8462e-04	5.7669e-05	2.8155e-04
(0.7, 0.7)	4.6842e-05	2.5129e-04	7.8493e-05	3.8322e-04
(0.8, 0.8)	6.1182e-05	3.2821e-04	1.0252e-04	5.0053e-04
(0.9, 0.9)	7.7433e-05	4.1540e-04	1.2975e-04	6.3349e-04

TABLE 2. Absolute errors $\underline{E}_i^m(x, y, r)$ and $\overline{E}_i^m(x, y, r)$, for Example 1, with $m = 5, 10, 15, 20$.

m	$\underline{E}_1^m(0.1, 0.1, 0.5)$	$\overline{E}_1^m(0.1, 0.1, 0.5)$	$\underline{E}_2^m(0.1, 0.1, 0.5)$	$\overline{E}_2^m(0.1, 0.1, 0.5)$
5	1.5525e-05	8.1646e-05	1.5355e-04	3.2478e-04
10	3.8262e-06	2.0525e-05	6.4166e-06	3.1331e-05
15	1.7020e-06	9.1105e-06	1.6772e-05	3.6101e-05
20	9.5597e-07	5.1283e-06	1.6019e-06	7.8208e-06

Example 5.2. Consider the system of linear two-dimensional fuzzy Fredholm integral equations

$$\left\{ \begin{array}{l} \underline{u}_1(x, y, r) = \underline{g}_1(x, y, r) + \int_0^1 \int_0^1 xs\underline{u}_j(s, t, r)dsdt + \int_0^1 \int_0^1 yt^2\underline{u}_j(s, t, r)dsdt, \\ \overline{u}_1(x, y, r) = \overline{g}_1(x, y, r) + \int_0^1 \int_0^1 xs\overline{u}_j(s, t, r)dsdt + \int_0^1 \int_0^1 yt^2\overline{u}_j(s, t, r)dsdt, \\ \underline{u}_2(x, y, r) = \underline{g}_2(x, y, r) + \int_0^1 \int_0^1 ys^2\underline{u}_j(s, t, r)dsdt + \int_0^1 \int_0^1 xt\underline{u}_j(s, t, r)dsdt, \\ \overline{u}_2(x, y, r) = \overline{g}_2(x, y, r) + \int_0^1 \int_0^1 ys^2\overline{u}_j(s, t, r)dsdt + \int_0^1 \int_0^1 xt\overline{u}_j(s, t, r)dsdt, \end{array} \right.$$

with

$$\begin{aligned} \underline{g}_1(x, y, r) &= (r^2 + r + 1)(xy - \frac{x}{6}) - \frac{y}{3}(r + 1)(e - 1), \\ \overline{g}_1(x, y, r) &= (4 - r)(xy - \frac{x}{6}) - y(3 - r)(e - 2), \\ \underline{g}_2(x, y, r) &= e^x(r + 1) - \frac{y}{8}(r^2 + r + 1) - \frac{x}{2}(r + 1)(e - 1), \\ \overline{g}_2(x, y, r) &= e^y(3 - r) - \frac{y}{8}(4 - r) - x(3 - r). \end{aligned}$$

One can easily verify that,

$$\begin{aligned} (\underline{u}_1(x, r), \overline{u}_1(x, r)) &= xy(r^2 + r + 1, 4 - r), \\ (\underline{u}_2(x, r), \overline{u}_2(x, r)) &= (e^x(r + 1), e^y(3 - r)), \end{aligned}$$

is an exact solution of the given problem.

The results are shown in Table 3 and Figs. 4 and 5. Table 3 shows the absolute errors $\underline{E}_i^m(x, y, r)$ and $\overline{E}_i^m(x, y, r)$, with $x = y = 0.1$, $m = 10$. We see that the proposed method is accurate for this example. Fig. 4 shows a comparison between the exact solution and the approximate solution by the presented method for $m = 5, 10, r = 0.5$. The dependence of the error function on m is depicted in Fig. 5. We see that the absolute error converges to zero as $m \rightarrow \infty$ (see the absolute error function of $u_1(x, y, r)$ in Fig. 5).

TABLE 3. Absolute errors $\underline{E}_i^m(x, y, r)$ and $\overline{E}_i^m(x, y, r)$, for Example 2, with $x = y = 0.1$, $m = 10$.

r	$\underline{E}_1^{10}(x, y, r)$	$\overline{E}_1^{10}(x, y, r)$	$\underline{E}_2^{10}(x, y, r)$	$\overline{E}_2^{10}(x, y, r)$
0.0	5.4686e-04	1.7266e-03	3.0589e-04	1.0481e-03
0.1	6.0175e-04	1.6698e-03	3.3714e-04	1.0153e-03
0.2	6.5705e-04	1.6129e-03	3.6970e-04	9.8258e-04
0.3	7.1276e-04	1.5560e-03	4.0358e-04	9.4984e-04
0.4	7.6887e-04	1.4991e-03	4.3877e-04	9.1710e-04
0.5	8.2539e-04	1.4423e-03	4.7528e-04	8.8435e-04
0.6	8.8232e-04	1.3854e-03	5.1311e-04	8.5161e-04
0.7	9.3965e-04	1.3285e-03	5.5225e-04	8.1887e-04
0.8	9.9739e-04	1.2716e-03	5.9271e-04	7.8613e-04
0.9	1.0555e-03	1.2148e-03	6.3448e-04	7.5338e-04

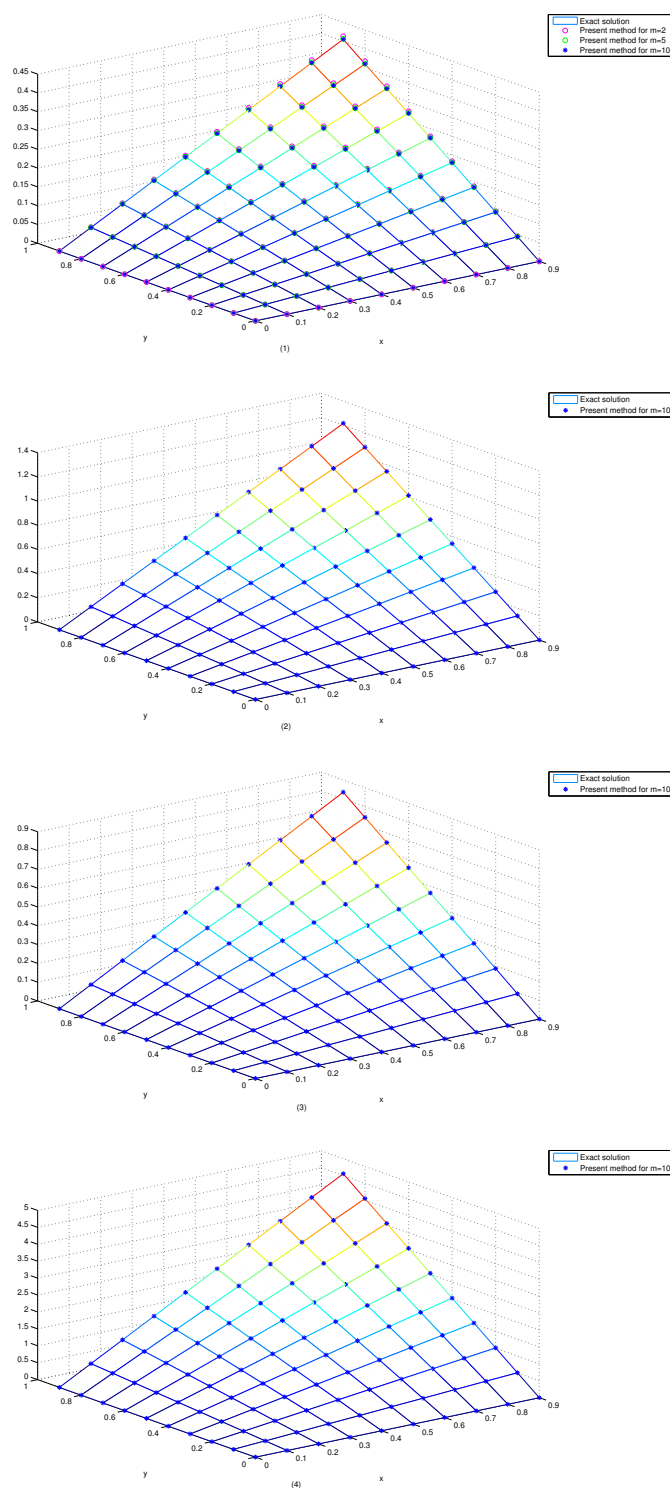


FIGURE 1. Comparison between the exact solution and the approximate solution by the present method of Example 1, with $r = 0.5$: (1): $\underline{u}_1(x, y, r)$. (2): $\overline{u}_1(x, y, r)$. (3): $\underline{u}_2(x, y, r)$. (4): $\overline{u}_2(x, y, r)$.

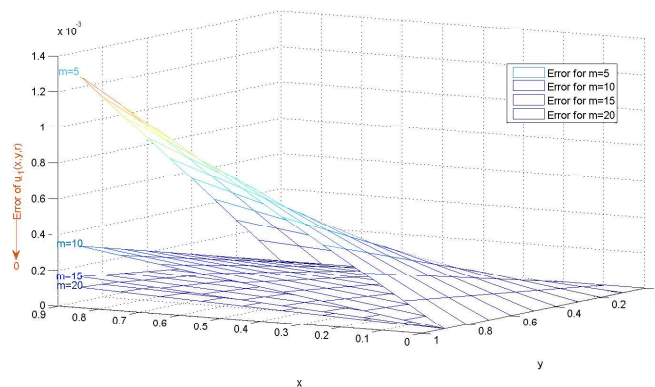


FIGURE 2. Compare the absolute error for $m = 5, 10, 15, 20$ of Example 1, for $\underline{u}_1(x, y, r)$ and $r = 0.5$.

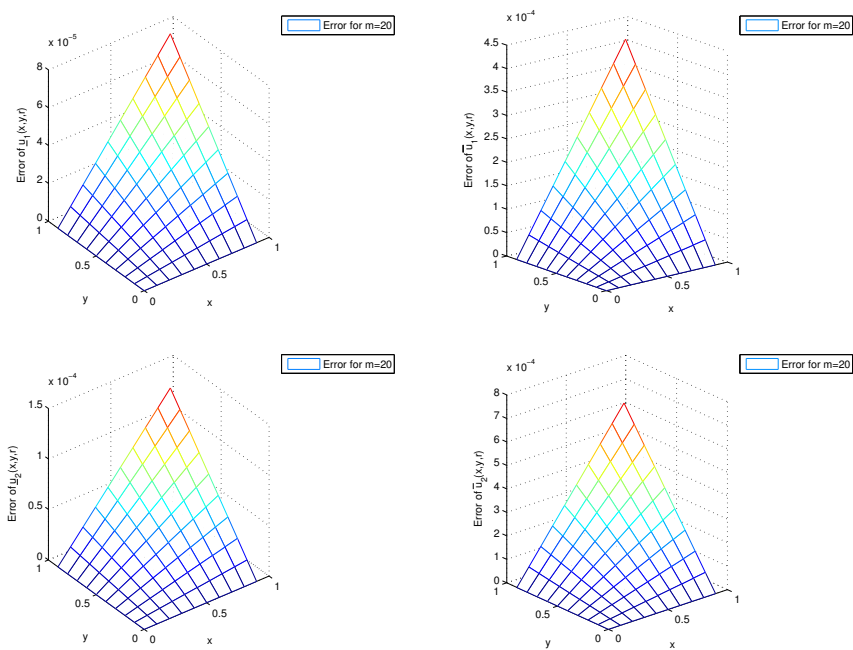


FIGURE 3. Absolute error functions by the present method of Example 1, with $r = 0.5$.

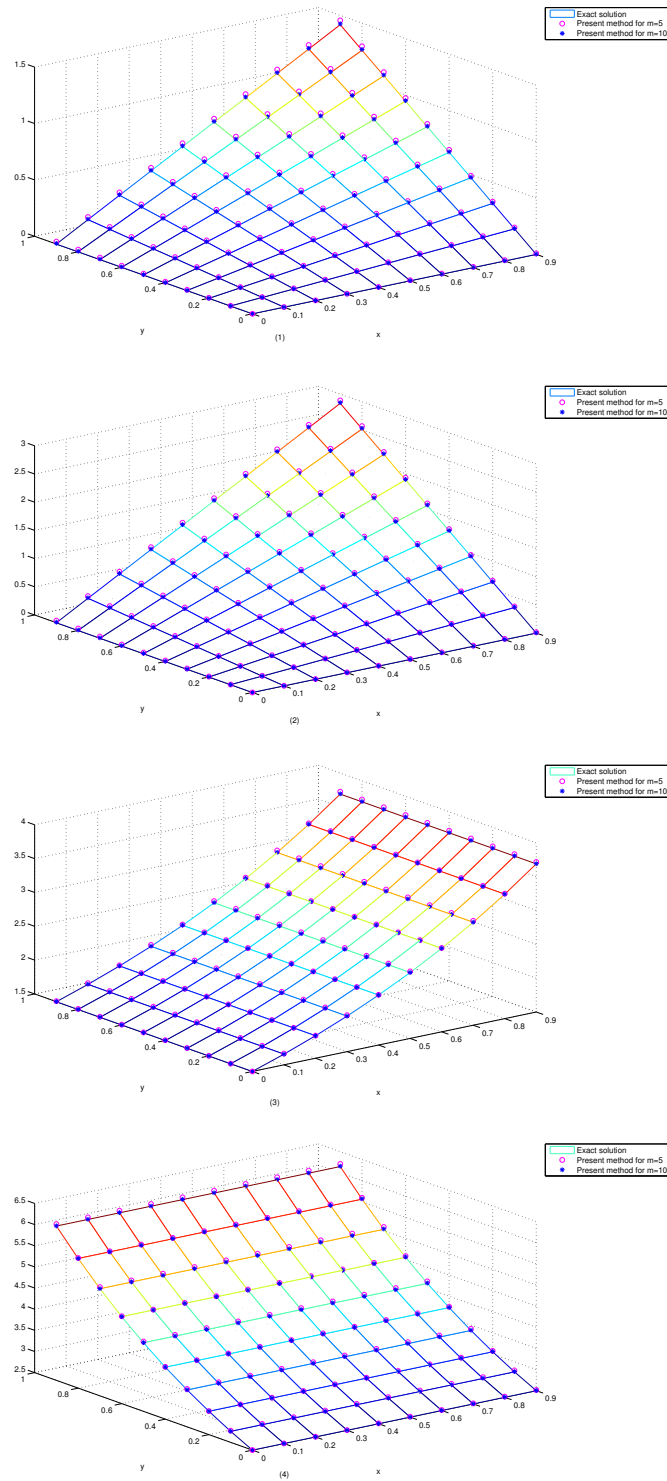


FIGURE 4. Comparison between the exact solution and the approximate solution by the present method in Example 2, with $r = 0.5$: (1): $\underline{u}_1(x, y, r)$. (2): $\overline{u}_1(x, y, r)$. (3): $\underline{u}_2(x, y, r)$. (4): $\overline{u}_2(x, y, r)$.

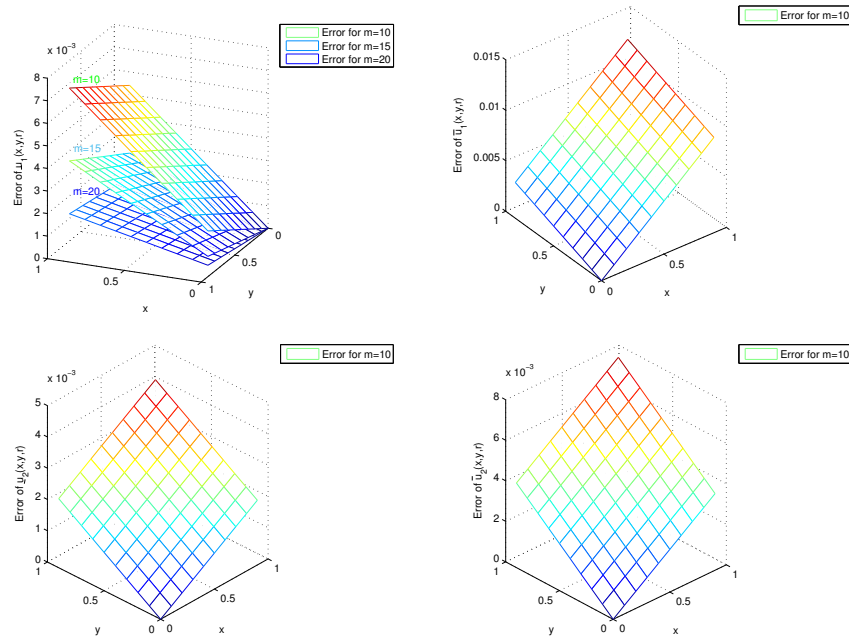


FIGURE 5. Absolute error functions by the present method of Example 2, with $r = 0.5$.

6. CONCLUSION

In this paper, we introduce TFs method for approximating the solution of the linear 2D-FFIES-2. The structural properties of TFs are utilized to reduce the 2D-FFIES-2 to a system of algebraic equations. The most important advantage of this method is low cost of setting up the equations without using any projection method such as Galerkin method, Collocation method, etc., and any integration. In the above presented numerical examples one can see that the proposed method well performs for linear 2D-FFIES-2. Furthermore, the proposed method can be run with increasing m until the results settle down to a suitable accuracy. Another direction for further research would be to extend the presented method to the systems of nonlinear 2D-FFIES-2, nonlinear mixed fuzzy Volterra–Fredholm integral equations, and fuzzy integro-differential equation.

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