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ON THE EXISTENCE RESULTS FOR (p,q)-KIRCHHOFF TYPE SYSTEMS WITH MULTIPLE PARAMETERS

SALEH SHAKERI [(1)] AND ARMIN HADJIAN [(2)]

ABSTRACT. In this paper, we are interested in the existence of positive solutions for the following nonlocal p-Kirchhoff problem of the type

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = \lambda a(x)v^\alpha - \mu & \text{in } \Omega,\\ -M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\Delta_q v = \lambda b(x)u^\beta - \mu & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $p,q>1$, $0<\alpha<p-1$, $0<\beta<q-1$, $M_i:\mathbb{R}_0^+\to\mathbb{R}^+$, $i=1,2$, are two continuous and increasing functions, λ,μ are two positive parameters, and $a,b\in C(\overline{\Omega})$.

1. Introduction

This paper is concerned with nonlocal Kirchhoff equation

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = \lambda a(x)v^\alpha - \mu & \text{in } \Omega,\\ -M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\Delta_q v = \lambda b(x)u^\beta - \mu & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $p,q>1$, $0<\alpha<p-1$, $0<\beta<q-1$, λ,μ are two positive parameters and the functions M_i,a,b satisfy the following conditions:

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(H_1) $M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$, $i = 1, 2$, are two continuous and increasing functions and $0 < m_i \le M_i(t) \le m_{i,\infty}$ for all $t \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ := [0, +\infty)$;

(H_2) $a, b \in C(\overline{\Omega})$ and $a(x) \ge a_0 > 0$, $b(x) \ge b_0 > 0$ for all $x \in \overline{\Omega}$.

problem (1.1) is called nonlocal because of the term $-M(\int_\Omega |\nabla u|^r dx)$ which implies that the first two equations in (1.1) are no longer pointwise equalities. This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Also, such a problem has physical motivation. Moreover, system (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \Big) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff [15]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of equation (1.2) is that the equation has a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L} \int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx$; hence the equation is no longer a pointwise identity. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of cross section, E is the Youngs modulus of the material, ρ is the mass density, and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert's equation. Later on, Woinowsky-Krieger (Nash-Modeer) incorporated this correction in the classical Euler-Bernoulli equation for the beam (plate) with hinged ends. See, for example, [5, 6] and the references therein. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [1, 8, 9, 13, 16, 21, 22], in which the authors have used different methods to prove the existence of solutions.

Nonlocal problems also appear in other fields, for example, biological systems where u and v describe a process which depends on the average of itself (for instance, population density); see [3, 4, 10, 17, 20] and the references therein.

We study problem (1.1) in the semipositone case. More precisely, under the conditions (H_1) and (H_2), using the sub- and supersolutions techniques, we establish positive constants μ^* and λ_* such that problem (1.1) has a positive solution when $\mu \le \mu^*$ and $\lambda \ge \lambda_*$. Our result in this note improves the previous one [2] in which $M_1(t) = M_2(t) \equiv 1$. It is clear that our situation in this present paper is different from [1] in which we studied the existence of positive solution for nonlocal equation with one parameter λ. To our best knowledge, this is an interesting and new research topic for singular (p,q)-Kirchhoff type systems.

To precisely state our existence result we first consider the following eigenvalue problem for the r-Laplace operator $-\Delta_r u$ (see [19]):

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Let $\phi_{1,r} \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (1.3) such that $\phi_{1,r} > 0$ in Ω and $\|\phi_{1,r}\|_\infty = 1$. It can be shown that $\frac{\partial \phi_{1,r}}{\partial \eta} < 0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m, δ, σ such that

$$\begin{cases} |\nabla \phi_{1,r}|^r - \lambda_{1,r}\phi_{1,r}^r \ge m & \text{on } \overline{\Omega}_\delta, \\ \phi_{1,r} \ge \sigma & \text{on } \Omega \backslash \overline{\Omega}_\delta, \end{cases} \tag{1.4}$$

where $\overline{\Omega}_\delta := \{x \in \Omega : \ d(x, \partial\Omega) \le \delta\}$.

We will also consider the unique solution $e_r \in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r e_r = 1 & \text{in } \Omega, \\ e_r = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

to discuss our result. It is known that $e_r > 0$ in Ω and $\frac{\partial e_r}{\partial \eta} < 0$ on $\partial\Omega$.

2. Existence of solutions

In this section, we will prove our result by using the method of sub- and supersolutions; we refer the readers to a recent paper [14] on the topic.

A pair of functions (ψ_1, ψ_2) is said to be a subsolution of problem (1.1) if it is in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx \le \int_\Omega (\lambda a(x)\psi_2^\alpha - \mu)\,w\,dx, \quad \forall w \in W,$$

and

$$M_2\left(\int_\Omega |\nabla\psi_2|^q\,dx\right)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx \le \int_\Omega (\lambda b(x)\psi_1^\alpha - \mu)\,w\,dx, \quad \forall w \in W,$$

where $W := \{w \in C_0^\infty(\Omega) :\ w \ge 0 \text{ in } \Omega\}$. A pair of functions $(z_1, z_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is said to be a supersolution if

$$M_1\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx \ge \int_\Omega (\lambda a(x) z_2^\alpha - \mu)\,w\,dx, \quad \forall w \in W,$$

and

$$M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx \ge \int_\Omega (\lambda b(x) z_1^\alpha - \mu)\,w\,dx, \quad \forall w \in W.$$

The following result plays an important role in our arguments. The readers may consult the papers [1, 11, 12] for details.

Lemma 2.1. *Assume that $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and increasing, and there exists $m_0 > 0$ such that $M(t) \ge m_0$ for all $t \in \mathbb{R}_0^+$. If the functions $u, v \in W_0^{1,r}(\Omega)$ satisfy*

$$M\left(\int_\Omega |\nabla u|^r\,dx\right)\int_\Omega |\nabla u|^{r-2}\nabla u\cdot\nabla\varphi\,dx \le M\left(\int_\Omega |\nabla v|^r\,dx\right)\int_\Omega |\nabla v|^{r-2}\nabla v\cdot\nabla\varphi\,dx,$$

for all $\varphi \in W_0^{1,r}(\Omega)$, $\varphi \ge 0$, then $u \le v$ in Ω.

From Lemma 2.1, we can establish the basic principle of the sub- and supersolutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = h(x,u,v) & \text{in } \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\Delta_q v = k(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h, k : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions

$(HK1)$ $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s,t belong to bounded sets.

$(HK2)$ There exists a function $g : \mathbb{R} \to \mathbb{R}$ being continuous, nondecreasing, with $g(0) = 0$, $0 \le g(s) \le C(1 + |s|^{\min\{p,q\}-1})$ for some $C > 0$, and applications $s \mapsto h(x,s,t) + g(s)$ and $t \mapsto k(x,s,t) + g(t)$ are nondecreasing, for a.e. $x \in \Omega$.

If $u, v \in L^\infty(\Omega)$, with $u(x) \le v(x)$ for a.e. $x \in \Omega$, we denote by $[u,v]$ the set $\{w \in L^\infty(\Omega) : u(x) \le w(x) \le v(x) \text{ for a.e. } x \in \Omega\}$. Using Lemma 2.1 and the method as in the proof of Theorem 2.4 of [18] (see also Section 4 of [7]), we can establish a version of the abstract lower and upper-solution method for our class of the operators as follows.

Proposition 2.1. *Let $M_1, M_2 : \mathbb{R}_0^+ \to \mathbb{R}^+$ be two functions satisfying the condition (H_1). Assume that the functions h, k satisfy the conditions (HK1) and (HK2). Assume that $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, are respectively, a weak subsolution and a weak supersolution of system (2.1) with $\underline{u}(x) \le \overline{u}(x)$ and $\underline{v}(x) \le \overline{v}(x)$ for a.e. $x \in \Omega$. Then there exists a minimal (u_*, v_*) (and, respectively, a maximal (u^*, v^*)) weak solution for system (2.1) in the set $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. In particular, every weak solution $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of system (2.1) satisfies $u_*(x) \le u(x) \le u^*(x)$ and $v_*(x) \le v(x) \le v^*(x)$ for a.e. $x \in \Omega$.*

Theorem 2.1. *Under the conditions (H_1)-(H_2), there exist positive constants μ^* and λ_* such that problem (1.1) has a positive solution when $0 < \mu \le \mu^*$ and $\lambda \ge \lambda_*$.*

Proof. Let $\lambda_{1,r}$, $\phi_{1,r}(r = p, q)$, and δ, m, σ, Ω_δ be as described in Section 1. We now construct our positive subsolution. Let

$$\psi_1 := \frac{p-1}{p}\phi_{1,p}^{\frac{p}{p-1}}, \qquad \psi_2 := \frac{q-1}{q}\phi_{1,q}^{\frac{q}{q-1}}.$$

We will verify that (ψ_1, ψ_2) is a subsolution of system (1.1). Indeed, we have by (H_1),

$$\begin{aligned}
\int_\Omega |\nabla \psi_1|^{p-2}\nabla\psi_1 \cdot \nabla w \, dx &= \int_\Omega |\nabla \phi_{1,p}|^{p-2}\phi_{1,p}\nabla\phi_{1,p} \cdot \nabla w \, dx \\
&= \int_\Omega |\nabla \phi_{1,p}|^{p-2}\nabla\phi_{1,p} \cdot \nabla(\phi_{1,p} w) \, dx - \int_\Omega |\nabla\phi_{1,p}|^p w \, dx \\
&= \int_\Omega \left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] w \, dx.
\end{aligned}$$

Similarly, we also have

$$\int_\Omega |\nabla \psi_2|^{q-2}\nabla\psi_2 \cdot \nabla w \, dx = \int_\Omega \left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] w \, dx.$$

Now, by (1.4), we have in $\overline{\Omega}_\delta$,

$$\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p \leq -m,$$

and

$$\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q \leq -m.$$

Therefore, if we choose $\mu \leq \mu^* := m \cdot \min\{m_1, m_2\}$, then it implies that in $\overline{\Omega}_\delta$,

$$\begin{aligned}
M_1\left(\int_\Omega |\nabla\psi_1|^p \, dx\right)\left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] &\leq m_1\left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] \\
&\leq \lambda a(x)\psi_2^\alpha - \mu,
\end{aligned}$$

and

$$\begin{aligned}
M_2\left(\int_\Omega |\nabla\psi_2|^q \, dx\right)\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] &\leq m_2\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] \\
&\leq \lambda b(x)\psi_1^\beta - \mu,
\end{aligned}$$

since $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$ for all $x \in \overline{\Omega}$.

On the other hand, in $\Omega\backslash\overline{\Omega}_\delta$, we have $\phi_{1,p} \geq \sigma > 0$ and $\phi_{1,q} \geq \sigma > 0$. Thus,

$$M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] \leq m_{1,\infty}\lambda_{1,p} \leq \lambda a(x)\psi_2^\alpha - \mu,$$

if

$$\lambda \geq \widehat{\lambda} := \frac{\left(m_{1,\infty}\lambda_{1,p} + m.\min\{m_1,m_2\}\right)\left(\frac{q}{q-1}\right)^\alpha}{a_0\sigma^{\frac{\alpha q}{q-1}}}. \tag{2.2}$$

Similarly,

$$M_2\left(\int_\Omega |\nabla\psi_2|^q\,dx\right)\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] \leq m_{2,\infty}\lambda_{1,q} \leq \lambda b(x)\psi_1^\beta - \mu,$$

if

$$\lambda \geq \widetilde{\lambda} := \frac{\left(m_{2,\infty}\lambda_{1,q} + m.\min\{m_1,m_2\}\right)\left(\frac{p}{p-1}\right)^\beta}{b_0\sigma^{\frac{\beta p}{p-1}}}. \tag{2.3}$$

From (2.2) and (2.3), we deduce that (ψ_1,ψ_2) is a subsolution if $\mu \leq \mu^*$ and $\lambda \geq \lambda_* := \max\{\widehat{\lambda},\widetilde{\lambda}\}$.

Next, we construct a supersolution (z_1,z_2) of system (1.1) such that $0 \leq \psi_i \leq z_i$ for $x \in \Omega$ and $i = 1,2$. We denote $(z_1,z_2) = (Ae_p, Be_q)$, where e_p, e_q are defined by (1.5), and constants $A, B > 0$ are large and to be chosen later. We will verify that (z_1,z_2) is a supersolution of system (1.1). To this end, let $w \in W$. We have

$$\begin{aligned}
&M_1\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx\\
&= A^{p-1}M_1\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega |\nabla e_p|^{p-2}\nabla e_p\cdot\nabla w\,dx\\
&= A^{p-1}M_1\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega w\,dx\\
&\geq m_1 A^{p-1}\int_\Omega w\,dx,
\end{aligned}$$

and

$$\begin{aligned}
&M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx\\
&= B^{q-1}M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |\nabla e_q|^{q-2}\nabla e_q\cdot\nabla w\,dx\\
&= B^{q-1}M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega w\,dx\\
&\geq m_2B^{q-1}\int_\Omega w\,dx.
\end{aligned}$$

Set $l_p = \|e_p\|_\infty$ and $l_q = \|e_q\|_\infty$, where e_p, e_q are defined by (1.5). Since $0 < \alpha < p-1$ and $0 < \beta < q-1$, it is easy to prove that there exist positive constants A, B such that

$$A \geq \left(\frac{\lambda\|a\|_\infty(Bl_q)^\alpha}{m_1}\right)^{\frac{1}{p-1}},$$

and

$$B \geq \left(\frac{\lambda\|b\|_\infty(Al_p)^\beta}{m_2}\right)^{\frac{1}{q-1}}.$$

Then, we have

$$\begin{aligned}
m_1A^{p-1} &\geq \lambda\|a\|_\infty(Bl_q)^\alpha\\
&\geq \lambda a(x)(Bl_q)^\alpha - \mu\\
&\geq \lambda a(x)(Be_q)^\alpha - \mu\\
&= \lambda a(x)z_2^\alpha - \mu,
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
m_2B^{q-1} &\geq \lambda\|b\|_\infty(Al_p)^\beta\\
&\geq \lambda b(x)(Al_p)^\beta - \mu\\
&\geq \lambda b(x)(Ae_p)^\beta - \mu\\
&= \lambda b(x)z_1^\beta - \mu.
\end{aligned} \tag{2.5}$$

From (2.4) and (2.5), it implies that

$$
\begin{aligned}
M_1\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx &\geq m_1A^{p-1}\int_\Omega w\,dx\\
&\geq \int_\Omega \left(\lambda a(x)z_2^\alpha-\mu\right)dx,
\end{aligned}\tag{2.6}
$$

and

$$
\begin{aligned}
M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx &\geq m_2B^{q-1}\int_\Omega w\,dx\\
&\geq \int_\Omega \left(\lambda b(x)z_1^\beta-\mu\right)dx.
\end{aligned}\tag{2.7}
$$

Relations (2.6) and (2.7) say that (z_1,z_2) is a supersolution of system (1.1). Moreover, from the definition of the subsolution (ψ_1,ψ_2) and supersolution (z_1,z_2) we can choose $A,B>0$ large enough so that $z_i\geq\psi_i$ for all $x\in\Omega$, $i=1,2$. Hence, using Proposition 2.1, it follows that system (1.1) has a positive solution $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ such that $\psi_1\leq u\leq z_1$ and $\psi_2\leq v\leq z_2$. The proof is complete. $\square$

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(1) Department of Mathematics, Ayatollah Amoli Branch, Islamic Azad University, Amol, Iran

E-mail address: s.shakeri@iauamol.ac.ir

(2) (Corresponding author) Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

E-mail address: hadjian83@gmail.com, a.hadjian@ub.ac.ir